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A counter example of Gross' star theorem

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1. Introduction.

Let R be an open Riemann surface. Let $w=f(p)$ be a meromorphic function on the sufrace R. We denote by Φ_f the covering surface generated by the inverse function of $w=f(p)$ over the extended w-plane. Let $q_0 \in \Phi_f$ be a regular point on Φ_f lying over the basic point $w_0=f(p_0) \neq \infty$ and let l_θ be the longest segment on Φ_f that starts from q_0, consists of only regular points of Φ_f and lies over the half straight line $\arg(w-w_0)=\theta$ $(0\leq\theta<2\pi)$ on the w-plane. Here a regular point of Φ_f is a point of Φ_f not being an algebraic branch point. If l_θ has finite length, then l_θ is said to be a singular segment with its argument θ of Φ_f. The set $\Omega=\bigcup_{0\leq\theta<2\pi} l_\theta$ is clearly a domain and is called a Gross' star region with the center q_0 on Φ_f. We call the set $S_\Omega=\{e^{i\theta} \mid l_\theta \text{ is singular}\}$ the singular set of Ω. Since the set $S_n=\{e^{i\theta} \mid \text{the length of } l_\theta \leq n\}$ is a closed set on the unit circle $\Gamma=\{|w|=1\}$ and $S_\Omega=\bigcup_{n=1}^{\infty} S_n$, S_Ω is an F_σ set on Γ.

If for any Gross' star region Ω on Φ_f the linear measure $(dm=d\theta)$ of S_Ω equals zero, then we say that the function $f(p)$ or Φ_f has the Gross' property. Further, if any meromorphic function on R has the Gross' property, then we say that the Riemann surface R has the Gross' property. W. Gross [1] proved that $R=\{|z|<\infty\}$ has the Gross' property. And M. Tsuji [cf. 5] extended this Gross' theorem in the following: If R is a domain on the z-plane and the boundary of R is of logarithmic capacity zero, then R has the Gross' property. And Z. Yûjôbô [cf. 5] proved a Riemann surface $R \in O_g$ has the Gross' property. And Z. Kuramochi [2], [3] proved that there exists a Riemann surface $R \in O_{HP}$ such that R has not the Gross' property and also there exists a domain $R \in O_{AB}-O_g$ on the z-plane such that R has not the Gross' property.

Further Z. Kuramochi considered the next K. Noshiro's problem: *Is the singular set of any Gross' star region of a covering surface belonging to O_g a set of capacity zero?* Let H be an F_σ set on Γ. If there exists a sequence of closed sets $F_n (n=1, 2, \cdots)$ such that $H=\bigcup_{n=1}^{\infty} F_n$ and $F_n \cap \overline{H-F_n}=\phi$ for every n, then the F_σ set H is called a *discrete* F_σ set. Z. Kuramochi proved the following:

THEOREM [4]. *Let H be an arbitrary discrete F_σ set of linear measure zero on Γ. Then there exists a planar covering surface belonging to O_g over the w-plane which has a Gross' star region with the singular set H.*

In the present paper we show that the above theorem is valid for an arbitrary F_σ set of linear measure zero.

THEOREM. *Let H be an arbitrary F_σ set of linear measure zero on Γ. Then there exists a planar covering surface belonging to O_g over the w-plane which has a Gross' star region with the singular set H.*

But the connectivity of covering surfaces of the above theorems is infinite. It still remain to solve the problem: Let Φ be a covering surface which is conformal equivalent to $\{|z|<\infty\}$. Then, is the singular set of any Gross' star region of Φ a set of capacity zero?

2. Kuramochi's lemma.

Let I be a closed interval in Γ such that $I=\{e^{i\theta}\,|\,0\leq a\leq\theta\leq b<2\pi\}$. Then $m(I)=b-a$. Take real numbers $R>0$ and $\alpha>0$ such that $R\exp(-\alpha)>1$. Let $I_i\,(i=0, 1, \cdots, n)$ be closed intervals such that

$$I_i=\left\{w\,\middle|\,|w|=R\exp\left(-\alpha+\frac{i\alpha}{2n}\right),\ a\leq\arg w\leq b\right\}.$$

We cut slits in the w-plane. Set

$$\mathscr{L}_i=\left\{|w|\leq\infty\right\}-I_{i-1}\cup I_i\,(i=1, 2, \cdots, n) \text{ and } \mathscr{L}_{n+1}=\left\{|w|\leq\infty\right\}-I_n\,.$$

