# A generalization of Whitney Lemma

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§0. In this paper we study the elimination of the intersections of manifolds which is a generalization of WHITNEY LEMMA as follows.

WHITNEY LEMMA (*simply connected version*) (*see* [*R* & *S*]): *Let* $P^p$, $Q^q$ *be a pair of connected compact locally flat submanifolds of* $M^m$ *which are transverse, so that* $p+q=m$. *Suppose* (1) $p\geqq 3$, $q\geqq 3$ *and* $\pi_1(M)=0$ *or* (2) $p\geqq 2$, $q\geqq 3$ *and* $\pi_1(M-Q)=0$. *If the intersection number of* $P$ *and* $Q$, $\varepsilon(P, Q)$, *is zero, we can ambient isotope* $P$ *off* $Q$, *by an isotopy which has compact support.*

We work in the *PL* category ([Z]) throughout the paper.

MAIN RESULT I (BOUNDED VERSION) (COROLLARY TO THEOREM 1). *Let* $P$ *be a compact* $p$-*manifold and* $M$ *be an* $m$-*manifold. Let* $Q$ *be a compact* $q$-dim. *submanifold of* $M$ *and* $f: P\to \mathrm{Int}\, M$ *be an embedding, so that* $p+q=m+k$. *If* (1) $\partial P\neq\phi$, $P$ *is* $k$-*connected,* $k\leqq p-3$ *and* $f(P)\cap Q \subset f(\mathrm{Int}\, P)$ *or* (2) $\partial Q\neq\phi$, $Q$ *is* $k$-*connected,* $k\leqq q-3$ *and* $f(P)\cap Q\subset \mathrm{Int}\, Q$, *then there is an embedding* $g: P\to \mathrm{Int}\, M$ *which is ambient isotopic to* $f$ *and* $g(P)\cap Q=\phi$.

MAIN RESULT II (CLOSED VERSION) (THEOREM 2.) *Let* $P$, $M$ *be a connected closed* $p$- *and* $m$-*manifolds and* $Q$ *be a connected closed* $q$-*submanifold of* $M$. *Let* $f: P\to M$ *be an embedding and let* $p+q=m+k$. *Put* $N=f(P)\cap Q$.

(1) *If* $P$, $Q$ *are* $k$-*connected and* $M$ *is* $(k+1)$-*connected and if* $k+3\leqq p$, $k+3\leqq q$ *then* $P$-*side and* $Q$-*side intersection classes* $\varepsilon_P(N)$ *and* $\varepsilon_Q(N)$ *are defined* (§2 *for definition*).

(2) *Suppose* $p, q\leqq m-3$ *and* $\varepsilon_P(N)=0$ *or* $\varepsilon_Q(N)=0$ *provided* $\min(p, q)\geqq 2k+3$ *or* $\varepsilon_i(N)=0$ *provided* $\max(p, q)\geqq 2k+3$ *where* $i=Q$ *if* $\max(p, q)=p$ *and* $i=P$ *if* $\max(p, q)=q$. *Then there is an embedding* $g: P\to M$ *so that* $g$ *is ambient isotopic to* $f$ *and* $g(P)\cap Q=\phi$.

(3) *If* $P$, $Q$ *are* $(k+1)$-*connected and* $M$ *is* $(k+2)$-*connected and if* $k+4\leqq p$, $k+4\leqq q$, $\varepsilon_P(N)$ *and* $\varepsilon_Q(N)$ *are uniquely determined for* $N$ (*i.e. they do not depend on the choice of* $K$, $L$ *and* $J$ *at the definition of* $\varepsilon_P(N)$, $\varepsilon_Q(N)$).

(4) *Let $P, Q, M$ and $k, p, q$ are all the same as above* (3). *Then $\varepsilon_P(N)$ and $\varepsilon_Q(N)$ are isotopy invariants* (*i.e. if $f_0, f_1$: $P\to M$ are isotopic embeddings, $\varepsilon_P(N_0)=\varepsilon_P(N_1)$ and $\varepsilon_Q(N_0)=\varepsilon_Q(N_1)$ where $N_i=f_i(P)\cap Q$ $(i=0, 1)$*).

$D^p, S^p$ denote standard $p$-ball and $p$-sphere. $\Sigma^p$ means an embedded $p$-sphere in some manifold. $|K|$ means the underlying space of a complex $K$. For compact manifolds $M, N$ a proper embedding $f: M\to N$ means an embedding such that $f(\partial M)\subset\partial N$ and $f(\mathrm{Int}\, M)\subset\mathrm{Int}\, N$.

## § 1. Bounded version

It is easy to prove the following lemma.

