Complex parallelizable manifolds

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§ 1. **Introduction.** H. C. Wang studied a complex parallelizable manifold, which can be considered as the natural counterpart, in the theory of complex manifolds, of an ordinary completely parallelizable manifold, and obtained the following result [2][1]:

A compact complex parallelizable manifold can be regarded, up to a holomorphic homeomorphism, as the the compact coset space of a complex Lie group by a discrete isotropic subgroup and becomes Kählerian if and only if it is a complex torus.

The purpose of the present paper is to study the geometric properties of complex parallelizable manifolds without assumption of compactness. The method of investigation is the same as in the theory of extended Lie systems [5], [6], that is to express all the geometric quantities in terms of the scalars of structure C_{bc}^{a} and their conjugates (see § 2). We introduce, in § 3, a Riemannian metric g on a complex parallelizable manifold M and show that a condition for M to be Kählerian is that $C_{bc}^{a}=0$. Consequently the above result of Wang can be stated as follows:

A connected complete complex parallelizable manifold M can be regarded as the coset space of a complex Lie group by a discrete subgroup if and only if all the scalars C_{bc}^{a} are constant. And M is Kählerian with respect to the g if and only if $C_{bc}^{a}=0$.

The real version of the above first statement, of course, holds good [3]. In § 4, we deal with the special vector fields i. e. Killing, conformal Killing, divergence-free and harmonic vector fields. As a consequence we have that some special parallelizations give rise to $C_{bc}^{a}=0$. Finally we prove in § 5 that any holomorphic sectional curvature is non-positive at every point of M and so is the scalar curvature.

§ 2. **Complex parallelization.** Let M be an n-dimensional complex manifold with a complex structure J. Then M is called a complex parallelizable manifold if there exist, on M, n holomorphic vector fields linearly independent everywhere. We denote the vector fields and their complex conjugates by $Z_1, Z_2, \cdots, Z_n$ and $\bar{Z}_1, \bar{Z}_2, \cdots, \bar{Z}_n$ respectively. We set

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1) Numbers in brackets refer to the references at the end of the paper.

$$(2.1) \qquad\qquad \bar{Z}_a = Z_{\bar{a}} \qquad (a = 1, 2, \cdots, n),$$

and so we can use later the convention that $\bar{a} = a$.

As $\{Z_1, Z_2, \cdots, Z_n, Z_{\bar{1}}, Z_{\bar{2}}, \cdots, Z_{\bar{n}}\}$ is clearly a basis for the complex tangent space $T_x^C(M)$ at each point $x \in M$, we call it the complex basic frame of M. Further, we denote the dual of the above basis by $\{\omega^1, \omega^2, \cdots, \omega^n, \omega^{\bar{1}}, \omega^{\bar{2}}, \cdots, \omega^{\bar{n}}\}$, where $\bar{\omega}^a = \omega^{\bar{a}}$, and call it the complex basic coframe of M.

Since the set of all holomorphic vector fields on M forms a complex Lie algebra over the field of complex numbers, there exist holomorphic functions C_{bc}^a on M such that

$$(2.2) \qquad\qquad [Z_b, Z_c] = C_{bc}^a Z_a,$$

where the usual summation convention is used. In the sequel, this convention will be used unless otherwise stated. We call the above-mentioned functions C_{bc}^a the scalars of structure for the complex parallelization. It follows from the definition of Poisson bracket and (2.2) that

$$(2.3) \qquad\qquad C_{bc}^a + C_{cb}^a = 0,$$

$$(2.4) \qquad \partial_b C_{cd}^a + \partial_c C_{db}^a + \partial_d C_{bc}^a - C_{bc}^e C_{ed}^a - C_{cd}^e C_{eb}^a - C_{db}^e C_{ec}^a = 0,$$

where $\partial_b C_{cd}^a = Z_b C_{cd}^a$. These identities are fundamental.

From now on, we use indices as follows:
Small Latin indices $a, b, c, \cdots$ run from 1 to n, while capital indices $A, B, C, \cdots$ run through $1, 2, \cdots, n, \bar{1}, \bar{2}, \cdots, \bar{n}$.

