Hokkaido Mathematical Journal Vol. 6 (1977) p. 69–73

A remark on the index of G-manifolds in the representation theory

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(Received May 15, 1976)

1. Introduction

Fix a finite group G. Let $\mathscr{F}$ be a family of subgroups of G which satisfies if $H \in \mathscr{F}$ and $H' \subset H$, then $H' \in \mathscr{F}$. Then G-bordism group of G-manifolds is denoted by $\Omega_*(G, \mathscr{F})$. And its elements are the bordism classes $[G, M]$ where M is a differentiable closed manifold and all isotropy groups G_x are in $\mathscr{F}$. Now we consider the index of G-manifolds. It is well known that the index I is a bordism invariant of Ω_{4k}. And it is extended naturally to the G-bordism invariant: $I: \Omega_{4k}(G, \mathscr{F}) \longrightarrow RO(G)$, where $RO(G)$ is the Grothendieck group of G over R.

In this paper we compute the index of G-manifolds with $\mathscr{F} = \{1\}$ in $RO(G)$ in the sense of R. Lee [5].

2. The homomorphism I: $\Omega_{4k}(G, \mathscr{F}) \longrightarrow RO(G)$

Let M be a compact oriented differentiable G-manifold without boundary and $\mathscr{F}$-free. The bilinear form $\Phi: H^{2k}(M; R) \times H^{2k}(M; R) \longrightarrow R$ is defined by $\Phi(x, y) = \langle x \cup y, [M] \rangle$, where $[M]$ is the orientation class of M. Then by the Poincaré duality, Φ is non-singular, symmetric and G-invariant. In $H^{2k}(M; R)$, we set G-invariant maximal subspaces

$$V_+ = \left\{ x \in H^{2k}(M; R) \,\middle|\, \Phi(x, x) > 0 \text{ if } x \neq 0 \right\}$$

$$V_- = \left\{ x \in H^{2k}(M; R) \,\middle|\, \Phi(x, x) > 0 \text{ if } x \neq 0 \right\}, \quad \text{then}$$

$I: \Omega_{4k}(G, \mathscr{F}) \longrightarrow RO(G)$ is defined by $I[G, M] = [V_+] - [V_-]$ (see [4] pp. 578), where $[V_\pm]$ is the equivalence class of $V_\pm$ in $RO(G)$. Now by the well known result (see [4] pp. 85–86), it is proved that

(2. 1) The correspondence $I: \Omega_{4k}(G, \mathscr{F}) \longrightarrow RO(G)$ is the well-defined homomorphism.

In particular, $G = \{1\}$, since $\Omega_{4k}(G, \mathscr{F}) = \Omega_{4k}$ and $RO(G) = Z[K]$, where K is a trivial representation, $I: \Omega_{4k} \longrightarrow RO(G)$ is $I[M] = I(M)[K]$, where $I(M)$ is the index of M.

Let H be a proper subgroup of G, then the extension homomorphism $i_*: \Omega_*(H, \mathscr{F}) \longrightarrow \Omega_*(G, \mathscr{F})$ is defined by $i_*[H, M]=[G, G\underset{H}{\times} M]$. And similarly the extension homomorphism $i_*: RO(H) \longrightarrow RO(G)$ is defined by $i_*[V]=[RG\underset{RH}{\otimes} V]$, where RG(rep. RH) is the group ring of G(rep. H) over R. And by the restriction of the action of G to that of H, the restriction homomorphism is defined.
Then the following diagram is commutative.

$$\begin{array}{ccc} \Omega_{4k}(G, \mathscr{F}) & \xrightarrow{\quad i^* \quad} & \Omega_{4k}(H, \mathscr{F}) \\ \Big\downarrow I & & \Big\downarrow I \\ RO(G) & \xrightarrow{\quad i^* \quad} & RO(H) \end{array} \tag{2.2}$$

Here i^* is the restriction homomorphism.

$$\begin{array}{ccc} \Omega_{4k}(H) & \xrightarrow{\quad i_* \quad} & \Omega_{4k}(G) \\ \Big\downarrow I & & \Big\downarrow I \\ RO(H) & \xrightarrow{\quad i_* \quad} & RO(G) \end{array} \tag{2.3}$$