LEMMA 1. *Let $I^n, \tilde{I}^k, \tilde{I}^k_0$ be the cartesian products of closed intervals as follows where $I^m=\overbrace{[0, 1]\times\cdots\times[0, 1]}^{m}$.*

$$I^n\supset\tilde{I}^k\equiv I^{k-1}\times\left[0, \frac{3}{4}\right]\times\left[-\frac{1}{2}, \frac{1}{2}\right]^{n-k}\supset\left[\frac{1}{4}, \frac{3}{4}\right]^{k-1}$$
$$\times\left[\frac{1}{4}, \frac{3}{4}\right]\times\left[-\frac{1}{2}, \frac{1}{2}\right]^{n-k}\equiv\tilde{I}^k_0$$

*Then there is an ambient isotopy of $I^n$ keeping the boundary fixed and carrying $\tilde{I}^k$ onto $Cl(\tilde{I}^k-\tilde{I}^k_0)$.*

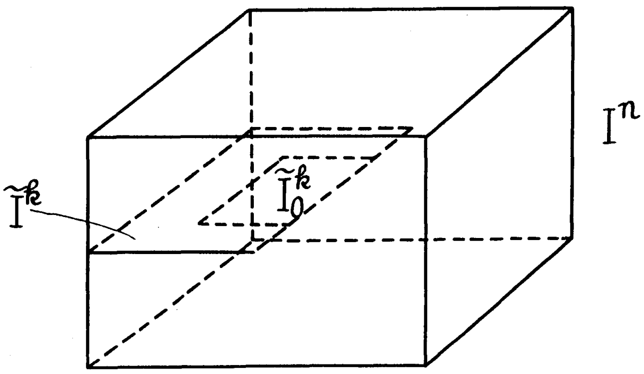


**Fig. 1.**

DEFINITION. We say that $k$-dim. complex $K$ has singular dim. $p$ (briefly $S$-dim. $p$) if for any $x\in|K-K^{(p)}|$ there is a neighborhood $U(x)$ of $x$ in $|K|$ so that $U(x)\cong R^k$ and if for some $y\in|K-K^{(p-1)}|$ there is no neighborhood $U(y)$ so that $U(y)\cong R^k$ where $K^{(p)}$ is the $p$-skeleton of $K$. In particular if for any $x\in|K|$ there is a neighborhood $U(x)$ of $x$ in $|K|$ so that $U(x)\cong R^k$ we say that $K$ has $S$-dim. $-1$. So if $|K|$ is a manifold, $K$ has $S$-dim. $-1$. On the other hand any $k$-complex has $S$-dim$\leqq k-1$. We denote

$$S(K)=\left\{x\in|K|\ |^{\exists}U(x)\ni U(x)\cong R^{k}\right\}.$$

If $K$ has $S$-dim. $p$, $S(K)\subset K^{(p)}$. Let $F(K)$ be the frontier set of $K$ i.e. $F(K)=\{\varDelta\in K|\varDelta$ *has at least one free face*$\}$.

THEOREM 1. *Let $M^m$ be an $m$-manifold and $K$ be a $k$-*dim. *complex with $S$-*dim. *$p$. Let $L$ be an $l$-*dim. *complex with $S$-*dim. *$q$ which is a subcomplex in* Int *$M$ and $f; K\to$* Int *$M$ be an embedding. Put $r=k+l-m$ and $W=f(K)\cap L$.*

(1) *$F(K)\neq\phi$, $W\cap f(F(K))=\phi$ and a connected component, say $X$, of $f(K-S(K))$ which contains a connected component of $W$ is $r$-connected where $r<\min(k-p, k-2)$ and $X\cap f(F(K)-S(K))\neq\phi$.*

(2) *$F(L)\neq\phi$, $W\cap F(L)=\phi$ and a connected component, say $X_1$, of $L-S(L)$ which contains a connected component of $W$ is $r$-connected where $r<\min(l-p, l-2)$ and $X_1\cap(F(L)-S(L))\neq\phi$. If the above* (1) *or* (2) *is satisfied, there is an embedding $g: K\to$* Int *$M$ which is ambient isotopic to $f$ and so that $g(K)\cap L=\phi$.*

PROOF. We will show (1). In (2) we may change $f(K)$ with $L$. Since $\dim(f(K)\cap L)\leqq k+l-m=r$, we can translate $f$ by $\varepsilon$-isotopy to $\tilde{f}$ so that $|\tilde{f}(K)\cap L|\cap\tilde{f}(|K^{(p)}|)=\phi$ because $r+p-k<0$. Let $\widetilde{W}=|\tilde{f}(K)\cap L|$ then a connected component $\widetilde{X}$ of $\tilde{f}(K-S(K))$ containing a connected component of $\widetilde{W}$ is $r$-connected ($r\leqq k-3$) by the assumption. And $K-S(K)$ is a $k$-manifold. By ZEEMAN's ENGULFING THEOREM ([Z. CHAP. 7]) there is a collapsible polyhedron $A^{r+1}$ in $\widetilde{X}\subset\tilde{f}(K-S(K))$ such that $A\supset|\tilde{f}(K)\cap L|=\widetilde{W}$. Since $\widetilde{X}\cap\tilde{f}(F(K)-S(K))\neq\phi$, we can take a simple are $\alpha$ in $\tilde{f}(K-S(K))$ joining $a$ and $b(a\in A, b\in\tilde{f}(F(K)-S(K))$.