We know that

$$\overline{[Z_b, Z_c]} = [Z_{\bar{b}}, Z_{\bar{c}}], \qquad \overline{C_{bc}^a} = C_{\bar{b}\bar{c}}^{\bar{a}}, \qquad \overline{\partial_b C_{cd}^a} = \partial_{\bar{b}} C_{\bar{c}\bar{d}}^{\bar{a}},$$

from which it follows that the conjugates of (2.2), (2.3) and (2.4) hold good. And we have $[Z_b, Z_{\bar{c}}] = 0$. Thus we can set

$$(2.5) \qquad\qquad [Z_B, Z_C] = C_{BC}^A Z_A,$$

where the only non-zero elements of C_{BC}^A are those of the types C_{bc}^a and $C_{\bar{b}\bar{c}}^{\bar{a}}$. Now we shall give a real parallelization on M. Since Z_a is of type $(1, 0)$ and $Z_{\bar{a}}$ of type $(0, 1)$, if we put

$$(2.6) \qquad\qquad X_a = Z_a + Z_{\bar{a}}$$

we have

$$(2.7) \qquad\qquad JX_a = \sqrt{-1}(Z_a - Z_{\bar{a}})$$

Then it is easily verified that the vector fields $X_1, X_2, \cdots, X_n, JX_1, JX_2, \cdots$.

JX_n are all infinitesimal automorphisms of J [1] and are linearly independent everywhere, and consequently they define a real parallelization. We call $\{X_a, JX_a\}$ and its dual $\{\eta^a, J\eta^a\}$ $(a = 1, 2, \cdots, n\}$ the real basic frame and coframe respectively. In this case, from the above definition η^a and (2.6), (2.7) it turns out that

$$(2.8) \qquad \eta^a = \frac{1}{2}(\omega^a + \omega^{\bar{a}}), \qquad J\eta^a = -\frac{\sqrt{-1}}{2}(\omega^a - \omega^{\bar{a}}).$$

§3. Riemannian connection.

First, we shall introduce a Riemannian metric g on M so that the real basic frame be orthonormal. For this, we define the g as follows:

$$(3.1) \qquad \begin{aligned} g(X_a, X_b) &= g(JX_a, JX_b) = \delta_{ab}, \\ g(X_a, JX_b) &= g(JX_a, X_b) = 0. \end{aligned}$$

From (2.6) and (2.7) we have

$$(3.2) \qquad Z_a = \frac{1}{2}(X_a - \sqrt{-1}\,JX_a), \qquad Z_{\bar{a}} = \frac{1}{2}(X_a + \sqrt{-1}\,JX_a).$$

Put $g_{AB} = g(Z_A, Z_B)$. Then it follows from (3.1) and (3.2) that

$$(3.3) \qquad g_{ab} = g_{\bar{a}\bar{b}} = 0, \qquad g_{\bar{a}b} = g_{a\bar{b}} = \frac{1}{2}\delta_{ab}.$$

In this case, it is easily seen by virtue of (3.1) that the g is a Hermitian metric. Thus we have

PROPOSITION 1. *A metric* g *defined by* (3.1) *becomes a Kähler metric if and only if all the scalars of structure* C^a_{bc} *are equal to zero.*

PROOF. The fundamental 2-form Φ of M is defind by

$$(3.4) \qquad \Phi(X, Y) = g(JX, Y) \quad \text{for all vector fields } X \text{ and } Y.$$

In this case, a necessary and sufficient condition for the g to be Kählerian is given by

$$d\Phi(Z_A, Z_B, Z_C) = 0 \quad \text{for all indices } A, B \text{ and } C.$$

On use of a well-known fromula [1], we have

$$(3.5) \qquad 3d\Phi(Z_A, Z_B, Z_C) = Z_A \Phi(Z_B, Z_C) + Z_B \Phi(Z_C, Z_A) + Z_C \Phi(Z_A, Z_B)$$
$$-\Phi\big([Z_A, Z_B], Z_C\big) - \Phi\big([Z_B, Z_C], Z_A\big) - \Phi\big([Z_C, Z_A], Z_B\big).$$

From (2.5), (3.3), (3.4) and (3.5), we have

(3. 6) $$3d\Phi(Z_{\bar{a}}, Z_b, Z_c) = -\frac{\sqrt{-1}}{2} C^a_{bc}.$$

Similarly we see that $d\Phi(Z_a, Z_b, Z_c)$ and $d\Phi(Z_{\bar{a}}, Z_{\bar{b}}, Z_{\bar{c}})$ vanish and the others are of the same type as (3.6). This proves our assertion.