Rroof. (2.2) is trivial and for (2.3), let $[H, M]$ be a element of $\Omega_{4k}(H)$, since M is a free H-manifold, it follows that $H^{2k}(G\underset{H}{\times} M)=RG\underset{RH}{\otimes} H^{2k}(M)$. And by the definition, $I[H, M]=[V_+]-[V_-]$, then $H_{2k}(M)=V_+\oplus V_-$. And so $H^{2k}(G\underset{H}{\times} M)=(RG\underset{RH}{\otimes} V_+)\oplus(RG\underset{RH}{\otimes} V_-)$. $RG\underset{RH}{\otimes} V_\pm$ are the G-invariant maximal subspaces for symmetric bilinear form on

$$H^{2k}(G\underset{H}{\times} M), \quad \text{that is} \quad Ii_*[H, M]=[RG\underset{RH}{\otimes} V_+]-[RG\underset{RH}{\otimes} V_-].$$

Hence $Ii_*[H, M]=i_*I[H, M]$.

3. The index of G-manifolds in the case of $\mathscr{F}=\{1\}$

Fix a finite group G and $\mathscr{F}=\{1\}$. Let RG denote by the group ring of G over R.

THEOREM. *$I: \Omega_{4k}(G) \longrightarrow RO(G)$ and if $[G, M]\in\Omega_{4k}(G)$, then $I[G, M]=I(M/G)[RG]$.*

Proof. For the augmentation $\varepsilon_*: \Omega_n(G) \longrightarrow \Omega_n$, $\varepsilon_*[G, M]=[M/G]$, the reduced bordism group $\tilde{\Omega}_n(G)$ is denoted by Ker $[\varepsilon_*: \Omega_n(G) \longrightarrow \Omega_n]$. Since $\varepsilon_*[H, M]=[M/H]=[G\underset{H}{\times} M/G]=\varepsilon_* i_*[H, M]$ if $[H, M]\in\Omega_n(H)$, where i_* is the extension $\Omega_n(H) \longrightarrow \Omega_n(G)$, there is a following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{\Omega}_n(H) & \longrightarrow & \Omega_n(H) & \xrightarrow{\varepsilon_*} & \Omega_n & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \tilde{\Omega}_n(G) & \longrightarrow & (\Omega_n G) & \xrightarrow{\varepsilon_*} & \Omega_n & \longrightarrow & 0 \end{array}$$

In particular, $H=\{1\}$, then $\tilde{\Omega}_*(1)=0$, ε_* is identity. If $[G, M]\in\Omega_n(G)$, since $\varepsilon_*[G, M]=[M/G]=\varepsilon_* i_*[M/G]$, $[G, M]-i_*[M/G]\in\operatorname{Ker}\varepsilon_*=\tilde{\Omega}_n(G)$, and hence we have $I[G, M]\equiv Ii_*[M/G]$ (mod $I(\tilde{\Omega}_{4k}(G))$). By (2. 3), for $H=\{1\}$, $Ii_*[M/G]=I(M/G)[RG]$, therefore $I[G, M]\equiv I(M/G)[RG]$ (mod $I(\tilde{\Omega}_{4k}(G))$). (1)
Now, let C denote the class of torsion group consisting of the elements of odd order.
Then there exists the following theorem in [6] (pp. 41).

THEOREM. *For any CW-pair (X, A), there is an isomorphism*

$$\theta:\ \Omega_n(X, A)\cong \sum_{p+q=n} H_p(X, A\,;\Omega_q)\quad (mod\ C)$$

And the reduced bordism group $\tilde{\Omega}_n(X)$ is denoted by $\operatorname{Ker}\varepsilon_*[\Omega_n(X)\longrightarrow \Omega_n(pt)]$, where ε is a collapsing map $\varepsilon: X\rightarrow pt$. In particular $\tilde{\Omega}_n(G)=\tilde{\Omega}_n(BG)$. Let X be connected, then by the construction of θ, the following diagram is commutative.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{\Omega}_n(X) & \longrightarrow & \Omega_n(X) & \xrightarrow{\varepsilon_*} & \Omega_n & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \theta} & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Ker}\varepsilon_* & \longrightarrow & \sum\limits_{p+q=n} H_p(X\,;\Omega_q) & \xrightarrow{\varepsilon_*} & \sum\limits_{p+q=n} H_p(pt\,;\Omega_q) & \longrightarrow & 0 \end{array}$$

And $\operatorname{Ker}\varepsilon_*=\sum\limits_{p+q=n}\tilde{H}_p(X\,;\Omega_q)$, $\sum\limits_{p+q=n} H_p(pt\,;\Omega_q)\cong\Omega_n$, and so Mod C isomorphism θ induces the homomorphism θ_1; $\tilde{\Omega}_n(X)\longrightarrow \sum\limits_{p+q=n}\tilde{H}_p(X\,;\Omega_q)$. By the above commutativity, $\operatorname{Ker}\theta_1\in C$.

