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Primitive extensions of rank 3 of $2^{n}\cdot GL(n, 2)$

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1. Introduction.

As is well known, $(n+1)$-dimensional general linear group $GL(n+1,2)$ $=PSL(n+1,2)$ over $GF(2)$, the field with two elements is simple for $n\geqq 2$ and acts doubly transitively on $P(n, 2)$, the set of the points of n-dimensional projective space over $GF(2)$. Taking a point p in $P(n, 2)$, set $\Delta=P(n, 2)-\{p\}$ and let H be the stabilizer of p in $GL(n+1,2)$. Then H is the semi-direct product of an elementary abelian group of order 2^{n} and $GL(n, 2)$. The transitive permutation group (H, Δ) has rank 2 extension $(GL(n+1,2), P(n, 2))$. In this note, we determine primitive extensions of rank 3 of (H, Δ).

THEOREM. *Let* (G, Ω) *be a primitive extension of rank* 3 *of* (H, Δ). *Then*

(i) $n=1$ *and* (G, Ω) *is isomorphic to the dihedral group of order* 10 *acting on* 5 *letters, or*

(ii) $n=2$ *and* (G, Ω) *is isomorphic to the alternating group* A_{6} *acting on the unordered pairs of* $\{1,2,3,4,5,6\}$.

The idea of the proof of our Theorem is due to Bannai [2], which determined primitive extensions of rank 3 of $(PSL(n, 2^{f}), P(n-1,2^{f}))$, and the author thanks Dr. E. Bannai. He is also grateful to the referee for setting Lemma 7 a better form.

NOTATION. We follow the notation of Higman [4] mostly and use [4] frequently. In a transitive permutation group G on a finite set Ω, we denote by a^{g} the image of $a\in\Omega$ under $g\in G$, and for a subset X of Ω, G_{X} denotes the pointwise stabilizer of X, $G_{X}=\{g\in G|x^{g}=x$ for all $x\in X\}$. If $X=\{a, b, \cdots\}$, G_{X} is written $G_{ab\cdots}$. For a subset Y of G and $g\in G$, we let $Y^{g}=g^{-1}Yg$, $g^{Y}=\{g^{y}=y^{-1}gy|y\in Y\}$ and $a^{Y}=\{a^{y}|y\in Y\}$. The number of G_{a}-orbits $(a\in\Omega)$ counting $\{a\}$, is called the rank of (G, Ω).

The following notation will be fixed throughout this note. Let (G, Ω) be a primitive extension of rank 3 of (H, Δ), that is, 1) (G, Ω) is a primitive permutation group of rank 3, and 2) there exists an orbit $\Delta(a)$ of the stabilizer G_{a} of a point $a\in\Omega$ such that G_{a} acts faithfully on $\Delta(a)$ and $(G_{a}, \Delta(a))$ is isomorphic to (H, Δ).

Let $\Gamma(a)$ be another non-trivial orbit of G_a and we may assume $\Delta(a)^g = \Delta(a^g)$ and $\Gamma(a)^g = \Gamma(a^g)$ for all $a \in \Omega$ and all $g \in G$. Set $k = |\Delta(a)|$ $(=|\Delta| = 2(2^n-1))$ and $l = |\Gamma(a)|$. The intersection numbers λ, μ for G are defined by

$$|\Delta(a) \cap \Delta(b)| = \begin{cases} \lambda & \text{if} \quad b \in \Delta(a) \\ \mu & \text{if} \quad b \in \Gamma(a). \end{cases}$$

Then the relation $\mu l = k(k-\lambda-1)$ holds by [4, Lemma 5].

2. Proof of Theorem.

In case $n=1$, we easily obtain (i) of Theorem and so we assume $n \geqq 2$ (so $k = 2(2^n-1) \geqq 6$) in the following. Since $(G_a, \Delta(a))$, (H, Δ) and $(GL(n+1, 2)_{(10\cdots0)}, P(n, 2)-\{(10\cdots0)\})$ are isomorphic to one another, we may assume $G_a = H = GL(n+1, 2)_{(10\cdots0)}$ and $\Delta(a) = \Delta = P(n, 2) - \{(10\cdots0)\}$. Take $b = (010\cdots0) \in \Delta(a)$. It is easily seen that G_{ab} has the orbits-length $1, 1, k-2$ on Δ. As G has even order, $\Delta(a)$ and $\Gamma(a)$ are self-paired by Wielandt [6, Theorem 16.5]. By [5], one of the following holds.