Next, we shall express concretely the components of the Riemannian connection with respect to the complex basic frame in terms of the scalars of structure and their conjugates. We denote by ∇ the covariant differentiation and set

(3. 7) $$\nabla_{Z_B} Z_C = \Gamma^A_{BC} Z_A.$$

We requier

(3. 8) $$\bar{\Gamma}^A_{BC} = \Gamma^{\bar{A}}_{\bar{B}\bar{C}}$$

with the convention that $\bar{\bar{a}} = a$. Since the connection Γ is torsion-free, the torsion tensor T together with (2.5) and (3.7) yields

$$T(Z_B, Z_C) = \Gamma^A_{BC} Z_A - \Gamma^A_{CB} Z_A - C^A_{BC} Z_A = 0,$$

which implies

(3. 9) $$\Gamma^A_{BC} - \Gamma^A_{CB} = C^A_{BC}.$$

Besides, as the Γ is a metric connection, i.e. $\nabla g = 0$, we have

(3. 10) $$g_{AE}\Gamma^E_{CB} + g_{BE}\Gamma^E_{CA} = 0,$$

which is, by virtue of (3.3), equivalent to

(3. 11) $$\Gamma^A_{CB} + \Gamma^{\bar{B}}_{CA} = 0.$$

If we denote by (g^{AB}) the inverse of the matrix (g_{AB}), it is easily seen that the matrix (g^{AB}) has the following components:

(3. 12) $$g^{ab} = g^{\bar{a}\bar{b}} = 0, \qquad g^{\bar{a}b} = g^{a\bar{b}} = 2\delta_{ab}.$$

Using (3.9) cyclically and applying (3.10), we have

(3. 13) $$\Gamma^A_{BC} = \frac{1}{2} g^{AE}(g_{CD}C^D_{EB} + g_{ED}C^D_{BC} - g_{BD}C^D_{CE}).$$

Further, noticing that $C^A_{BC} = 0$ except elements of the types C^a_{bc} and $C^{\bar{a}}_{b\bar{c}}$ and making use of (3.3), (3.12), (3.13), we calculate $\Gamma^a_{bc}, \Gamma^{\bar{a}}_{bc}$ and get

$$\Gamma^a_{bc} = \frac{1}{2} C^a_{bc}, \qquad \Gamma^{\bar{a}}_{bc} = 0.$$

In this case, the other types of Γ^A_{BC} are obtained from (3.8), (3.9) and (3.11). Thus we have the following list:

(3.14)

$$\Gamma^a_{bc} = \frac{1}{2}C^a_{bc}, \qquad \Gamma^a_{\bar{b}\bar{c}} = 0, \qquad \Gamma^a_{\bar{b}c} = \frac{1}{2}\bar{C}^c_{ab}, \qquad \Gamma^a_{b\bar{c}} = \frac{1}{2}\bar{C}^b_{ac}$$

$$\Gamma^{\bar{a}}_{bc} = 0, \qquad \Gamma^{\bar{a}}_{\bar{b}\bar{c}} = \frac{1}{2}\bar{C}^a_{bc}, \qquad \Gamma^{\bar{a}}_{\bar{b}c} = \frac{1}{2}C^b_{ac}, \qquad \Gamma^{\bar{a}}_{b\bar{c}} = \frac{1}{2}C^c_{ab}.$$

§4. Special vector fields. Let V be a real vector field whose contravariant components with respect to the complex basic frame are (v^A) i.e. $V = v^A Z_A$. Put $v_A = g_{AB} v^B$. Then a necessary and sufficient condition for the V to be a Killing vector field is given by [4]

(4.1) $$\nabla_B v_A + \nabla_A v_B = Z_B v_A - \Gamma^D_{BA} v_D + Z_A v_B - \Gamma^D_{AB} v_D = 0.$$

In view of (2.6) and (2.7), the contravariant components of X_a and JX_a (a; fixed) with respect to the complex basic frame are given by $(\delta^i_a, \delta^{\bar{b}}_{\bar{a}})$ and $(\sqrt{-1}\,\delta^b_a, -\sqrt{-1}\,\delta^{\bar{b}}_{\bar{a}})$ respectively.

Since the real basic frame is orthonormal with respect to the metric g, for the covariant components we, from (2.8), have $\left(\frac{1}{2}\delta^a_b, \frac{1}{2}\delta^{\bar{a}}_{\bar{b}}\right)$ and $\left(-\frac{\sqrt{-1}}{2}\delta^a_b, -\frac{\sqrt{-1}}{2}\delta^{\bar{a}}_{\bar{b}}\right)$. These components of X_a and JX_a will be used through this section.