And connect $\mathscr{L}_i$ and $\mathscr{L}_{i+1}$ crosswise on the slit $I_i\,(i=1, 2, \cdots, n)$. Thus we have an $(n+1)$-sheeted planar covering surface of the w-plane. We denote this resulting surface by $\mathfrak{R}_n(I;\,R\exp(-\alpha))$. Every branch point of $\mathfrak{R}_n(I;\,R\exp(-\alpha))$ lies over the segments:

$$\left\{w\,\middle|\,R\exp(-\alpha)\leq|w|\leq R\exp\left(-\frac{\alpha}{2}\right),\ \arg w=a \textit{ and } b\right\}.$$

And the border of $\mathfrak{R}_n(I;\,R\exp(-\alpha))$ is the set I_0 in $\mathscr{L}_1$. Let $\omega(z)=\omega(z;\,I, R\exp(-\alpha))$ be the continuous function on $\mathfrak{R}_n(I;\,R\exp(-\alpha))$ such that the boundary value 0 on I_0 in $\mathscr{L}_1$, harmonic in $\mathfrak{R}_n(I;\,R\exp(-\alpha))-\mathscr{L}_{n+1}$ and $\omega(z)=1$ on $\mathscr{L}_{n+1}$. Let $D(\omega)$ be the Dirichlet integral of ω on $\mathfrak{R}_n(I;\,R\exp(-\alpha))$. Then Z. Kuramochi proved the following result:

LEMMA ([4]). $D(\omega)\leq\dfrac{4\,m(I)}{\alpha}+\dfrac{\pi}{n}$.

As the consequence, if $n \geq \frac{\alpha\pi}{m(I)}$, *then* $D(\omega) \leq \frac{5\,m(I)}{\alpha}$.

For any given closed interval I and any given $\alpha > 0$, we fix the integer $n = n(I, \alpha)$ such that $\frac{\alpha\pi}{m(I)} \leq n < \frac{\alpha\pi}{m(I)} + 1$. Then we write $\mathfrak{R}(I;\, R\exp(-\alpha)) = \mathfrak{R}_n(I;\, R\exp(-\alpha))$. And we denote by $\mathscr{L}_1(I;\, R\exp(-\alpha))$ the first sheet $\mathscr{L}_1$ of $\mathfrak{R}(I;\, R\exp(-\alpha))$ and denote by $\mathscr{L}(I;\, R\exp(-\alpha))$ the last sheet $\mathscr{L}_{n+1}$ of $\mathfrak{R}(I;\, R\exp(-\alpha))$.

Let P be a point of Γ. We denote by $\mathfrak{R}(P;\, R\exp(-\alpha))$ the surface which is obtained from $\{|w| \leq \infty\}$ by deleting the slit

$$\left\{w = re^{i\theta} \middle| R\exp(-\alpha) \leq r \leq R\exp\left(-\frac{\alpha}{2}\right),\ \theta = \arg P\right\}.$$

3. A partition of an F_σ set H.

Let H be an F_σ set of linear measure zero on Γ. Then the F_σ set H is a countable union of closed sets $F_n (n = 1, 2, \cdots)$; $H = \bigcup_n F_n$. Take F_1. Since the complement CF_1 of F_1 is open, CF_1 is the union of a countable collection of disjoint open intervals $I(i)$ $(i = 1, 2, \cdots)$; $CF_1 = \bigcup_i I(i)$. By $F_1 = \bigcap_{n=1}^{\infty} (C(\bigcup_{i=1}^{n} I(i)))$ and $m(F_1) = 0$, $\lim_{n\to\infty} m(C(\bigcup_{i=1}^{n} I(i))) = 0$. Let $a(j)$ be an integer such that $m(C(\bigcup_{i=1}^{a(j)} I(i)) < \frac{1}{4^j}$ and $1 \leq a(1) \leq a(2) \leq \cdots$. $C(\bigcup_{i=1}^{a(j)} I(i))$ is the union $J(j)$ and $P(j)$, where $J(j)$ is the union of a finite collection of disjoint closed intervals $J_i(j)$ $(i = 1, 2, \cdots, N(J(j)))$ and $P(j)$ is an isolated and finite set. Then $J(j) \supset J(j+1)$ and $P(j) \subset P(j+1) \subset F_1$. We set $J(0) = \Gamma$ and $Q(j) = P(j) \cap J(j-1)$ for any $j \geq 1$. For an interval I on Γ we denoted by $e(I)$ the end points of I. Then we see the following:

(1) $J(j+1) \cup Q(j+1) \subset J(j)$ *and every component of* $J(j+1)$ *is contained in some component of* $J(j)$ $(j \geq 0)$.

(2) $m(J(j)) < \frac{1}{4^j}$ $(j \geq 1)$.