(*) $l > 1$ is a divisor of k, and $\lambda = 0$ or $k-2$,

(**) $l > k-2$ and l is a divisor of $k(k-2)$, and $\lambda = 0$ or 1.

LEMMA 1. The cases (**) with $\lambda = 0$ and (*) do not occur.

PROOF. Since $\mu l = k(k-\lambda-1)$ and $0 \leqq |\Gamma(a) \cap \Gamma(c)| = l - k + \mu - 1$ for $c \in \Gamma(a)$, we have

$\mu = k-1$ and $l = k$ in case (*) with $\lambda = 0$ or (**) with $\lambda = 0$,

$\mu = 1$ and $l = k$ in case (*) with $\lambda = k-2$.

In all the cases, by Higman [4, Lemma 7], $(\lambda-\mu)^2 + 4(k-\mu) = (k-1)^2 + 4$ must be a square, say $e^2, e > 0$. But, since $4 = (e+k-1)(e-(k-1))$, we have $2(k-1) = (e+k-1) - (e-(k-1)) = 3$ or 0, a contradiction.

So we are left with the case (**) with $\lambda = 1$ and throughout the rest of the paper we consider this case in detail. $\Delta(a) \cap \Gamma(b)$ is a G_{ab}-orbit of length $k-2$ and take a point $c \in \Delta(a) \cap \Gamma(b)$. As $\Delta(a)$ is self-paired, G contains an element g interchanging a and b by [6, Theorem 16.4]. Set $d = c^g \in \Delta(b) \cap \Gamma(a)$. Then $|G_{ab} : G_{abd}| = |G_{ab} : G_{abc}^g| = |G_{ab} : G_{abc}| = k-2$.

Now we want to know the possible values of μ, for then the possible values of l are known from $\mu l = k(k-2)$ and we can apply Higman [4, Lemma 7]. Since $\mu = |\Delta(a) \cap \Delta(d)|$ is a sum of lengths of some G_{ad}(or G_{abd})-orbits on $\Delta(a)$, it is sufficient to know the structure of G_{abd} and the lengths of G_{abd}-orbits on $\Delta(a)$. Let us set

$$G^{(n,i)}=\left\{\left(\begin{array}{c|c} \overbrace{*}^{i} & 0 \\ \hline * & * \end{array}\right)\}i \in GL(n, 2)\right\},$$

$$R^{(n,i)}=\left\{\left(\begin{array}{c|c} I_i & 0 \\ \hline * & I_{n-i} \end{array}\right) \in GL(n, 2)\right\} \text{ where } I_i \text{ denotes } i\times i \text{ identity matrix,}$$

$$S^{(n,i)}=\left\{\left(\begin{array}{c|c} I_i & 0 \\ \hline 0 & * \end{array}\right) \middle| * \in GL(n-i, 2)\right\}.$$

Moreover, set $K=G_{ab}$, $M=G_{abd}$, $R=R^{(n+1,2)}$ and $S=S^{(n+1,2)}$. Then we have $K=RS\rhd R$, $R\cap S=1$ and $|K:M|=k-2$. We denote by π and ρ, the natural homomorphism $K\to S$ and the natural isomorphism $S\to GL(n-1, 2)$, respectively. Furthermore, set $N=\rho\pi(M)$ and $m=|S:\pi(M)|=|GL(n-1, 2):N|$. Then we obtain $m=(k-2)|M\cap R|/|R|$. Note that $|R|=2^{2(n-1)}$ and m is a divisor of $k-2=2^2(2^{n-1}-1)$.

LEMMA 2. If $n\geqq 6$, then $N^t\subseteq G^{(n-1,1)}$ or $G^{(n-1,n-2)}$ for some $t\in GL(n-1, 2)$.