Since dim. $A\leqq r+1\leqq$ dim. $(K-S(K))-2$ and $K-S(K)$ is a manifold, by the general position technique we may assume $A\cap\alpha=a$. So $A\cup\alpha$ is a collapsible polyherdron and $N(A\cup\alpha, K'')\equiv B_0^k$, $N(B_0^k, K'''')\equiv B^k$ and $N(B_0^k, M'''')\equiv B^m$ are all balls. Then by LEMMA 1 there is an ambient isotopy $H$ of $B^m$ keeping $\partial B^m$ fixed and carrying $B^k$ onto $Cl(B^k-B_0^k)$ and so $H_1\tilde{f}(K)\cap L=\phi$. We may put $g=H_1\tilde{f}$.

PROOF OF COROLLARY. Any manifold has $S$-dim. $-1$. So $S(P)=S(Q)=\phi$. Hence by THEOREM 1 we obtain a required embedding $g: P\to$Int $M$.

## § 2. Closed version.

DEFINITION 2. A link $L=(S^n; \Sigma^p, \Sigma^q)$ is weak homotopically trivial if $\Sigma^p$ is homotopic to zero in $S^n-\Sigma^q$ or if $\Sigma^q$ is homotopic to zero in $S^n-\Sigma^p$. A link $L=(S^n; \Sigma^p, \Sigma^q)$ is strong slice if there are disjoint locally flat $(p+1)$-

and $(q+1)$-balls $B^{p+1}$, $B^{q+1}$ in $B^{n+1}$ so that $\partial B^{p+1}=\Sigma^p$ and $\partial B^{q+1}=\Sigma^q$ where $\partial B^{n+1}=S^n$.

LEMMA 2. *Let* $L=(S^n\,;\,\Sigma^p,\Sigma^q)$ *be a link with* $p+q=n+k-1$ $(k\geqq 1)$, $p, q\leqq n-3$.

(1) $p, q\geqq 2k+2$ *and* $L$ *is weak homotopically trivial.*

(2) $\max(p, q)\geqq 2k+2$ *and* $\Sigma^{\min(p,q)}$ *is homotopic to zero in* $S^n-\Sigma^{\max(p,q)}$.

*If* (1) *or* (2) *is satisfied,* $L$ *is a trivial link i.e. there are disjoint locally flat* $(p+1)$-*and* $(q+1)$-*balls* $B^{p+1}$, $B^{q+1}$ *in* $S^n$ *such that* $\partial B^{p+1}=\Sigma^p$, $\partial B^{q+1}=\Sigma^q$.

REMARK 1. By LICKORISH ([L]) any strong slice link is a trivial link when $p, q\leqq n-3$.

PROOF OF LEMMA 2. We will show that $L$ is trivial when $\max(p, q)=q$ $\Sigma^p\sim 0$ in $S^n-\Sigma^q$. Since $q\leqq n-3$, $(S^n, \Sigma^q)$ is a trivial knot and so $S^n-\Sigma^q$ $=S^{n-q-1}\times R^{q+1}=S^{p-k}\times R^{n-p+k}$. Since $\Sigma^p\sim 0$ in $S^n-\Sigma^q$, there is a $PL$ map $\tilde{f}: D^{p+1}\to S^n-\Sigma^q$ such that $\tilde{f}(\partial D^{p+1})=\Sigma^p$ and $\dim. S(\tilde{f})\leqq 2(p+1)-n\leqq p-1$. There is a $(p+1)$-ball $D_1^{p+1}$ in $\operatorname{Int} D^{p+1}$ such that $D_1^{p+1}\cap S(f)=\phi$ and $\tilde{f}(\partial D^{p+1})$ is homotopic to $\tilde{f}(\partial D_1^{p+1})$ in $S^n-\Sigma^q$. And the link $L_1=(S^n\,;\tilde{f}(\partial D_1^{p+1}), \Sigma^q)$ is trivial because $\tilde{f}(\partial D_1^{p+1})$ bounds a non-singular $(p+1)$-ball $\tilde{f}(D_1^{p+1})$ in $S^n-\Sigma^q$. Since $S^n-\Sigma^q$ is $(p-k-1)$-connected and $p-k-1\geqq 2p-n+2$, $\tilde{f}(\partial D^{p+1})$ is ambient isotopic to $\tilde{f}(\partial D_1^{p+1})$ in $S^n-\Sigma^q$ by ISOTOPY THEOREM ([Z. CHAP. 8]). Hence $L=(S^n\,;\,\Sigma^p, \Sigma^q)$ is the same link type of $(S^n\,;\,\tilde{f}(\partial D_1^{p+1}), \Sigma^q)$ and $L$ is a trivial. The other cases are followed by the same ways.