(3) $F_1 = \overline{\bigcup_{j=1}^{\infty} \bigcup_{i=1}^{N(J(j))} e(J_i(j))} \cup (\bigcup_{j=1}^{\infty} Q(j))$.

(*The poorf of* (3).) Since $e(J_i(j)) \cup P(j) \subset (\bigcup_k e(I(k))) \subset F_1$ and F_1 is closed, we have $\overline{\bigcup_j \bigcup_i e(J_i(j))} \cup (\bigcup_j Q(j)) \subset F_1$. Suppose $x \in F_1$. By $x \notin I(i)$ for every i, $x \in J(j) \cup P(j)$ for every $j (j \geq 1)$. Then $x \in P(j_0)$ for some j_0 or $x \in J(j)$ for every j. In the former case, $x \in Q(j)$ for some j. And in the latter case, there is some $i(j)$ such that $x \in J_{i(j)}(j)$ for every j. Then by (1),

$J_{i(1)}(1)\supset J_{i(2)}(2)\supset\cdots$ and $x\in\bigcap_j J_{i(j)}(j)$. Since $m(J(j))\to 0$ and $\bigcap_j J_{i(j)}(j)$ is connected, we have $\{x\}=\bigcap_j J_{i(j)}(j)\subset\overline{\bigcup_j e(J_{i(j)}(j))}$.

We set $J(j,0)=\overline{J(j)-J(j+1)}$ $(j\geq 0)$. Then $J(j,0)$ is the union of a finite collection of disjoint closed intervals. Take F_2. Set $F(2;j)=F_2\cap J(j,0)$. Then $F(2;j)$ is closed and we have

(4) $F_1\cup F_2=F_1\cup(\bigcup_{j=0}^{\infty}F(2;j))$

(*The proof of* (4).) We show the following: If $x\notin F_1$, then $x\in J(j,0)$ for some $j\geq 0$. Let $x\notin F_1$. Suppose $x\in I(k)$. If $1\leq k\leq a(1)$, then $x\notin J(1)\cup P(1)$. If $a(j)<k\leq a(j+1)$ for some $j\geq 1$, then $x\in J(j)\cup P(j)-J(j+1)\cup P(j+1)$. Since $P(j)\subset F_1$, by $x\notin F_1$, we see $x\notin J(1)$ or $x\in J(j)-J(j+1)$ for some $j\geq 1$, that is $x\in J(j,0)$ for some $j\geq 0$. Hence if $x\in F_2-F_1$, then $x\in F(2;j)$.

$J(j,0)-F(2;j)$ is the union of a countable collection of disjoint intervals $I(j,i)$ $(i=1,2,\cdots)$; $J(j,0)-F(2;j)=\bigcup_i I(j,i)$. These intervals are open intervals except for a finite number of half-open intervals. By $m(F(2;j))=0$, $\lim_{n\to\infty}m(J(j,0)-\bigcup_{i=1}^{n}I(j,i))=0$. Then for every $j\geq 0$ and every $k\geq 1$ there is an integer $a(j,k)$ such that $m(\bigcup_{\substack{j+k=n\\ j\geq 0,k\geq 1}}(J(j,0)-\bigcup_{i=1}^{a(j,k)}I(j,i)))<\frac{1}{4^{n+1}}$ and $1\leq a(j,1)\leq a(j,2)\leq\cdots$. $J(j,0)-\bigcup_{i=1}^{a(j,k)}I(j,i)$ is the union $J(j,k)$ and $P(j,k)$, where $J(j,k)$ is the union of a finite collection of disjoint closed intervals $J_i(j,k)$ $(i=1,2,\cdots,N(J(j,k)))$ and $P(j,k)$ is an isolated and finite set. Then $J(j,k)\supset J(j,k+1)$ and $P(j,k)\subset P(j,k+1)\subset F(2;j)$. We set $Q(j,k)=P(j,k)\cap J(j,k-1)$ $(k\geq 1)$ Then we see the following:

(5) $J(j,k+1)\cup Q(j,k+1)\subset J(j,k)$ *and every component of* $J(j,k+1)$ *is contained in some component of* $J(j,k)$ $(j\geq 0,k\geq 0)$.

(6) $\sum_{\substack{j+k=n\\ j\geq 0,k\geq 1}}m(J(j,k))<\frac{1}{4^{n+1}}$.

(7) $F(2;j)=\overline{\bigcup_{k=1}^{\infty}\bigcup_{i=1}^{N(J(j,k))}e(J_i(j,k))}\cup(\bigcup_{k=1}^{\infty}Q(j,k))$.

