U-rational extension of a ring

By Kenji NISHIDA

(Received June 26, 1975)

Introduction.

Let R be a ring with identity and U be a right R-module such that $R \subset \Pi E(U) = C$ where $E(U)$ is the injective hull of U. Then the double centralizer of C is a ring S and is a U-rational extension of R as a right R-module. A ring S is regarded as a subring of a maximal right quotient ring of R.

In [5], K. Masaike states a characterization of a ring of which a canonical inclusion of R into a maximal quotient ring is a right flat epimorphism. We will generalize this result for a canonical inclusion of R into S.

Throughout this paper, a ring R has always an identity element and an R-module is unital. An injective hull of an R-module M is written by $E(M)$. Let X and Y be the right R-modules. We say X is Y-torsionless if X is embeddable into some product of Y, i.e., $X \subset \Pi Y$. This is equivalent that for any nonzero $x \in X$ there exists an R-homomorphism f of X into Y such that $f(x) \neq 0$.

1. U-rational extension of a ring

Let U be a right R-module such that $E(U)$ is faithful. Then we have $R \subset \Pi E(U)$. We put $C = \Pi E(U)$, $H = \mathrm{Hom}_R(C, C)$. Then C becomes a bimodule ${}_H C_R$, thus we get $S = \mathrm{Hom}_H(C, C)$ the double centralizer of C_R.

PROPOSITION 1. *C is injective as a right S-module,* $\mathrm{Hom}_R(C, C) = \mathrm{Hom}_S(C, C)$, *and if B_R is a direct summand of C_R, then B is a right S-module and also a direct summand of C as a right S-module.*

PROOF. This is well-known (see [3], [4] for example), but for the completeness, we state the proof.

Let $0 \to X \to Y$ be an exact sequence of right S-modules, and f be an S-homomorphism of X into C. Since C_R is injective, f can be extended to $g: Y_R \to C_R$. We will show that g is an S-homomorphism.

For any $y \in Y$, define the mapping $k_y: S \to C$ by $k_y(s) = g(ys) - g(y)s$ for $s \in S$. This is clearly an R-homomorphism and can be extended to $k'_y \in H$ by injectivity of C_R. Then $k'_y(R) = k_y(R) = 0$, therefore, $k_y(s) = k'_y(s) = k'_y((1)s) = (k'_y(1))s = 0$ (here we use the canonical embedding of S_R into C_R; $s \mapsto (1)s$).

Thus g is an S-homomorphism, so C is injective as a right S-module.

Next, we obtain trivially $\mathrm{Hom}_S(C, C)\subset\mathrm{Hom}_R(C, C)$, and because ${}_HC_S$ is a bimodule, equality holds.

Finally, we shall show if B_R is a direct summand of C_R, then B is a right S-module and also a direct summand of C as a right S-module. Let $C=B\oplus A$ where A is an R-submodule of C. Take any $b\in B$, $s\in S$ and let p be a canonical projection from C onto A. Then $p((b)s)=(p(b))s=0$, thus, $(b)s\in B$ for any $b\in B$ and $s\in S$. This means that B is a right S-module. By the same way, A is also a right S-module, therefore, B is a direct summand of C as a right S-module.

Let M be a right R-module and N be a submodule of M. Following Findlay and Lambek [2], we call M a U-rational extension of N if $\mathrm{Hom}_R(M'/N, U)=0$ for any submodule M' of M that contains N. This is equivalent to $\mathrm{Hom}_R(M/N, E(U))=0$ by Proposition 2. 1 of [2].

PROPOSITION 2. *S is a U-rational extension of R as a right R-module. If an R-submodule T of C is a U-rational extension of R, then $T\subset S$.*

PROOF. The first assertion follows easily from the proof of Theorem 2 of [3].

In order to prove the second part, we shall show that $h(t)=0$ for any $t\in T$ and $h\in H$ such that $h(R)=0$. If $h(t)\neq 0$, then there exists $f: C\to E(U)$ such that $f(h(t))\neq 0$. Put $g=fh|_T: T\to E(U)$. Then we have

$$g(t)=f(h(t))\neq 0$$

and

$$g(R)=f(h(R))=0.$$

Since T is a U-rational extension of R, this is a contradiction. Thus, $h(t)=0$ and then we have $t\in S$.