DEFINITION 3. Let $P^p, M^m$ be closed $p$- and $m$-manifold and $Q^q$ be a closed $q$-submanifold of $M^m$ with $p+q=m+k$. Let $f: P^p\to M^m$ be an embedding. Put $N=f(P)\cap Q$. If $p, q\leqq m-3$, by ([A & Z]) we may assume $f(P)$ intersect $Q$ transversally, and so $N$ is a closed $k$-manifold. (It may not be connected). Now suppose that there are subcomplexs $K, L, J$ as follows;

(1) $K\subset P$ and $f^{-1}(N)\subset K\searrow 0$ in $P$.

(2) $L\subset Q$ and $N\subset L\searrow 0$.

(3) $J\subset M$, $J\cap f(P)=f(K)$, $J\cap Q=L$ and $J\searrow 0$ in $M$.

Then we defined $P$-side & $Q$-side intersection classes $\varepsilon_P(N)$, $\varepsilon_Q(N)$ of $f(P)$ and $Q$ as follows. Let $B(K)=U(K, P'')$, $B(L)=U(L, Q'')$ and $B(J)=U(J, M'')$ be second derived neighborhood of $K, L$ and $J$ respectively then they are all $p$-, $q$- and $n$-balls and $f(B(K))$, $B(L)$ are properly embedded in $B(J)$. So $\partial(B(J)\,;\,f(B(K)), B(L))=(\partial B(J)\,;\,f(\partial B(K)), \partial B(L))$ is a link such as $(S^{m-1}\,;$ $\Sigma^{p-1}, \Sigma^{q-1})$. We define $\varepsilon_P(N)=\{f(\partial B(K))\}\in\pi_{p-1}(S^{p-k-1})\cong\pi_{p-1}(\partial B(J)-\partial B(L))$ and $\varepsilon_Q(N)=\{\partial B(L)\}\in\pi_{q-1}(S^{q-k-1})\cong\pi_{q-1}(\partial B(J)-f(\partial B(K)))$.

PROOF OF THEOREM 2. (1) Since $P$ is $k$-connected and $p\geqq k+3$, by ZEEMAN'S ENGULFING THEOREM ([Z. CHAP. 7]) there is a $(k+1)$-dim. collapsible polyhedron $K_1$ in $P$ containing $f^{-1}(N)$. And since $Q$ is $k$-connected ($q \geqq k+3$), there is a $(k+1)$-dim. collapsible polyhedron $L_1$ in $Q$ containing $N$. Furthermore since $m \geqq k+4$ and $M$ is $(k+1)$-connected, there is a $(k+2)$-dim. collapsible polyhedron $J_1$ in $M$ containing $f(K_1) \cup L_1$. Let $F_1 = Cl(((J_1 \cap f(P)) - f(K_1))$ and $G_1 = Cl((J_1 \cap Q) - L_1$ then $\dim F_1 \leqq (k+2)+p-m \leqq k-1$ and $\dim G_1 \leqq k+2+q-m \leqq k-1$. Now we will proceed induction as follows.