Suppose $F(k-1;i_1,\cdots,i_{k-2})$, $J(i_1,\cdots,i_{k-1})$ and $Q(i_1,\cdots,i_{k-1})$ are defined, where $J(i_1,\cdots,i_{k-1})$ $(i_1\geq 0,\cdots,i_{k-1}\geq 0)$ is the union of a finite collection of disjoint closed intervals, $Q(i_1,\cdots,i_{k-1})$ $(i_1\geq 0,\cdots,i_{k-2}\geq 0,i_{k-1}\geq 1)$ is an isolated and finite set and $J(i_1,\cdots,i_{k-2},i_{k-1}+1)\cup Q(i_1,\cdots,i_{k-2},i_{k-1}+1)\subset J(i_1,\cdots,i_{k-1})$. And $F(k-1;i_1,\cdots,i_{k-2})=F_{k-1}\cap J(i_1,\cdots,i_{k-2},0)$. We define $F(k;i_1,\cdots,i_{k-1})$, $J(i_1,\cdots,i_k)$ and $Q(i_1,\cdots,i_k)$. We set

$J(i_1,\cdots,i_{k-1},0)=\overline{J(i_1,\cdots,i_{k-1})-J(i_1,\cdots,i_{k-2},i_{k-1}+1)}$

Then $J(i_1, \cdots, i_{k-1}, 0)$ is the union of a finite collection of disjoint closed intervals. And we set

$$F(k\,;\, i_1, \cdots, i_{k-1}) = J(i_1, \cdots, i_{k-1}, 0) \cap F_k\,.$$

Then $F(k\,;\, i_1, \cdots, i_{k-1})$ is closed and we have

(8) $F_1 \cup F_2 \cup \cdots \cup F_k = F_1 \cup (\bigcup_{j\geq 0} F(2\,;\, j)) \cup \cdots \cup (\bigcup_{i_1\geq 0, \cdots, i_{k-1}\geq 0} F(k\,;\, i_1, \cdots, i_{k-1}))$

(*The proof of* (8)) We suppose

$$F_1 \cup F_2 \cup \cdots \cup F_{k-1} = F_1 \cup (\bigcup_j F(2\,;\, j)) \cup \cdots \cup (\bigcup_{i_1, \cdots, i_{k-2}} F(k-1\,;\, i_1, \cdots, i_{k-2})).$$

$x \notin F_1 \cup (\bigcup_j F(2\,;\, j)) \cup \cdots \cup (\bigcup_{i_1, \cdots, i_{k-2}} F(k-1\,;\, i_1, \cdots, i_{k-2}))$. Since $x \notin F_1$, $x \in J(i_1, 0)$ for some $i_1 \geq 0$. Since $x \notin F(2\,;\, i_1)$, $x \in J(i_1, i_2, 0)$ for some $i_2 \geq 0$. $\cdots$ Since $x \notin F(k-1\,;\, i_1, \cdots, i_{k-2})$, $x \in J(i_1, \cdots, i_{k-1}, 0)$ for some $i_{k-1} \geq 0$. Hence, if $x \in F_k - (F_1 \cup \cdots \cup F_{k-1})$, then $x \in F(k\,;\, i_1, \cdots, i_{k-1})$.

$J(i_1, \cdots, i_{k-1}, 0) - F(k\,;\, i_1, \cdots, i_{k-1})$ is the union of a countable collection of disjoint intervals $I(i_1, \cdots, i_{k-1}, i)$ $(i=1, 2, \cdots)$;

$$J(i_1, \cdots, i_{k-1}, 0) - F(k\,;\, i_1, \cdots, i_{k-1}) = \bigcup_i I(i_1, \cdots, i_{k-1}, i)\,.$$

These intervals are open intervals except for a finite number of half-open intervals. By $m(F(k\,;\, i_1, \cdots, i_{k-1}))=0$, $\lim_{n\to\infty} m(J(i_1, \cdots, i_{k-1}, 0) - \bigcup_{i=1}^{n} I(i_1, \cdots, i_{k-1}, i))=0$. Then for every $i_1 \geq 0, \cdots, i_{k-1} \geq 0$ and every $i \geq 1$, there is an integer $a(i_1, \cdots, i_{k-1}, i)$ such that

$$m(\bigcup_{\substack{i_1+\cdots+i_{k-1}+j=n \\ i_1\geq 0, \cdots, i_{k-1}\geq 0, j\geq 1}} (J(i_1, \cdots, i_{k-1}, 0) - \bigcup_{i=1}^{a(i_1, \cdots, i_{k-1}, j)} I(i_1, \cdots, i_{k-1}, i))) < \frac{1}{4^{n+(k-1)}} \text{ and}$$