2. Flat epimorphism

We shall begin this section with stating some definitions and notations. In what follows, let R, S and U be as in the previous section.

DEFINITIONS. *Let $A(B)$ be a right ideal of $R(S)$. Then we call $A(B)$ U-dense if* $\mathrm{Hom}_R(R/A, E(U))=0$ $(\mathrm{Hom}_S(S/B, E(U))=0)$.

When A is a right ideal of R, A is U-dense if and only if R is a U-rational extension of A as right R-module, but since U is not a right S-module, we take above definitions.

For any right $R(S)$-module $M(N)$, we put

$$V_R(M)=\{m\in M\,;\ mA=0 \text{ for some } U\text{-dense right ideal } A \text{ of } R\}$$

$$(V_S(N)=\{n\in N;\ nB=0 \text{ for fome } U\text{-dense right ideal } B \text{ of } S\}).$$

LEMMA 1. *If A is a U-dense right ideal of R, then*

$$A:s=\{r\in R\,;\ sr\in A\}$$

is a U-dense right ideal of R for any $s\in S$.

PROOF. Consider a map $f:R\to S$ that is defined by $f(r)=sr$, $r\in R$. Then f is an R-homomorphism. Thus, $A:s=f^{-1}(A)$. Since S is a U-rational extension of R,S is also a U-rational extension of A by Pproposition 1. 3 of [2]. Therefore, $R=f^{-1}(S)$ is a U-rational extension of $f^{-1}(A)=A:s$ by Proposition 2. 2 of [2]. Thus, $A:s$ is U-dense.

LEMMA 2. (i) *B is a U-dense right ideal of S if and only if $B\cap R$ is a U-dense right ideal of R.*

(ii) *If A is a U-dense right ideal of R, then AS is a U-dense right ideal of S.*

PROOF. (i) Assume that B is a U-dense right ideal of S. If there exists nonzero R-homomorphism $f:R/(B\cap R)\to E(U)$, then it can be extended to $f':S/B\to E(U)$ by injectivity of $E(U)$. By the same way as in Proposition 1, f' becomes an S-homomorphism and nonzero. This is a contradiction. Thus, $\mathrm{Hom}_R(R/(B\cap R), E(U))=0$.

The converse is trivial by $R/(B\cap R)\cong(R+B)/B$.

(ii) Trivial by (i) and $A\subset AS\cap R$.

LEMMA 3. *If M is a right S-module, then $V_S(M)=V_R(M)$.*

PROOF. This follows from lemma 3.

Now, next Proposition 3 and 4 are generalization of K. Masaike ([5]. Proposition 1 and 3).

PROPOSITION 3. *A right R-module M is $E(U)$-torsionless if and only if $V_R(M)=0$.*

PROOF. Assume that M is $E(U)$-torsionless. Let $0\neq x\in E(U)$ and A be a U-dense right ideal of R. Consider an R-homomorphism $f:R\to E(U)$ such that $f(r)=xr\,(r\in R)$. If $xA=0$, then f induces a nonzero homomorphism $f':R/A\to E(U)$. This is a contradiction. Thus, $V_R(M)=0$. Conversely, assume $V_R(M)=0$, then, for any nonzero $x\in M$, $A=\{r\in R\,;\ xr=0\}$ is not a U-dense right ideal of R. Thus there exists a nonzero homomorphism $g:R\to E(U)$ such that $g(A)=0$. On the other hand, $R/A\cong xR$, so there exists canonically a homomorphism $h:xR\to E(U)$ such that $h(xr)=g(r)$. It can be extended to $h': M\to E(U)$ by injectivity of $E(U)$. Thus, $h'(x)=h(x)=g(1)\neq 0$. Hence, M is $E(U)$-torsionless.

Let T be a ring extension of R. Then we call a canonical inclusion of R into T a right flat epimorphism if ${}_RT$ is flat and $T\otimes T\cong T$ canonically (we always form a tensor product as R-modules).