$\Phi(i)$: There exist collapsible polyhedra $K_i, L_i, J_i$ so that $f^{-1}(N) \subset K_i \subset P$, $N \subset L_i \subset Q$ and $f(K_i) \cup L_i \subset J_i \subset \mathrm{Int}\, M$. And $\dim F_i \leqq k-i$ $\dim G_i \leqq k-i$ where $F_i = Cl((J_i \cap f(P) - f(K_i))$ and $G_i = Cl(J_i \cap Q) - L_i)$. We already showed the case $i=1$ in the above. We will show $\Phi(i+1)$ by assuming $\Phi(i)$. Since $\dim F_i \leqq k-i$, by ENGULFING THEOREM there is a $(k-i+1)$-dim. polyhedron $\tilde{K}_{i+1}$ in $P$ containing $f^{-1}(F_i)$ so that $K_i \cup \tilde{K}_{i+1} \searrow 0$. And there is a $(k-i+1)$-dim. polyhedron $\tilde{L}_{i+1}$ in $Q$ containing $G_i$ so that $L_i \cup \tilde{L}_{i+1} \searrow 0$. Furthermore there is a $(k-i+2)$-dim. polyhedron $\tilde{J}_{i+1}$ in $M$ so that $f(\tilde{K}_{i+1}) \cup \tilde{L}_{i+1} \subset \tilde{J}_{i+1}$ and $J_i \cup J_{i+1} \searrow 0$. Let $K_{i+1} = K_i \cup \tilde{K}_{i+1}$, $L_{i+1} = L_i \cup \tilde{L}_{i+1}$ and $J_{i+1} = J_i \cup \tilde{J}_{i+1}$. As $F_i \subset f(\tilde{K}_{i+1}) \subset \tilde{J}_{i+1}$, $(J_{i+1} \cap f(P)) - f(K_{i+1}) \subset (\tilde{J}_{i+1} \cap f(P)) - f(K_{i+1})$ and so $\dim F_{i+1} = \dim Cl((J_{i+1} \cap f(P)) - f(K_{i+1})) \leqq k-i+2+p-m = k-i+2+k-q \leqq k-i-1$. Similarly since $G_i \subset \tilde{I}_{i+1} \subset \tilde{J}_{i+1}$, $(J_{i+1} \cap Q) - L_{i+1} \subset (\tilde{J}_{i+1} \cap Q) - L_{i+1}$ and so $\dim G_{i+1} = \dim Cl(J_{i+1} \cap Q) - L_{i+1} \leqq k-i+2+q-m = k-i+2+k-p \leqq k-i-1$. This completes the proof of $\Phi(i+1)$. $\Phi(k+1)$ tell us that there are collapsible polyhedra $K_{k+1}, L_{k+1}, J_{k+1}$ such that $f^{-1}(N) \subset K_{k+1} \subset P$, $N \subset L_{k+1} \subset Q$ and $f(K_{k+1}) \cup L_{k+1} \subset J_{k+1} \subset \mathrm{Int}\, M$. And $F_{k+1} = Cl((J_{k+1} \cap f(P)) - f(K_{k+1})) = \phi$, $G_{k+1} = Cl((J_{k+1} \cap Q) - L_{k+1}) = \phi$. So we may put $K = K_{k+1}$, $L = L_{k+1}$ and $J = J_{k+1}$ for $K, L, J$ of the definition of $P$-side and $Q$-side intersection classes.

(2) We will show the case $\min(p, q) = p \geqq 2k+3$ and $\varepsilon_P(N) = 0$. Let $K, L$ and $J$ be polyhedra of the definition of $P$-side intersection class. Then $(\partial B(J); f(\partial B(K)), \partial B(L))$ is a link such as $(S^{m-1}; \Sigma^{p-1}, \Sigma^{q-1})$ where $B(J) = U(J, M'')$, $B(K) = U(K, P'')$ and $B(L) = U(L, Q'')$ are all second derived neighborhood. (See DEF. 3). Since $\varepsilon_P(N) = 0$, $\Sigma^{p-1}$ is homotopic to zero in $S^{m-1} - \Sigma^{q-1}$. Hence by (1) of LEMMA 2 $(S^{m-1}; \Sigma^{p-1}, \Sigma^{q-1})$ is trivial i.e. $(\partial B(J); f(\partial B(K)), \partial B(L))$ is a trivial link.

So there is a locally flat embedding $g_1: B^p \to \partial B(J)$ so that $g_1(\partial B^p) = f(\partial B(K))$ and $g_1(B^p) \cap \partial B(L) = \phi$. Using the collar of $\partial B(J)$ in $B(J)$ we can deform $g_1$ to a locally flat proper embedding $g_2: B^p \to B(J)$ so that $g_2(\partial B^p) = f(\partial B(K))$ and $g_2(B^p) \cap B(L) = \phi$. Since $p \leqq m-3$, by ISOTOPY THEOREM ([Z, CHAP. 8]) $f(B(K))$ and $g_2(B^p)$ is ambient isotopic in $B(J)$ keeping the boundary $f(\partial B(K))$ fixed. Let $g: P \to M$ be an embedding defined by

$$g=\begin{cases} f \text{ on } P-\operatorname{Int} B(K) \\ g_2 \text{ on } B(K). \end{cases}$$

Then $g$ is ambient isotopic to $f$ and $g(P)\cap Q=\phi$.
Similarly we can prove the other cases.

(3) Let $(K_i, L_i, J_i)$ $(i=1, 2)$ be two systems of subpolyhedra with the following properties,

(a) $K_i\subset P$ and $f^{-1}(N)\subset K_i\searrow 0$,

(b) $L_i\subset Q$ and $N\subset L_i\searrow 0$,

(c) $J_i\subset M$, $J_i\cap f(P)=K_i$, $J_i\cap Q=L_i$ and $J_i\searrow 0$.