$1 \leq a(i_1, \cdots, i_{k-1}, 1) \leq a(i_1, \cdots, i_{k-1}, 2) \leq \cdots$. $J(i_1, \cdots, i_{k-1}, 0) - \bigcup_{i=1}^{a(i, \cdots, i_{k-1}, j)} I(i_1, \cdots, i_{k-1}, i)$ is the union $J(i_1, \cdots, i_{k-1}, j)$ and $P(i_1, \cdots, i_{k-1}, j)$, where $J(i_1, \cdots, i_{k-1}, j)$ is the union of a finite collection of disjoint closed intervals $J_i(i_1, \cdots, i_{k-1}, j)$ $(i=1, 2, \cdots, N(J(i_1, \cdots, i_{k-1}, j))$ and $P(i_1, \cdots, i_{k-1}, j)$ is an isolated and finite set. We set $Q(i_1, \cdots, i_{k-1}, j) = P(i_1, \cdots, i_{k-1}, j) \cap J(i_1, \cdots, i_{k-1}, j-1)$ $(j \geq 1)$. Then $J(i_1, \cdots, i_{k-1}, j) \supset J(i_1, \cdots, i_{k-1}, j+1)$ and $Q(i_1, \cdots, i_{k-1}, j) \subset Q(i_1, \cdots, i_{k-1}, j+1) \subset F_k$. Then we see the following:

(9) $J(i_1, \cdots, i_{k-1}, i_k+1) \cup Q(i_1, \cdots, i_{k-1}, i_k+1) \subset J(i_1, \cdots, i_k)$ and every component of $J(i_1, \cdots, i_{k-1}, i_k+1)$ is contained in some component of $J(i_1, \cdots, i_k)$ $(i_1 \geq 0, \cdots, i_k \geq 0)$.

(10) $$\sum_{\substack{i_1+\cdots+i_k=n \\ i_1\geq 0, \cdots, i_{k-1}\geq 0, i_k\geq 1}} m(J(i_1, \cdots, i_k)) < \frac{1}{4^{n+(k-1)}}$$

(11) $$F(k\,;\, i_1, \cdots, i_{k-1}) = \overline{\bigcup_{i_k=1}^{\infty} \bigcup_{i=1}^{N(J(i_1, \cdots, i_k))} e(J_i(i_1, \cdots, i_k))} \cup (\bigcup_{i_k=1}^{\infty} Q(i_1, \cdots, i_k)).$$

4. The construction of the Riemann surface $\mathfrak{R}_H$.

In § 3, we defined $J(i_1, \cdots, i_k)$ and $Q(i_1, \cdots, i_k)$ for every k, where $J(i_1, \cdots, i_k)$ is the union of a finite collection of disjoint intervals $J_i(i_1, \cdots, i_k)$ on Γ ($i=1, 2, \cdots, N(J(i_1, \cdots, i_k))$ and $Q(i_1, \cdots, i_k)$ is a finite set on Γ. We set $Q(i_1, \cdots, i_k) = \{Q_i(i_1, \cdots, i_k) \in \Gamma\, ;\, i=1, 2, \cdots, N(Q(i_1, \cdots, i_k))\}$. We take a sequence of real numbers $\{R_n\}_{n=1}^{\infty}$ which satisfies the conditions $1 < R_n < R_{n+1} \exp\left(-\frac{1}{2}\right)$ and

$$(12) \qquad \frac{1}{\log \dfrac{R_n}{R_{n-1}}} \sum_{\substack{i_1+\cdots+i_k \le n-k \\ i_1 \ge 0, \cdots, i_{k-1} \ge 0, i_k \ge 1 \\ 1 \le k \le n-1}} N(J(i_1, \cdots, i_k)) \to 0 \ (\text{as } n \to \infty)$$

We consider the following Riemann surfaces ;

$$\mathfrak{R}(0) = \left\{|w| \le \infty\right\},$$

$$\mathfrak{R}\left(J_i(i_1, \cdots, i_k)\right) = \mathfrak{R}\left(J_i(i_1, \cdots, i_k)\, ;\, R_{i_1+\cdots+i_{k-1}+k} \exp\left(-\frac{1}{2^{i_1+\cdots+i_k}}\right)\right)$$

$$\mathfrak{R}\left(Q_i(i_1, \cdots, i_k)\right) = \mathfrak{R}\left(Q_i(i_1, \cdots, i_k)\, ;\, R_{i_1+\cdots+i_{k-1}+k} \exp\left(-\frac{1}{2^{i_1+\cdots+i_k}}\right)\right)$$