PROPOSITION 4. *A canonical inclusion of R into S is a right flat epimorphism if and only if $M\otimes S$ is $E(U)$-torsionless as right S-module for every (finitely generated) $E(U)$-torsionless right R-module M.*

PROOF. Assume that a canonical inclusion of R into S is a right flat epimorphism. We have $E(U)_S\cong E(U)\otimes S_S$ by Corollary 1.3 of [6].

Now, we shall prove that a canonical mapping $M\to M\otimes S$ ($m\mapsto m\otimes 1$) is a monomorphism for any $E(U)$-torsionless module M_R. If some nonzero $m\in M$, $m\otimes 1=0$, then there exists $f: M\to E(U)$ such that $f(m)\neq 0$. The homomorphism f induces $f\otimes\mathrm{Id}: M\otimes S\to E(U)\otimes S\cong E(U)$. Then $0\neq f(m)\otimes 1=(f\otimes\mathrm{Id})(m\otimes 1)=0$. This is a contradiction. Thus, by Proposition 1.7 of [6] $M\otimes S$ is an essential extension of M as an R-module. Therefore, $M\otimes S$ is $E(U)$-torsionless as an R-module. By assumption, for any right S-modules K and K', $\mathrm{Hom}_R(K, K')=\mathrm{Hom}_S(K, K')$. Thus, $M\otimes S$ is $E(U)$-torsionless as a right S-module.

For the converse, we shall show that ${}_RS$ is flat and the canonical mapping $S\otimes S\to S$ is an isomorphism.

If we show $A\otimes S\cong AS$ canonically for any finitely generated right ideal A of R, then the flatness of ${}_RS$ follows from section 5.4 Proposition 1 of [4]. Thus, we will show that a canonical mapping $i: A\otimes S\to S$ is a monomorphism. Let $u=\sum a_k\otimes s_k\in A\otimes S$ and $\sum a_k s_k=0$. Put $B=\bigcap_k R: s_k$. Then by lemma 2, B is a U-dense right ideal of R. For any $b\in B$ $ub=\sum a_k\otimes s_k b=\sum a_k s_k b\otimes 1=0$. Thus, $uB=0$ so $u\in V_S(A\otimes S)$. But $A\subset R\subset \Pi E(U)$ implies that $A\otimes S$ is $E(U)$-torsionless as an S-module. By Proposition 3 $V_S(A\otimes S)=0$. Therefore, $u=0$. Thus, ${}_RS$ is flat.

Next we will show $V_S(S\otimes S)=0$. Let $\sum s_k\otimes s'_k\in V_S(S\otimes S)$, and $K=s_1R+s_2R+\cdots+s_nR$. By the flatness of ${}_RS$, we have $K\otimes S\subset S\otimes S$. Therefore $\sum s_k\otimes s'_k\in V_S(K\otimes S)$. On the other hand, $K\subset S\subset \Pi E(U)$ and K is finitely generated, so by assumption $V_S(K\otimes S)=0$. Thus, $\sum s_k\otimes s'_k=0$. Therefore, $V_S(S\otimes S)=0$. If $u=\sum s'_k\otimes s_k\in S\otimes S$ and $\sum s_k s'_k=0$, then as above $u\in V_S(S\otimes S)=0$. Therefore, $u=0$. Thus, the canonical mapping of $S\otimes S$ onto S is a monomorphism, whence an isomorphism.

References

[1] C. FAITH: Lectures on injective modules and quotient rings, Springer-Verlag (1967).

[2] G. D. FINDLAY and J. LAMBEK: A generallized ring of quotients I, Can. Math. Bull. 1 (1958), 77–85.

[3] T. KATO: Rings of U-dominant dimension$\geqq 1$, Tohoku Math. J. 21 (1969), 321–327.

[4] J. LAMBEK: Lectures on rings and modules, Blaisdell (1966).

[5] K. MASAIKE: On quotient rings and torsionless modules, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A 11 (1971), 26–31.

[6] L. SILVER: Noncommutative localizations and applications, J. Algebra 7 (1967), 44–76.

Department of Mathematics
Hokkaido University