On the finite sum of Kronecker sets

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Let G be a LCA group with dual $\hat{G}$, and E a compact subset of G. Following Rudin [5], we say that E is a Kronecker set if, to each $f \in C(E)$ with $|f|=1$ and $\varepsilon>0$, there exists $\gamma \in \hat{G}$ such that $\|f-\gamma\|_E<\varepsilon$. Suppose G is a metrizable I-group, K is a compact subset of G, and E is a perfect totally disconnected, compact metric space. Then, as is well-known, there exist two Kronecker sets K_1 and $K_2 \subset G$, both homeomorphic to E, such that $K_1+K_2 \supset K$ (cf. [3; Lemma 3.4], [6], and [7; Lemma 7.2]).

In this note we prove two analogs to the above result. I thank Professor S. Saeki for his useful advices.

THEOREM 1 (*cf.* [7]) *Let $T=\{|z|=1\}$ be the circle group, and T^∞ the countable cartesian product thereof. Let also E be a compact metric space with a perfect subset. Then there exist two Kronecker sets K_1 and $K_2 \subset T^\infty$, both homeomorphic to E, such that $K_1+K_2=T^\infty$.*

THEOREM 2 *Let G be a metrizable LCA I-group, $E \subset G$ a compact set, and $N \geq 2$ a natural number. Then there exist disjoint Kronecker sets $K_1, \cdots, K_N$, all homeomorphic to $D_2=\{0,1\}^\infty$, such that*

(i) *the sum $K_1+\cdots+K_N$ contains E, and*

(ii) *the union of any $N-1$ sets of the K_j's is a Kronecker set.*

Theorem 1 is an easy consequence of the following.

LEMMA 1 *Let E be a compact metric space, and $C(E; T^\infty)$ the space of all continuous mappings from E into T^∞. Then, given $f_0 \in C(E; T^\infty)$, there exists $f \in C(E; T^\infty)$ such that both $f(E)$ and $(f_0-f)(E)$ are Kronecker sets in T^∞ homeomorphic to E.*

PROOF Our proof follows Kaufman's idea in [2] (see also [1; pp. 184–185]). First notice that $C(E; T^\infty)$ forms a complete metric, topological group under the topology of uniform convergence. Since E is a compact metric space, $C(E; T)$ is separable. Let $\{g_n\}_{n=1}^\infty$ be a countable dense set in $C(E; T)$. We write $N=\bigcup_{m,n} A(m,n)$, where

$$A(m,n)=\left\{f \in C(E; T^\infty):\ \|g_m-\chi(f)\|_E \geq 1/n \text{ for all } \chi \in \hat{T}^\infty\right\}.$$

($\hat{T}^\infty$ denotes the dual group of T^∞). It is obvious that every $A(m,n)$ is closed in $C(E;\ T^\infty)$. Moreover we claim that $A(m, n)$ has no interior point. In fact, let $g \in C(E;\ T)$, $f \in C(E;\ T^\infty)$, and U a neighborhood of f in $C(E;\ T^\infty)$. We write $f=(f_1, f_2, \cdots)$, where $f_n \in C(E;\ T)$ is the nth component of f. By the definition of the product topology of T^∞, there exists a natural number n such that $h \in U$, where

$$h=(f_1, \cdots, f_n, g, f_{n+2}, f_{n+3}, \cdots).$$

Then, denoting by $\chi \in \hat{T}^\infty$ the canonical projection onto the $n+1$th factor, we have $\chi(h)=g$ on E. This proves that on $A(m, n)$ has interior points and N is therefore of first category in $C(E;\ T^\infty)$.

We now prove that every $f \in C(E;\ T^\infty)\backslash N$ is one-to-one. Choose any distinct points a_1 and $a_2 \in E$. Since $\{g_n\}_{n=1}^\infty$ is dense in $C(E;\ T)$, there exists $g_m \in \{g_n\}_{n=1}^\infty$ such that $g_m(a_1) \neq g_m(a_2)$. Since $f \notin N$, we can find $\chi \in \hat{T}^\infty$ so that

$$\left\| g_m - \chi(f) \right\|_E < 3^{-1} \left| g_m(a_1) - g_m(a_2) \right|.$$

Then

$$\begin{aligned} \left| g_m(a_1) - g_m(a_2) \right| &\leq 2 \left\| g_m - \chi(f) \right\|_E + \left| \chi(f(a_1)) - \chi(f(a_2)) \right| \\ &\leq (2/3) \left| g_m(a_1) - g_m(a_2) \right| + \left| \chi(f(a_1)) - \chi(f(a_2)) \right| \end{aligned}$$

which confirms the one-to-oneness of f. Hence we obtain that $f(E)$ is homeomorphic to E. Finally it is easy to see that $f(E)$ is a Kronecker set whenever $f \notin N$ by the definition of N. Since $C(E;\ T^\infty)$ is a complete topological group, $C(E;\ T^\infty)\backslash((f_0 - N) \cup N)$ is non-empty and every element of this set has the required property.