We denote by $\mathscr{L}_1(J_i(i_1, \cdots, i_k))$ and $\mathscr{L}(J_i(i_1, \cdots, i_k))$ the first sheet and the last sheet of $\mathfrak{R}(J_i(i_1, \cdots, i_k))$ respectively (See § 2). And we concider the following segments ;

$$K_i(i_1, \cdots, i_k) = \left\{w = re^{i\theta} \middle| r = R_{i_1+\cdots+i_{k-1}+k} \exp\left(-\frac{1}{2^{i_1+\cdots+i_k}}\right),\ e^{i\theta} \in J_i(i_1, \cdots, i_k)\right\}$$

$$L_i(i_1, \cdots, i_k) = \left\{w = re^{i\theta} \middle| r = R_{i_1+\cdots+i_{k-1}+k} \exp\left(-\frac{1}{2^{i_1+\cdots+i_k}}\right) \le r \le \right.$$

$$\left. \le R_{i_1+\cdots+i_{k-1}+k} \exp\left(-\frac{1}{2^{i_1+\cdots+i_k+1}}\right),\ \theta = \arg Q_i(i_1, \cdots, i_k)\right\}.$$

We cut slits $\bigcup_i K_i(1)$, $\bigcup_i L_i(1)$, $\bigcup_i K_i(\underbrace{0, \cdots, 0}_{m}, 1)$ and $\bigcup_i L_i(\underbrace{0, \cdots, 0}_{m}, 1)$ $(m=1, 2, \cdots)$ in $\mathfrak{R}(0)$ and we denote the resulting surface by $\mathfrak{R}'(0)$.

Next we cut slits $K'(i_1, \cdots, i_{k-1}, i_k+1)$, $L'(i_1, \cdots, i_{k-1}, i_k+1)$, $K'(i_1, \cdots, i_k, \underbrace{0, \cdots, 0}_{m}, 1)$ and $L'(i_1, \cdots, i_k, \underbrace{0, \cdots, 0}_{m}, 1)$ $(m=1, 2, \cdots)$ in the last sheet $\mathscr{L}(J_i(i_1, \cdots, i_k))$ of $\mathfrak{R}(J_i(i_1, \cdots, i_k))$, where $K'(\alpha)$ $(\alpha = (i_1, \cdots, i_{k-1}, i_k+1)$ or $(i_1, \cdots, i_k, \underbrace{0, \cdots, 0}_{m}, 1)$, $m=1, 2, \cdots)$ is the union of all components $K_j(\alpha)$ such that $J_j(\alpha) \subset J_i(i_1, \cdots, i_k)$ and $L'(\alpha)$ is the union of all components $L_j(\alpha)$ such that $Q_j(\alpha) \in J_i(i_1, \cdots, i_k)$. We denote the resulting surface by $\mathfrak{R}'(J_i(i_1, \cdots, i_k))$.

We shall connect all Riemann surfaces $\mathfrak{R}'(0)$,

$\mathfrak{R}'(J_i(i_1, \cdots, i_k))$ $(i_1 \geq 0, \cdots, i_{k-1} \geq 0,\ i_k \geq 1;\ i=1, 2, \cdots, N(J(i_1, \cdots, i_k);\ k=1, 2, \cdots)$ and

$\mathfrak{R}'(Q_i(i_1, \cdots, i_k))$ $(i_1 \geq 0, \cdots, i_{k-1} \geq 0,\ i_k \geq 1;\ i=1, 2, \cdots, N(Q(i_1, \cdots, i_k);\ k=1, 2, \cdots)$