Now we consider $X=BG$ and $\theta_1:\ \tilde{\Omega}_n(G)\longrightarrow \sum\limits_{p+q=n}\tilde{H}_p(G\,;\Omega_q)$. According to the proposition of Cartan-Eilenberg, $\tilde{H}_*(G\,;Z)$ is a torsion group. (see [7] prop. 2.5 pp. 236) And is also $\sum\limits_{p+q=n}\tilde{H}_p(G\,;\Omega_q)$ since each Ω_q is finitely generated abelian group. And so Im θ_1 is a torsion group. Therefore $\tilde{\Omega}_{4k}(G)$ is a torsion group.

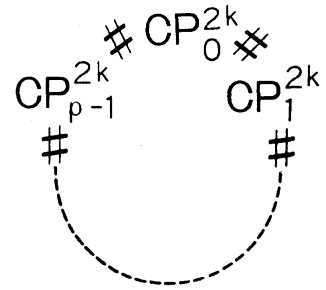
In the case I: $\tilde{\Omega}_{4k}(G)\longrightarrow RO(G)$, $RO(G)$ is the free abelian group, and its basis consist of the equivalence classes of irreducible representations.

Hence $I(\tilde{\Omega}_{4k}(G))=0$. And by (1), $I[G, M]=I(M/G)[RG]$.

COROLLARY. (*J. A. Schafer* [1], *H-T-Ku and M-C-Ku* [2]) *Let G be a finite group acting freely on M^{4k}. Then if G acts trivially on $H^{2k}(M\,;$*

R), the index $I(M)$ is zero.

EXAMPLE. Let CP^{2k} be a $4k$-dimensional complex projective space. We have $I(CP^{2k})=1$. Let p be an odd integer. Denote $4k$-dimensional closed manifold M by connected sum of $S^1\times S^{4k-1}$ and p-disjoint copies of CP^{2k}. Then M is diffeomorphic to L : L is



There exists an orientation preserving free Z_p-action on L by the cyclic permutation of each component. And so there exists an orientation preserving free Z_p-action on M via the diffeomorphism from M to L. And hence we have $[Z_p, M]\in \Omega_{4k}(Z_p)$.

In the case p: odd, $RO(Z_p)=Z[K]+Z[V_1]+\cdots+Z[V_{\frac{p-i}{2}}]$, where K is a trivial representation and each $V_i\left(i=1\cdots\frac{p-1}{2}\right)$ is the representation: $Z_p=\langle\zeta\rangle$ ζ: generator,

$$\zeta\longrightarrow\begin{pmatrix}\cos\frac{2\pi i}{p} & -\sin\frac{2\pi i}{p}\\ \sin\frac{2\pi i}{p} & \cos\frac{2\pi i}{p}\end{pmatrix}.$$

Since M/Z_p is diffeomorphic to the manifold attached one handle to CP^{2k}, it is cobordant to CP^{2k}. By the easy computation, $I[Z_p, M]=[K]+[V_1]+\cdots+[V_{\frac{p-1}{2}}]$.

REMARK. We consider the case where $G=Z_2$ and $\mathscr{T}$ is non-trivial. Let $Z_2=\langle T\rangle$ and M^T be fixed points set and denote self-intersection by $(M^T)^2$. Then

$\operatorname{Sign}(T, M)=I((M^T)^2)$, where $\operatorname{Sign}(T, M)=\operatorname{trace}(T^*|V_+)-\operatorname{trace}(T^*|V_-)$. (This is the proposition 6.15 in [3].) Using this result, it follows that if $[Z_2, M]\in\Omega_{4k}(Z_2, \mathscr{T})$, then

$$I:\ \Omega_{4k}(Z_2, \mathscr{T})\longrightarrow RO(Z_2)\ \text{is}$$

$I[Z_2, M]=\frac{1}{2}(I(M)+I((M^T)^2))[K]+\frac{1}{2}(I(M)-I((M^T)^2))[K_-]$, where K_- is one dimensional representation $T\longrightarrow -1$. And so, if Z_2 acts as ± 1 on $H^{2k}(M; R)$, then $I(M)=\pm I((M^T)^2)$. (Of course it follows also from the prop. 6.15

in [3].)

NOTE. In the unoriented case, the similar result was proved by R. Stong in [8]. If Z_2 acts on a $2n$-dimensional unoriented manifold, then

$$\chi(M) \equiv \chi((M^T)^2) \pmod 2.$$

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