Now we seek for a condition for the X_a (a; fixed) to be Killing vector field. From (3.14) and (4.1) we have

  i) $C^a_{bc} + C^a_{cb} = 0$ ii) $C^b_{ac} + \bar{C}^c_{ab} = 0$ iii) $\bar{C}^a_{bc} + \bar{C}^a_{cb} = 0$,

where the conditions i) and iii) hold good because of (2.3) and its conjugate. In the same way, for the JX_a (a; fixed) we have i), iii) and ii)' $C^b_{ac} - \bar{C}^c_{ab} = 0$. Consequently we can state

PROPOSITION 2. *When a is a fixed index, a necessary and sufficient condition for the X_a or for the JX_a to be a Killing vector field is respectively given by*

$$C^b_{ac} + \bar{C}^c_{ab} = 0 \quad \text{or} \quad C^b_{ac} - \bar{C}^c_{ab} = 0.$$

From the above proposition, we have immediately

COROLLARY 2.1. *All the vector fields of the real basic frame are Killing vector ones if and only if all the scalars C^a_{bc} vanish.*

A necessary and sufficient condition for the V to be a conformal Killing vector field is that there exist a scalar ρ satisfying

(4.2) $$\nabla_B v_A + \nabla_A v_B = 2\rho g_{AB} \quad [4]$$

When $V = X_a$ (a; fixed), from (3.3), (3.14) and (4.2) we have

$$C_{ac}^b + \bar{C}_{ab}^c = -4\rho\delta_{bc} \quad \text{for all indices } b \text{ and } c,$$

which implies $\rho=0$, by substituting a for b and c. This is valid also when $V=JX_a$. Hence we have

PROPOSITION 3. *If a vector field of the real basic frame is a comformal Killing vector one, then it is necessarily a Killing vector field.*

The divergence of the vector field V is given by

$$(4.3) \qquad\qquad \text{div } V = \nabla_E v^E .$$

Then from (3.14) and (4.3) we have

$$(4.4) \qquad \text{div } X_a = \frac{1}{2}(C_{ea}^e + \bar{C}_{ea}^e), \qquad \text{div } JX_a = \frac{\sqrt{-1}}{2}(C_{ea}^e - \bar{C}_{ea}^e).$$

Consequently we have

PROPOSITION 4. *When an index a is fixed, both the X_a and JX_a are divergence-free if and only if $C_{ea}^e = 0$.*

The vector field V is called harmonic if $\text{div } V = 0$ and

$$(4.5) \qquad\qquad \nabla_B v_C = \nabla_C v_B \quad [4].$$

Then from (4.4), (4.5) and proposition 2 we have

PROPOSITION 5. *When an index a is fixed, the vector field X_a or JX_a is harmonic if and only if $C_{bc}^a = 0$, $C_{ea}^e = 0$ and the counterpart JX_a or X_a is a Killing vector field respectively.*

Making summary of the results obtained, we can state

PROPOSITION 6. *The following conditions are equivalent:*

(1) *All the vector fields X_a ($a=1, 2, \cdots, n$) are harmonic.*

(2) *All the vector fields JX_a ($a=1, 2, \cdots, n$) are harmonic.*

(3) *All the vector fields of the real basic frame are Killing vector ones.*

(4) *A metric g defined by (3.1) is Kählerian.*

(5) *The set of all holomorphic vector fields on M forms a complex commutative Lie algebra.*

FROOF. Each condition is equivalent to $C_{bc}^a = 0$ for all a, b, c.

§ 5. **Curvatures.** Let R and K_{ABCD} be the Riemannian curvature tensor and its components with respect to the complex basic frame. Then since

$$(5.1) \qquad R(Z_C, Z_D) Z_B = [\nabla_{Z_C}, \nabla_{Z_D}] Z_B - \nabla_{[Z_C, Z_D]} Z_B ,$$

if we put

(5. 2) $R(Z_C, Z_D) Z_B = K^A_{BCD} Z_A$, where $K^A_{BCD} = g^{AE} K_{EBCD}$,

it follows from (2.5), (3.7), (5.1) and (5.2) that

(5. 3) $K^A_{BCD} = Z_C \Gamma^A_{DB} - Z_D \Gamma^A_{CB} + \Gamma^E_{DB} \Gamma^A_{CE} - \Gamma^E_{CB} \Gamma^A_{DE} - C^E_{CD} \Gamma^A_{EB}$.

Noticing that $Z_{\bar c} C^a_{db} = 0$, $Z_d \bar{C}^b_{ac} = 0$ and using (3.14), (5.3), we calculate $K^a_{b\bar c d}$ and get

(5. 4) $K^a_{b\bar c d} = \dfrac{1}{4} \sum_e (C^a_{de} \bar{C}^b_{ce} + C^c_{be} \bar{C}^d_{ae} - C^e_{bd} \bar{C}^e_{ac})$.