First we connect $\mathfrak{R}'(0)$ with $\mathfrak{R}'(J_i(\beta))$ ($\beta=(1)$ or $\beta=(\underbrace{0, \cdots, 0}_{m}, 1)$, $i=1, 2, \cdots, N(J(\beta))$, $m=1, 2, \cdots$) crosswise across each slit $K_i(\beta)$ on $\mathfrak{R}'(0)$ and the slit $K_i(\beta)$ on the first sheet $\mathscr{L}_1(J_i(\beta))$ of $\mathfrak{R}'(J_i(\beta))$ and connect $\mathfrak{R}'(0)$ with $\mathfrak{R}(Q_j(\beta))$ ($j=1, 2, \cdots, N(Q(\beta))$) crosswise across each slit $L_j(\beta)$ on $\mathfrak{R}'(0)$ and the silt $L_j(\beta)$ on $\mathfrak{R}(Q_j(\beta))$. Next we connect $\mathfrak{R}'(J_i(i_1, \cdots, i_k))$ with $\mathfrak{R}'(J_j(\alpha))$ such that $J_j(\alpha) \subset J_i(i_1, \cdots, i_k)$ crosswise each slit $K_j(\alpha)$ on the last sheet $\mathscr{L}(J_i(i_1, \cdots, i_k))$ of $\mathfrak{R}'(J_i(i_1, \cdots, i_k))$ and the slit $K_j(\alpha)$ on $\mathscr{L}_1(J_j(\alpha))$ of $\mathfrak{R}'(J_j(\alpha))$ and connect $\mathfrak{R}'(J_i(i_1, \cdots, i_k))$ with $\mathfrak{R}(Q_j(\alpha))$ such that $Q_j(\alpha) \in J_i(i_1, \cdots, i_k)$ crosswise across each silt $L_j(\alpha)$ on the last sheet $\mathscr{L}(J_i(i_1, \cdots, i_k))$ of $\mathfrak{R}'(J_i(i_1, \cdots, i_k))$ and the slit $L_j(\alpha)$ on $\mathfrak{R}(Q_j(\alpha))$. We denote the resulting surface $\mathfrak{R}_H$.

The surface $\mathfrak{R}_H$ has the following properties:

(a) $\mathfrak{R}_H$ is a planar covering surface belonging to O_g over w-plane.

(b) Let Ω be a Gross' star region with center $0 \in \mathfrak{R}'(0)$ on $\mathfrak{R}_H$. Then the singular set of Ω equals H.

(*The proof of* $\mathfrak{R}_H \in O_g$)

We define a exhaustion $\{\Omega_n\}_{n=0}^{\infty}$ of $\mathfrak{R}_H$ in the following.

$$\Omega_0 = \left\{ w \in \mathfrak{R}'(0) \middle| |w| < R_1 \exp\left(-\frac{1}{2}\right)\right\}$$

$$\begin{aligned}\Omega_1 = &\left\{ w \in \mathfrak{R}'(0) \middle| |w| < R_2 \exp\left(-\frac{1}{2}\right)\right\} \\ &\cup \left(\bigcup_i \left(\mathfrak{R}'\left(J_i(1)\right) - \left\{ w \in \mathscr{L}\left(J_i(1)\right) \middle| |w| \geq R_1 \exp\left(-\frac{1}{2^2}\right)\right\}\right)\right) \\ &\cup \left(\bigcup_i \mathfrak{R}\left(Q_i(1)\right)\right)\end{aligned}$$

$$\begin{aligned}\Omega_2 = &\left\{ w \in \mathfrak{R}'(0) \middle| |w| < R_3 \exp\left(-\frac{1}{2}\right)\right\} \\ &\cup \left(\bigcup_i \left(\mathfrak{R}'\left(J_i(1)\right) - \left\{ w \in \mathscr{L}\left(J_i(1)\right) \middle| |w| \geq R_3 \exp\left(-\frac{1}{2^2}\right)\right\}\right)\right) \\ &\cup \left(\bigcup_i \left(\mathfrak{R}'\left(J_i(2)\right) - \left\{ w \in \mathscr{L}\left(J_i(2)\right) \middle| |w| \geq R_1 \exp\left(-\frac{1}{2^3}\right)\right\}\right)\right) \\ &\cup \left(\bigcup_i \left(\mathfrak{R}'\left(J_i(0, 1)\right) - \left\{ w \in \mathscr{L}\left(J_i(0, 1)\right| |w| \geq R_2 \exp\left(-\frac{1}{2^2}\right)\right\}\right)\right) \\ &\cup \left(\bigcup_i \mathfrak{R}\left(Q_i(1)\right)\right) \cup \left(\bigcup_i \mathfrak{R}\left(Q_i(2)\right)\right) \cup \left(\bigcup_i \mathfrak{R}\left(Q_i(0, 1)\right)\right).\end{aligned}$$

$$\Omega_n=\left\{w\in\Re'(0)\,\middle|\,|w|<R_{n+1}\exp\left(-\frac{1}{2}\right)\right\}$$

$$\cup\left(\bigcup_{\substack{1\le i\le N(J(i_1,\cdots,i_k))\\ i_1+\cdots+i_k\le n-k\\ 1\le k\le n-1}}\left(\Re'\left(J_i(i_1,\cdots,i_k)\right)\right.\right.$$

$$\left.\left.-\left\{w\in\mathscr{L}\left(J_i(i_1,\cdots,i_k)\right)\,\middle|\,|w|\ge R_{n+1}\exp\left(-\frac{1}{2^{i_1+\cdots+i_k+1}}\right)\right\}\right)\right)$$