PROOF. From the above remark, it follows that $m\neq 1$ and m is not divisible by $2^{(n-1)-2}$. Hence by Bannai [1, Lemma 2], N fixes some complete subspace W of dimension, say $i-1$ of $P(n-2, 2)$. Noting that $GL(n-1, 2)$ is transitive on the set of all $(i-1)$-dimensional complete subspaces of $P(n-2, 2)$, we have $N^t\subseteq G^{(n-1,i)}$ for some $t\in GL(n-1, 2)$. But, since $|GL(n-1, 2):G^{(n-1,i)}|=(2^{n-1}-1)(2^{n-2}-1)\cdots(2^{n-1-(i-1)}-1)/(2^i-1)(2^{i-1}-1)\cdots(2-1)$ and $|GL(n-1, 2):N^t|$ is a divisor of $2^2(2^{n-1}-1)$, i must be 1 or $n-2$.

LEMMA 3. For $n\geqq 4$, $G^{(n,1)}$ and $G^{(n,n-1)}$ have no proper subgroup of index$\leqq 6$.

PROOF. Let T be a subgroup of $G^{(n,1)}=R^{(n,1)}S^{(n,1)}$ with $|G^{(n,1)}:T|\leqq 6$. Then $T\supseteq S^{(n,1)}$, for otherwise simple group $S^{(n,1)}\cong GL(n-1, 2)$ would have a proper subgroup $T\cap S^{(n,1)}$ of index$\leqq 6$ and $S^{(n,1)}$ would be contained in the symmetric group of degree 6, which is a contradiction.
Hence $T=(R^{(n,1)}\cap T)S^{(n,1)}$ and $S^{(n,1)}$ normalizes $R^{(n,1)}\cap T$ as $G^{(n,1)}\rhd R^{(n,1)}$. Since $R^{(n,1)}\cap T\neq 1$, take

$$r=\left(\begin{array}{c|c} 1 & 0 \\ \hline r_2 & \\ \vdots & I_{n-1} \\ r_n & \end{array}\right)\in R^{(n,1)}\cap T-\{1\}$$

Noting that

$$\left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A \end{array}\right) r \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & A^{-1} \end{array}\right) = \left(\begin{array}{c|c} 1 & 0 \\ \hline A\begin{pmatrix} r_2 \\ \vdots \\ r_n \end{pmatrix} & I_{n-1} \end{array}\right)$$

and $GL(n-1, 2)$ is transitive on the set of non-zero elements of the $(n-1)$-dimensional vector space over $GF(2)$, we obtain $R^{(n,1)}-\{1\}\subseteq r^{S^{(n,1)}}\subseteq T$ and so $G^{(n,1)}=R^{(n,1)}S^{(n,1)}\subseteq T$. As for $G^{(n,n-1)}$, a similar argument yields the result.

Combining Lemmas 2 and 3, we have $N^t=G^{(n-1,1)}$ or $G^{(n-1,n-2)}$ for some $t\in GL(n-1, 2)$ if $n\geqq 6$, that is,

LEMMA 4. If $n\geqq 6$, then $\pi(M)=M_1^s$ or M_2^s for some $s\in S$ and $m=(k-2)/4$, where

$$M_1=\left\{\left(\begin{array}{c|c} I_2 & 0 \\ \hline 0 & \begin{array}{c} 10\cdots 0 \\ * \end{array} \end{array}\right)\in GL(n+1, 2)\right\} \quad \text{and}$$

$$M_2=\left\{\left(\begin{array}{c|c} I_2 & 0 \\ \hline 0 & \begin{array}{cc} & 0 \\ * & \vdots \\ & 0 \\ & 1 \end{array} \end{array}\right)\in GL(n+1, 2)\right\}$$

LEMMA 5. If $n\geqq 6$, then $\pi(M)\subseteq M$.

PROOF. Since $m=(k-2)|M\cap R|/|R|=(k-2)/4$, $|MR:M|=4$. On the other hand, $\pi(M)\subseteq MR$ and so $|\pi(M):M\cap\pi(M)|\leqq 4$. Hence, Lemmas 3 and 4 yield $\pi(M)=M\cap\pi(M)$.

Now immediate calculations show

LEMMA 6. The lengths of the orbits of M_1 and M_2 on $\Delta(a)$ are respectively

$$\underbrace{1, \cdots, 1}_{6}, \quad \underbrace{(k-6)/4, \cdots, (k-6)/4}_{4}, \quad \text{and}$$

$$1, 1, \underbrace{(k-6)/8, \cdots, (k-6)/8}_{4}, \underbrace{(k+2)/8, \cdots, (k+2)/8}_{4}.$$

LEMMA 7. If $n\geqq 6$, the lengths of the orbits of M on $\Delta(a)$ are

(1) $1, 1;\ \underbrace{1, 1, 1, 1;\ k-6}$, or
(A subsum of these may be an orbit-length of M).