Let $B(K_i)=U(K_i, P'')$, $B(L_i)=U(L_i, Q'')$ and $B(J_i)=U(J_i, M'')$ $(i=1, 2)$ be second derived neighborhood. Then $\ell_i=(\partial B(J_i)\colon f(\partial B(K_i)), \partial B(L_i)$ are links such as $(S^{m-1}, \Sigma^{p-1}, \Sigma^{q-1})$. It is sufficient to show that $\ell_1$ is ambient isotopic to $\ell_2$ i.e. for an orintation preserving homeomorphism $g: \partial B(J_1)\to \partial B(J_2)$ there is a level preserving homeomorphism $H\colon \partial B(J_2)\times I\to\partial B(J_2)\times I$ so that $H_0=id.$ and $H_1gf(\partial B(K_1))=f(\partial B(K_2))$, $H_1g(\partial B(L_1))=\partial B(L_2)$.

CASE 1. We first consider the case $B(J_1)\subset B(J_2)$, $B(K_1)\subset B(K_2)$, $B(L_1)\subset B(L_2)$. In the case we may assume $B(J_1)\subset\operatorname{Int} B(J_2)$, $B(K_1)\subset\operatorname{Int} B(K_2)$ and $B(L_1)\subset\operatorname{Int} B(L_2)$ using the collars. By the weak SCHOENFLIES THEOREM ([H & Z]$_1$, [H & Z]$_2$)

$$A(J)\equiv B(J_2)-\operatorname{Int} B(J_1)\cong S^{m-1}\times I$$

$$A(K)\equiv B(K_2)-\operatorname{Int} B(K_1)\cong S^{p-1}\times I \qquad \text{and}$$

$$A(L)\equiv B(L_2)-\operatorname{Int} B(L_1)\cong S^{q-1}\times I$$

Then $A(L)$ and $f(A(K))$ are properly embedded in $A(J)$. Now $f(A(K))\cap A(L)=\phi$ because $f(A(K))\cap A(L)\subset f(P)\cap Q=N\subset L_1\subset\operatorname{Int} B(L_1)$. So $\ell_1$ and $\ell_2$ are link cobordant. $\ell_1$ and $\ell_2$ are ambient isotopic by [L].

CASE 2. General case. As $\dim N=k$, we may assume $\dim K_i\leqq k+1$, $\dim L_i\leqq k+1$ and $\dim J_i\leqq k+2$. And $k+4\leqq p$, $k+4\leqq q$ and $k+5<m$. $P, Q$ are $(k+1)$-connected and $M$ is $(k+2)$-connected. So by ENGULFING THEOREM there are collapsible subspaces $K_3, L_3, J_3$ so that $K_1\cup K_2\subset K_3\subset P$, $L_1\cup L_2\subset L_3\subset Q$, $J_1\cup J_2\subset J_3\subset M$, $f(P)\cap J_3=K_3$, $Q\cap J_3=L_3$. Let $B(K_3)=U(K_3, P'')$, $B(L_3)=U(L_3, Q'')$ and $B(J_3)=U(J_3, M'')$. Then $B(K_1)\cup B(K_2)\subset B(K_3)$, $B(L_1)\cup B(L_2)\subset B(L_3)$ and $B(J_1)\cup B(J_2)\subset B(J_3)$. By case 1 $\ell_1\approx\ell_3\approx\ell_2$ where $\ell_3=(\partial B(J_3)\,;\, f(\partial B(K_3)), \partial B(L_3))$ and where $\approx$ means ambient isotopic.

(4) Let $F\colon P\times I\to M\times I$ be a level preserving embedding so that $F_0=f_0$ and $F_1=f_1$. $W=F(P\times I)\cap(Q\times I)$, $N_0=F_0(P)\cap Q$ and $N_1=F_1(P)\cap Q$. Since

$P, Q$ are $(k+1)$-connected and $M$ is $(k+2)$-connected, by ENGULFING THEOREM there are collapsible subspaces $K_i^{k+1}$, $L_i^{k+1}$ and $J_i^{k+2}$ $(i=0, i)$ so that

(d) $F^{-1}(N_i)\subset K_i\subset P\times\{i\}$

(e) $N_i\subset L_i\subset Q\times\{i\}$

(f) $F_i(K_i)\cup L_i\subset J_i\subset M\times\{i\}$, $\quad J_i\cap F_i\big(P\times\{i\}\big)=F(K_i)$

$J_i\cap\big(Q\times\{i\}\big)=L_i$.