PROOF OF THEOREM 1

If we construct a continuous mapping from E onto T^∞, we shall have Theorem 1 by an application of Lemma 1. Since E is a compact metric space which contains a perfect subset, it contains a compact subset homeomorphic to D_2. Since D_2 is homeomorphic to the countable product space of D_2 with itself, and since $[0, 1]$ is a continuous image of D_2, the Tietze's extention theorem guarantees that $[0, 1]^\infty$ (and hence T^∞) is a continuous image of E. This comletes the proof.

To prove Theorem 2, we need a lemma.

LEMMA 2 Let G be a LCA group, $E \subset G$ a compact set, and n a natural number. Let also $V_i (1 \leq i \leq n)$ be open sets such that $\cup_{i=1}^n V_i \supset E$. Then there exist $W_i (1 \leq i \leq n)$, compact neighborhoods of 0, such that $\cup_{k=1}^n (w_k + V_k) \supset E$ for all $w_k \in W_k$.

PROOF First choose compact sets $K_i \subset V_i (1 \leq i \leq n)$ so that $\cup_{i=1}^n K_i \supset$

E. Next take a compact neighborhood W of 0 so that $K_i - W \subset V_i$ for all $1 \leq i \leq n$. Then $w_i \in W$ for $1 \leq i \leq n$ imply

$$\bigcup_{i=1}^n (w_i + V_i) \supset \bigcup_{i=1}^n K_i \supset E,$$

as was required.

PROOF OF THEOREM 2

To make the proof simple, we shall only prove Theorem 2 for $N=3$. Our proof is similar to that of Lemma 3.4 of [3].

For $r=1, 2, \cdots$, we construct a finite collection $\mathscr{K}_r$ of distinct compact sets in G with non-empty interior. First choose any compact sets $K_1, K_2, K_3 \subset G$ so that

$$\operatorname{int} K_1 + \operatorname{int} K_2 + \operatorname{int} K_3 \supset E,$$

and put $\mathscr{K}_r = \{K_r\}$ for $r=1, 2, 3$. Suppose that $\mathscr{K}_n = \{K_{nj}\}_{j=1}^{p(n)}$ are constructed for all $1 \leq n \leq r+2$ and some $r \geq 1$, and that

$$\operatorname{int} K_r + \operatorname{int} K_{r+1} + \operatorname{int} K_{r+2} \supset E,$$

where $K_n = \bigcup_{j=1}^{p(n)} K_{nj}$ for all n. Since E is compact, there exist distinct points $x_m \in \operatorname{int} K_r$ and $x'_m \in \operatorname{int} K_{r+1} (1 \leq m \leq n(r))$ such that $\bigcup_{m=1}^{n(r)} (x_m + x'_m + \operatorname{int} K_{r+2}) \supset E$. There is no loss of generality in assuming that all the K_{rj} (resp. $K_{(r+1)j}$) contain at least two x_m's (resp. x'_m's). Applying Lemma 2, we obtain disjoint compact neighborhoods $W_{r1}, \cdots, W_{rn(r)} \subset \operatorname{int} K_r$ and $W'_{r1}, \cdots, W'_{rn(r)} \subset \operatorname{int} K_{r+1}$ of these points such that

$$\bigcup_{m=1}^{n(r)} (w_m + w'_m + \operatorname{int} K_{r+2}) \supset E$$

for all choices of $w_m \in W_{rm}$ and $w'_m \in W'_{rm}$. Since G is an I-group, we can find $v_m \in \operatorname{int} W_{rm}$ and $v'_m \in \operatorname{int} W'_{rm}$ so that $\{v_m, v'_m : 1 \leq m \leq n(r)\}$ is a Kronecker set (see [5 ; 5.2]). Choose a finite set F_r of $\hat{G}$ so that to any real numbers a_m and $a'_m (1 \leq m \leq n(r))$ there corresponds a $\gamma \in F_r$ satisfying

$$(1) \quad \left|\exp(ia_m) - \gamma(v_m)\right| < 1/r \qquad (1 \leq m \leq n(r))$$

$$(1)' \quad \left|\exp(ia'_m) - \gamma(v_m)\right| < 1/r \qquad (1 \leq m \leq n(r))$$