(2) $1, 1;\ \underbrace{(k-6)/2,\ (k+2)/2}$
(The sum may be an orbit-length of M).

PROOF. Using $s\in S(\subseteq K\subseteq G_a)$ in Lemma 4, set $M'=M^{s^{-1}}$. Then the lengths of the orbits of M on $\Delta(a)$ are equal to those of the orbits of M' on $\Delta(a)^{s^{-1}}=\Delta(a)$. Therefore it suffices to examine the orbits-structure of M' on $\Delta(a)$. Of course, $M'(\subseteq K)$ fixes $b=(010\cdots0)$ and $(110\cdots0)$.
Here we set

$$R_1=\left\{\left(\begin{array}{cc|c} \multicolumn{2}{c|}{I_2} & 0 \\ \hline * & 0 & \\ \vdots & \vdots & I_{n-1} \\ * & 0 & \end{array}\right)\in R\right\} \quad\text{and}\quad R_2=\left\{\left(\begin{array}{cc|c} \multicolumn{2}{c|}{I_2} & 0 \\ \hline 0 & * & \\ \vdots & \vdots & I_{n-1} \\ 0 & * & \end{array}\right)\in R\right\}.$$

Then $|M'\cap R_i|\geqq 2^{n-3}$, $i=1,2$ since $|K:M'|=2^2(2^{n-1}-1)$ and $|R_i|=2^{n-1}$.
Take elements

$$r_1=\left(\begin{array}{cc|c} \multicolumn{2}{c|}{I_2} & 0 \\ \hline \alpha_3 & 0 & \\ \vdots & \vdots & I_{n-1} \\ \alpha_{n+1} & 0 & \end{array}\right)\neq 1\in M'\cap R_1 \quad\text{and}\quad r_2=\left(\begin{array}{cc|c} \multicolumn{2}{c|}{I_2} & 0 \\ \hline 0 & \beta_3 & \\ \vdots & \vdots & I_{n-1} \\ 0 & \beta_{n+1} & \end{array}\right)\neq 1\in M'\cap R_2.$$

By Lemmas 4 and 5, $M'\supseteq\pi(M)^{s^{-1}}=M_1$ or M_2. Firstly, suppose that $M'\supseteq M_1$. Clearly the followings are M_1-orbits on $\Delta(a)$ of length $(k-6)/4=2(2^{n-2}-1)$;

$$(00010\cdots0)^{M_1}=\{(0,0,a_3,a_4,\cdots,a_{n+1})|(a_4,\cdots,a_{n+1})\neq(0,\cdots,0)\},$$
$$(10010\cdots0)^{M_1}=\{(1,0,a_3,a_4,\cdots,a_{n+1})|(a_4,\cdots,a_{n+1})\neq(0,\cdots,0)\},$$
$$(01010\cdots0)^{M_1}=\{(0,1,a_3,a_4,\cdots,a_{n+1})|(a_4,\cdots,a_{n+1})\neq(0,\cdots,0)\}$$

and

$$(11010\cdots0)^{M_1}=\{(1,1,a_3,a_4,\cdots,a_{n+1})|(a_4,\cdots,a_{n+1})\neq(0,\cdots,0)\}.$$

On the other hand, it is easily seen that r_1 carries an element of the first (resp. the third) to one of the second (resp. the fourth), and r_2 carries an element of the first to one of the third. Therefore, the above four M_1-orbits are contained in one M'-orbit. Also, though four points $(0010\cdots0)$, $(0110\cdots0)$, $(1010\cdots0)$ and $(1110\cdots0)$ are M_1-invariant, these may or may not be moved one another through r_1 and r_2. The case $M'\supseteq M_2$ is treated similarly.

Since μ is a subsum of the lengths of the orbits of M on $\Delta(a)$ and is a divisor of $k(k-2)$, from Lemma 7 we have (note that $\mu\neq 0$, k by [4, Corollary 3])

LEMMA 8. If $n\geqq 6$, μ is equal to one of the values; $1, 2, 3, 4, 5, 6$, $(k-2)/2$ and $k-2$.