Let $S=K_0^{k+1}\cup F^{-1}(W)\cup K_1^{k+1}$ and $T=L_0^{k+1}\cup W\cup L_1^{k+1}$. Then by BOUNDED VERSION OF ENGULFING THEOREM (see [I. Th. 4.3]) there are collapsible subspaces $K, L$ and $J$ so that

(g) $S\subset K\subset P\times I$, $\quad K\cap\big(P\times\{i\}\big)=K_i \quad (i=0, 1)$

(h) $T\subset L\subset Q\times I$, $\quad L\cap\big(Q\times\{i\}\big)=L_i \quad (i=0, 1)$

(i) $F(K)\cup L\subset J\subset M\times I$, $\quad J\cap F(P\times I)=F(K)$, $\quad J\cap(Q\times I)=L$,

$\big(J\cap\big(M\times\{i\}\big)\big)\cap F(K)=J_i\cap F(K)=F(K_i) \quad (i=0, 1)$,

$\big(J\cap\big(M\times\{i\}\big)\big)\cap L=J_i\cap L=L_i \quad (i=0, 1)$.

Then $(\partial B(J)\,;\ F(\partial B(K)),\ \partial B(L))=(S^m\,;\ \Sigma^p, \Sigma^q)$ where $B(J)=U(J, M\times I)$, $B(K)=U(K, P\times I)$ and $B(L)=U(L, Q\times I)$ are second derived neighborhoods. And $\partial(B(J);\ F(B(K)), B(L))-\mathrm{Int}\,(B(J_0);\ F(B(K_0)), B(L_0))\cup\mathrm{Int}(B(J_1);\ F(B(K_1)), B(L_1))\cong(S^{m-1}\times I;\ \Sigma^{p-1}\times I, \Sigma^{q-1}\times I)$ where $\partial(X;\ Y, Z)=(\partial X;\ \partial Y, \partial Z)$ and $\mathrm{Int}\,(X;\ Y, Z)=(\mathrm{Int}\,X;\ \mathrm{Int}\,Y, \mathrm{Int}\,Z)$. It gives a link cobordism between $\mathscr{L}_0=\partial(B(J_0)\,;\ F(B(K_0)),\ B(L_0))$ and $\mathscr{L}_1=\partial(B(J_1)\,;\ F(B(K_1)),\ B(L_1))$. Hence $\mathscr{L}_0$ and $\mathscr{L}_1$ are ambient isotopic by ([L]). This completes the theorem.

REMARK 2. We can define the intersection classes $\varepsilon_P(N_i)$, $\varepsilon_Q(N_i)$ at the connected componentwise $N_1, N_2, \cdots, N_n$ of $N=f(P)\cap Q$ same as WHITNEY LEMMA. And we can also prove (2) of THEOREM 2 under $\sum_i \varepsilon_p(N_i)=0$ or $\sum_i \varepsilon_Q(N_i)=0$ with other sutable conditions. But in particular (1) of THEOREM 2 requires a stronger assumption so that $p, q\geqq 2(k+1)$. So we defined the intersection classes $\varepsilon_P, \varepsilon_Q$ for the whole $N=f(P)\cap Q$.

REMARK 3. Let $P_i^p$ $(i=1, 2)$, $Q^q$ be closed submanifolds in a closed manifold $M^m$ with $p+q=m+k$ and $p, q\leqq m-3$. We may assume $P_i$ transversally intersect $Q$. So $N_i=P\cap Q$ is a closed $k$-manifold ($N_i$ may be empty). Let $[P_i]\in H_p(M)$, $[Q]\in H_q(M)$ be homology classes represented by $P_i, Q$. And let $[P_i]^*\in H^{m-p}(M)$ be a cohomology class corresponding to $[P_i]$ by POINCARÉ

DUALITY. Then $[P_i]^* \cap [Q] = [N_i] \in H_k(M)$. So if $P_1$ is homologous to $P_2$ in $M$, $[N_1] = [N_2]$. That is, the homology class of $N$ in $M$ is uniquely determined by the homology classes of $P$ and $Q$. But for $p+q=m+k$ ($k \geqq 1$), we can not obtain WHITNEY TYPE LEMMA from $[N]$ only.

Contrary to THEOREM 2, if $M$ has a high connectivity we can obtain a following by global deformation.

THEOREM 3. *Let $P^p$, $M^m$ be closed $p$- and $m$-manifold and $Q^q$ be a closed $q$-manifold of $M$. Let $f: P \to M$ be an embedding and* $\min(p, q) \leqq m-3$. *If $M$ is* $\min(p, q)$*-connected, there is a homeomorphism $g: M \to M$ which is isotopic to identity and $gf(P) \cap Q = \phi$.*

From (4) of THEOREM 2 and THEOREM 3 we obtain the following.

