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A uniqueness theorem for surfaces in the large

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1. Introduction

It has long been conjectured that among the compact (connected) oriented surfaces in E^3 only the sphere has constant mean curvature μ. This was established by H. Hopf [4] for surfaces of genus zero. For arbitrary genus a proof was given by A. D. Aleksandrov [1] under the assumption that there are no self intersections, *i.e.*, for surfaces embedded in E^3. Recently S.-S. Chern and the author [2] showed that a closed orientable surface of constant μ such that $K^2 \geqq C/2\mu^2$ is a sphere. (K is the Gaussian curvature and C is a nonnegative scalar invariant of the Gauss map of the immersed surface.) In this note the following theorem is obtained by a similar method.

THEOREM. *A closed orientable surface* in E^3 *of constant mean curvature* μ *with* $K(\mu^2 - K) \geqq -C/4\mu^2$ *is a sphere.*

T. Klotz and R. Osserman [6] proved that a complete surface immersed in E^3 with constant μ and $K \geqq 0$ is a sphere, a plane, or a right circular cylinder. Our method also yields a new proof of this since under the conditions K is a superharmonic function. (Observe that in the Theorem, K is not assumed to be nonnegative.)

2. Harmonic Mappings

To prove the theorem, the well-known fact that the Gauss map of a surface S in E^3 is harmonic if the mean curvature is constant, is used. A brief review of the theory as given in [3] is now presented.

Let M and N be smooth oriented Riemannian manifolds of dimensions m and n, respectively, with the metrics ds_M^2 and ds_N^2, and let $f: M \rightarrow N$ be a smooth mapping. Locally, then, $ds_M^2 = \sum \omega_i^2$ and $ds_N^2 = \sum \omega_a^{*2}$, where the ω_i, $i = 1, \cdots, m$ and ω_a^*, $a = 1, \cdots, n$ are linear differential forms in M and N, respectively. (Corresponding quantities in N are denoted with an asterisk.) We write $f^* \omega_a^* = \sum A_i^a \omega_i$. The covariant differential D of A_i^a is defined by

$$DA_i^a \equiv dA_i^a + \sum_j A_j^a \omega_{ji} + \sum_b A_i^b \omega_{ba}^* = \sum_j A_{ij}^a \omega_j \quad \text{(say)} \tag{2.1}$$

with $A^a_{ij}=A^a_{ji}$, where the ω_{ji} and ω^*_{ba} are the connection forms of ds^2_M and ds^2_N, respectively. (In the sequel, the symbol f^* is omitted from such formulas with no resulting confusion.) The mapping f is said to be *harmonic* if $\sum A^a_{ii}=0$.

Taking the exterior derivative of (2.1) and using the structure equations in M and N, we get if f is harmonic, the Laplacian

$$\sum_k A^a_{ikk}=\sum_k A^a_k R_{ki}-\sum_{b,c,d,k} R^*_{bacd}A^b_k A^c_k A^d_i,$$

where R_{ki} is the Ricci tensor of ds^2_M and R^*_{bacd} is the curvature tensor of ds^2_N. Let $u=\sum(A^a_i)^2$. Then, its Laplacian $\Delta u=\sum u_{kk}$ is given by

$$\text{(2.2)}\qquad \frac{1}{2}\Delta u=\sum_{a,i,j}(A^a_{ij})^2+\sum_{a,i,j}R_{ij}A^a_i A^a_j-\sum_{\substack{a,b,c,d\\ i,j}}R^*_{abcd}A^a_i A^b_j A^c_i A^d_j.$$

The details of the proof of this formula may be found in [3].

The last term in (2.2) may be expressed as

$$\text{(2.3)}\qquad \sum R^*_{abcd}A^a_i A^b_j A^c_i A^d_j=2\sum_{i<j}R^*(A_i,A_j)\|A_i\wedge A_j\|^2,$$

where A_i is the local vector field with components $(A^1_i,\cdots,A^n_i)$ and $R^*(A_i,A_j)$ denotes the sectional curvature of N along the section spanned by A_i and A_j at each point.

3. Proof of the Theorem

Let $M=S$, $N=$the unit sphere in E^3 with the constant curvature metric, and let f: $S\to N$ be the Gauss map. In this case, $A^a_i=h_{ia}$, $i,a=1,2$, where the h_{ia} are the coefficients of the second fundamental form of the immersion. Since $\mu=$const., f is a harmonic map. Noting that $2\mu=h_{11}+h_{22}$, $K=h_{11}h_{22}-(h_{12})^2$, $u=4\mu^2-2K$ and $A^a_{ij}=h_{ija}$, the formula (2.2) yields after an elementary computation,

$$\text{(3.1)}\qquad -\Delta K=4K(\mu^2-K)+\frac{C}{\mu^2},$$

where C is the nonnegative scalar invariant of f given by $C=\mu^2\sum(h_{ijk})^2$. For, the *r.h.s.* of (2.3) equals $2K^2$ since $R^*(A_i,A_j)=1$ and $\sum_{i<j}\|A_i\wedge A_j\|^2=K^2$. By hypothesis, $K(\mu^2-K)\geqq -C/4\mu^2$, so S being compact, K must be a positive constant. Thus, S is a sphere.

It was shown in [2] that on a surface of constant mean curvature in E^3 the function $\log K^2$ is superharmonic wherever $K\neq 0$. However, we were unable to obtain a geometrical interpretation of this fact.

If $K\geqq 0$ and $\mu=$const., formula (3.1) says K is a superharmonic function. We assume S is complete. If $K\equiv 0$, formula (3.1) says that

S is either a plane or a right circular cylinder. If $K \not\equiv 0$ and S is compact, it is a sphere by the theorem. If $K \not\equiv 0$ and S is not compact, Theorem 15 in A. Huber [5] says that S is parabolic. It follows that K is a constant greater than zero, so S is compact, and we have a contradiction.

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