LEMMA 9. The case (**) with $\lambda=1$ and $n\geqq 6$ does not occur.

PROOF. Noting that $\mu l=k(k-2)$, by Lemma 8 we can apply [4, Lemma 7] to conclude the result. For instance (set $D=(\mu-1)^2+4(k-\mu)$),

$\mu=2$: $D=4k-7$ is a square and divides $(2k+(1-\mu)(k+l))^2=(k(k-4)/2)^2$ and so does $7^2\cdot 3^4$, which is impossible since $k=2(2^n-1)$.

$\mu=3$: $D=4(k-2)$ and so $(k-2)/4=2^{n-1}-1$ is a square, say e^2, $e>0$. Hence $2(2^{n-2}-1)=(e-1)(e+1)$, which is a contradiction since $e-1$ and $e+1$ are even or odd simultaneously.

$\mu=(k-2)/2$: $D=(k/2)^2+8$ is a square, say e^2, $e>0$. Hence $8=(e-(k/2))(e+(k/2))$ and so $k=7$ or 2, a contradiction.

Now we are left with the (**) with $\lambda=1$ and $2\leqq n\leqq 5$.

LEMMA 10. The case (**) with $\lambda=1$ and $3\leqq n\leqq 5$ does not occur.

PROOF. Since μ is a divisor of $k(k-2)$, $k=2(2^n-1)$, we know the possible values of μ. In case $n=3$ with $\mu\neq 2, 4$ and $n=4, 5$, we have a contradiction by [4, Lemma 7]. When $n=3$, the order of any proper normal subgroup T of H is a divisor of 8. In fact, $H=G^{(4,1)}=R^{(4,1)}S^{(4,1)}$ and $T\cap S^{(4,1)}\lhd S^{(4,1)}$, hence $T\cap S^{(4,1)}=1$ or $S^{(4,1)}$. The former implies $|T|$ is a divisor of 8. The latter yields $T=S^{(4,1)}(R^{(4,1)}\cap T)$ and, since $R^{(4,1)}\cap T\neq 1$, as in the proof of Lemma 3, we obtain $T=H$. On the other hand, $|G|=2^6\cdot 3^3\cdot 7\cdot 11$ and $2^6\cdot 3^2\cdot 7\cdot 19$ in case $\mu=2$ and 4, respectively. In both cases G is not simple (e.g., Hall [3]). In the former case, for a minimal normal subgroup T of G, $|H\cap T|$ is a divisor of 8 by the above remark. Hence $|T|=2^i\cdot 3^2\cdot 11$, $0\leqq i\leqq 3$. Since T is characteristic simple and $|T|$ contains the prime 11 to the first power only, T must be simple. This is impossible from the order of T. Likewise we have a contradiction in case $\mu=4$.

LEMMA 11. In case (**) with $\lambda=1$ and $n=2$, (G, Ω) is isomorphic to the alternating group A_6 acting on the unordered pairs of $\{1, 2, 3, 4, 5, 6\}$.

PROOF. It is easily checked that (H, Δ) is isomorphic to the symmetric group S_4 acting on the unordered pairs of $\{1, 2, 3, 4\}$. By [4, Lemma 7], the case $\mu=3$, $|\Omega|=15$, $|G|=360$ remains. If G is not simple and has a minimal normal subgroup T, then $|T|=3\cdot 5, 2^2\cdot 3\cdot 5$ or $2^2\cdot 3^2\cdot 5$ and T is simple since T is characteristic simple. Hence $T\cong A_5$ and G is isomorphic to a subgroup of $Aut\ T\cong S_5$, which contradicts $|G|=360$. Thus G is simple and isomorphic to A_6. On the other hand, the following is checked: A_6 has two conjugate classes of elementary abelian subgroups of order 4, whose representatives are $V_1=\{1, (12)(34), (13)(24), (14)(23)\}$ and $V_2=\{1, (12)(34), (12)(56), (34)(56)\}$. A_6 has two conjugate classes of subgroups isomorphic to S_4, whose representatives are the normalizers $N_{A_6}(V_1)$ and

$N_{A_6}(V_2)$, whose 3-elements have one 3-cycle and two 3-cycles, respectively, and there exists an outer automorphism of A_6 taking one class into the other. This establishes the lemma.

Thus we complete the proof of Theorem.

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