COROLLARY. *Let $P, Q, M$ and $f$ be the same as* THEOREM 3 *and let $p+q=m+k$, $k+4 \leqq p, q$. If $P$ and $Q$ are $(k+1)$-connected, the intersection classes $\varepsilon_P(N) = \varepsilon_Q(N) = 0$ where $N = f(P) \cap Q$.*

PROOF OF THEOREM 3. We may assume $\min(p, q) = p$ without loss of generality.

Since $p \leqq m-3$ and $M$ is $p$-connected, by ENGULFING THEOREM there is an $m$-ball $B^m$ in $M$ so that $f(P) \subset \operatorname{Int} B^m$. We take another $m$-ball $B_1^m$ in $M$ so that $B_1 \cap Q = \phi$. Then by the homogeneity of the ball ([N]) there is a homeomorphism $g: M \to M$ which is isotopic to *id.* and $g(B^m) = B_1^m$. Hence $gf(P) \cap Q \subset g(B^m) \cap Q = B_1^m \cap Q = \phi$.

## § 3. Example with non-trivial intersection class.

Let $p, q, m, k$ be all non-negative integers with $p+q=m+k$ and suppose $m \geqq p+k+2$, $m \geqq p+3$ and $m \geqq q+3$. Given a non zero element $\varepsilon_P \in \pi_{p-1}(S^{p-k-1})$. We will first construct a link $\mathscr{L} = (S^{m-1}; \Sigma^{p-1}, \Sigma^{q-1})$ with $\{\Sigma^{p-1}\} = \varepsilon_P \in \pi_{p-1}(S^{p-k-1}) \cong \pi_{p-1}(S^{m-1} - \Sigma^{q-1})$. Let $f_0: S^{p-1} \to S^{p-k-1} \times \operatorname{Int} D^{m-p+k}$ be a map so that $\{f_0\} = \varepsilon_P \in \pi_{p-1}(S^{p-k-1})$. Since $m \geqq p+k+2$ and $m \geqq p+3$, by EMBEDDING THEOREM ([Z. CHAP. 8]) there is an embedding $f: S^{p-1} \to S^{p-k-1} \times \operatorname{Int} D^{m-p+k}$ which is homotopic to $f_0$. Since $S^{p-k-1} \times D^{m-p+k} = S^{m-q-1} \times D^q$, we paste the boundary $\partial(S^{p-k-1} \times D^{m-p+k})$ with the boundary $\partial(D^{m-q} \times S^{q-1})$ so that $(S^{p-k-1} \times D^{m-p+k}) \underset{\partial}{\cup} (D^{m-q} \times S^{q-1}) = S^{m-1}$. Then $(S^{m-1}; f(S^{p-1}), (O \times S^{q-1})) \equiv (S^{m-1}; \Sigma^{p-1}, \Sigma^{q-1})$ is required link $\mathscr{L}$ where $O$ is the center of $D^{m-q}$.

We can construct proper locally flat embeddings $\phi: D^p \to D^m$, $\psi: D^q \to D^m$ so that $\phi(\partial D^p) = \Sigma^{p-1}$, $\psi(\partial D^q) = \Sigma^{q-1}$ and $\phi$ intersects transversally to $\psi$ (see [A & Z]). Let $N = \phi(D^p) \cap \psi(D^q) \subset \operatorname{Int} D^m$ then $N$ is a closed $k$-dim. manifold.

Since $(S^{m-1}, \Sigma^{p-1})$ and $(S^{m-1}, \Sigma^{q-1})$ are trivial knots, we can extent $\phi|\partial D^p$, $\psi|\partial D^q$ to the embeddings $\Phi : \partial D^p \times D^{m-p} \to S^{m-1}$, $\Psi : \partial D^p \times D^{m-p} \to S^{m-1}$.
Let $W = D^m \cup_\Phi (D^p \times D^{m-q}) \cup_\Phi (D^q \times D^{m-q})$ and let $D(W)$ be the double of $W$. Then we may consider $M$, $f(P)$, $Q$, $N$ in the above theorem $D(W)$, $\phi(D^p) \underset{\partial\phi}{\cup} (D^p \times \{0\})$, $\psi(D^q) \underset{\partial\phi}{\cup} (D^q \times \{0\})$ and $\phi(D^p) \cap \psi(D^q)$. $\phi(D^p)$, $\psi(D^q)$ and $D^m$ are $f(B(K))$, $B(L)$ and $B(J)$ at the definition of the intersection classes. And $\varepsilon_P(N) = \varepsilon_P(\neq 0) \in \pi_{p-1}(S^{p-k-1})$.

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