First, we can state

LEMMA. *The sectional curvature of the holomorphic section involving X_a (a; fixed) is non-positive and equal to zero only when $C^a_{ae} = 0$ for $e = 1$, $2, \cdots, n$.*

PROOF. If we denote by p the holomorphic section involving X_a, it is clear that $\{X_a, JX_a\}$ is an orthonormal basis for p. Then the sectional curvature is given by

$$K(p) = R(X_a, JX_a, X_a, JX_a),$$

which is, by virtue of (2.6) and (2.7), reducible to

(5. 5) $K(p) = -4R(Z_{\bar a}, Z_a, Z_{\bar a}, Z_a)$.

On the other hand, we have

(5. 6) $R(Z_{\bar a}, Z_a, Z_{\bar a}, Z_a) = g\left(R(Z_{\bar a}, Z_a) Z_a, Z_{\bar a}\right)$.

On making use of (3.3), (5.2), (5.4), (5.5) and (5.6), we have

$$K(p) = -2K^a_{a\bar a a} = -\sum_e |C^a_{ae}|^2 \leqq 0.$$

Next, we have

PROPOSITION 7. *Any holomorrphic sectional curvature is non-positive at every point of M.*

PROOF. Let p be a holomorphic section at a point $x \in M$ and $\{v, Jv\}$ be an orthonormal basis for p. Then we can, in the tangent space $T_x(M)$, take an orthonormal frame $\{Y_a, JY_a\}$ $(a = 1, 2, \cdots, n)$ involving the $\{v, Jv\}$, provided that $Y_1 = v$ and $JY_1 = Jv$. And further, there exists an orthogonal transformation φ which maps the real basic frame to $\{Y_a, JY_a\}$. The matrix corresponding to φ is of the form : $\begin{pmatrix} l^a_b & m^a_b \\ -m^a_b & l^a_b \end{pmatrix}$, where l^a_b and m^a_b are real

constants, and so

(5. 7) $Y_a = l_a^e X_e + m_a^e JX_e , \qquad JY_a = -m_a^e X_e + l_a^e JX_e .$

If we put

$$u_b^a = l_b^a + \sqrt{-1}\, m_b^a , \qquad U_a = \frac{1}{2}(Y_a - \sqrt{-1}\, JY_a) ,$$

from (3.2) and (5.7) we have

(5. 8) $U_a = u_a^e Z_e , \qquad U_{\bar{a}} = \bar{u}_a^e Z_{\bar{e}} .$

Since u_a^e are constant and det. $(u_a^e) \neq 0$, if we consider on M, each U_a is a holomorphic vector field and $\{U_a\}$ $(a=1, 2, \cdots, n)$ are linearly independent at every point of M. Therefore the $\{U_a\}$ define a new complex paralleli-zation of M. It is easily seen that the matrix (u_a^e) is unitary. Consequently it follows from (3.3) and (5.8) that a Riemannian metric such that the real frame $\{Y_a, JY_a\}$ induced from the complex frame $\{U_a\}$ is orthonormal is identifiied with the metric defined by (3.1) and hence the respective Riemannian connections are the same. Thus we can apply the lemma to this case. That is to say,

$$K(p) = R(v, Jv, v, Jv) = R(Y_1, JY_1, Y_1, JY_1) \leqq 0 .$$

Finally we have

PROPOSITION 8. *The scalar curvature is non-positive and equal to zero onyl when $C_{bc}^a = 0$ for all indices a, b and c.*

PROOF. Let S and S_{AB} be the Ricci tensor and its components with respect to the complex basic frame. Then we have

$$S_{AB} = S(Z_A, Z_B) = K_{BEA}^E ,$$

from which it, in consideration of the cyclic property of the curvature tensor, follows that

(5. 9) $S_{\bar{a}b} = K_{b e \bar{a}}^e + K_{b \bar{e} \bar{a}}^{\bar{e}} = -K_{b \bar{a} e}^e + \overline{K_{e \bar{b} a}^e} - \overline{K_{a \bar{b} e}^e} .$

Substituting (5.4) in (5.9), we have

(5. 10) $S_{\bar{a}b} = \frac{1}{4}\sum_{e,f}(C_{ef}^a \bar{C}_{ef}^b - 2C_{bf}^a \bar{C}_{ef}^e - 2C_{ef}^e \bar{C}_{af}^b - 2C_{be}^f \bar{C}_{ae}^f) .$

The scalar curvature R is given by

$$R = g^{AB} S_{AB} ,$$

which is, by means of (3.12), reducible to

(5. 11) $$R = 4 \sum_a S_{\bar{a}a} .$$

Consequently from (5.10) and (5.11), we have

$$R = - \sum_{a,e,f} |C_{ef}^a|^2 - 4 \sum_f |C_{ef}^e|^2 \leq 0 .$$

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