Since F_r is a finite set, there exist disjoint compact neighborhoods $K_{(r+3)m}$ of v_m and $K_{(r+4)m}$ of v'_m such that $1 \leq m \leq n(r)$ imply

$$(2) \quad \left|\gamma(x) - \gamma(v_m)\right| < 1/r \qquad (x \in K_{(r+3)m} \text{ and } \gamma \in F_r)$$

$$(2)' \quad \left|\gamma(x) - \gamma(v'_m)\right| < 1/r \qquad (x \in K_{(r+4)m} \text{ and } \gamma \in F_r)$$

$$(3) \quad \operatorname{diam} K_{(r+3)m} < 1/r \text{ and } K_{(r+3)m} \subset W_{rm}$$

$$(3)' \quad \operatorname{diam} K_{(r+4)m} < 1/r \text{ and } K_{(r+4)m} \subset W'_{rm}.$$

Finally we define $\mathscr{K}_{r+k}=\{K_{(r+k)m}\}_{m=1}^{n(r)}$ for $k=3$ and 4, which completes our inductive construction of $\{\mathscr{K}_r\}_{r=1}^{\infty}$.

Putting

$$H_k=\bigcap_{q=0}^{\infty}\bigcup_m K_{(3q+k)m} \qquad (k=1,2,3),$$

we claim that these three sets have the required properties. It is obvious that H_1, H_2, H_3 are disjoint, perfect, and totally disconnected, and that $H_1+H_2+H_3\supset E$. Therefore we need only confirm that, say, $H_1\cup H_2$ is a Kronecker set.

Let $f\in C(H_1\cup H_2;T)$ and $\varepsilon>0$ be given. Choose a natural number q so that $1/(q-1)<\varepsilon/2$, and set $r=3q-2$. Since f is uniformly continuous, we can demand that there are real numbers a_m and a'_m $(1\leq m\leq n(r))$ satisfying

$$(4)\quad \left|f(x)-\exp(ia_m)\right|<\varepsilon/2 \qquad (x\in H_1\cap K_{(r+3)m} \text{ and } 1\leq m\leq n(r))$$

$$(4)'\quad \left|f(x)-\exp(ia'_m)\right|<\varepsilon/2 \qquad (x\in H_2\cap K_{(r+4)m} \text{ and } 1\leq m\leq n(r)).$$

(Notice $\bigcup_{m=1}^{n(r)}K_{(r+3)m}=\bigcup_{m=1}^{n(r)}K_{(3q+1)m}\supset H_1$ and similarly for H_2.)
Choose $\gamma\in F_r$ as in (1) and (1)'. We then have by (2) and (4) that

$$\left|f(x)-\gamma(x)\right|\leq\left|f(x)-\exp(ia_m)\right|+\left|\exp(ia_m)-\gamma(v_m)\right|+\left|\gamma(v_m)-\gamma(x)\right|$$
$$<\varepsilon/2+1/r+1/r<\varepsilon/2+1/(q-1)<\varepsilon$$

whenever $x\in H_1\cap K_{(r+3)m}$ for some $1\leq m\leq n(r)$.
Similarly we have by (2)' and (4)'

$$\left|f(x)-\gamma(x)\right|<\varepsilon \qquad (x\in H_2).$$

In other words, we have proved that $|f(x)-\gamma(x)|<\varepsilon$ for all $x\in H_1\cup H_2$ and some $\gamma\in\hat{G}$. This completes the proof.

REMARK After the first draft of this note was written, Professer S. Saeki pointed out that the following variance of Kaufman's theorem [2] yields an alternative and simple proof of Theorem 2.

Let G be a metrizable LCA I-group, H a σ-compact independent subset thereof, and D a totally disconnected compact metric space. Then quasi-all $f\in C(D;G)$ have the properties that

(i) f is one-to-one,
(ii) $f(D)$ is a Kronecker set, and
(iii) $Gp(f(D))\cap Gp(H)=\{0\}$.

If, in addtion, H is a totally disconnected Kronecker set, then (ii) can be strengthened to be (ii)′ $f(D)\cup H$ is a Kronecker set. (cf. [6; Lemma]). Theorem 2 follows from an inductive application of this result. We omit the details.

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