$$\cup\left(\bigcup_{\substack{1\le i\le N(J(i_1,\cdots,i_k))\\ i_1+\cdots+i_k=n-k+1\\ 1\le k\le n}}\left(\Re'\left(J_i(i_1,\cdots,i_k)\right)\right.\right.$$

$$\left.\left.-\left\{w\in\mathscr{L}\left(J_i(i_1,\cdots,i_k)\right)\,\middle|\,|w|\ge R_{i_1+\cdots+i_{k-1}+k}\exp\left(-\frac{1}{2^{i_1+\cdots+i_k+1}}\right)\right\}\right)\right)$$

$$\cup\left(\bigcup_{\substack{1\le i\le N(J(i_1,\cdots,i_k))\\ i_1+\cdots+i_k\le n-k+1\\ 1\le k\le n}}\Re\left(Q_i(i_1,\cdots,i_k)\right)\right)$$

Let $\omega_n(w)$ be the hamonic measure of $\partial\Omega_n$ with respect to $\Omega_n-\overline{\Omega_{n-1}}$. Let r_1 and r_2 be positive real numbers such that $r_1<r_2$. We denote by $\omega(w\,;\,r_1,r_2)$ the hamonic mesure of $\{|w|=r_2\}$ with respect to the ring domain $\{r_1<|w|<r_2\}$. And we set

$$\omega(w\,;\,J_i(i_1,\cdots,i_k))=\omega\left(w\,;\,J_i(i_1,\cdots,i_k),\ R_{i_1+\cdots+i_{k-1}+k}\exp\left(-\frac{1}{2^{i_1+\cdots+i_k}}\right)\right)$$ (See §2 for the definition of $\omega(w\,;\,I,R\exp(-\alpha))$). Then by Dirichlet principle we have

$$D\left(\omega_n(w)\right)\le D\left(\omega(w\,;\,R_n,R_{n+1}\exp\left(-\frac{1}{2}\right)\right)$$

$$+\sum_{\substack{i_1+\cdots+i_k\le n-k\\ 1\le k\le n-1}}D\left(\omega(w\,;\,R_n,R_{n+1}\exp\left(-\frac{1}{2}\right)\right)\times N\left(J(i_1,\cdots,i_k)\right)$$

$$+\sum_{\substack{1\le i\le N(J(i_1,\cdots,i_k))\\ i_1+\cdots+i_k=n-k+1\\ 1\le k\le n}}D\left(\omega\left(w\,;\,J_i(i_1,\cdots,i_k)\right)\right)$$

Since Kuramochi's lemma,

$$D\left(\omega(w\,;\,J_i(i_1,\cdots,i_k)\right)\le\frac{5\times m(J_i(i_1,\cdots,i_k))}{\dfrac{1}{2^{i_1+\cdots+i_k}}},$$

Then, by (10), we have

$$\sum_{\substack{i_1+\cdots+i_k=n-k+1\\ 1\le k\le n}}\sum_i D\left(\omega(w\,;\,J_i(i_1,\cdots,i_k)\right)\le\frac{5\times 2^{n+1}}{4^n}=\frac{5}{2^{n-1}}.$$

Hence we have

$$D(\omega_n) \leq \frac{5}{2^n} + \frac{2\pi}{\log \frac{R_{n+1}\exp\left(-\frac{1}{2}\right)}{R_n}} \left(1 + \sum_{\substack{i_1+\cdots+i\leq n-k \\ 1\leq k\leq n-1}} N\Big(J(i_1, \cdots, i_k)\Big)\right)$$

Then by (12), we have $D(\omega_n) \to 0$ as $n \to \infty$. Thus we have $\mathfrak{R}_H \in O_g$.

(*The proof of* (b))

We note that the set of all branch points of the covering surface $\mathfrak{R}_H$ is the set

$$\cup \Big(e\big(J_i(i_1, \cdots, i_k)\big) \cup Q_i(i_1, \cdots, i_k)\Big).$$

Let S_Ω be the singular set of Ω. Since $F(k\,;\, i_1, \cdots, i_{k-1}) = J(i_1, \cdots, i_{k-1}, 0) \cap F_k\, (k \geq 2)$, we have

$$S_\Omega \cap \Big\{|w| \leq R_1\Big\} = F_1$$

$$S_\Omega \cap \Big\{R_1 < |w| \leq R_2\Big\} = F(2\,;\, 0)$$

$$S_\Omega \cap \Big\{R_2 < |w| \leq R_3\Big\} = F(2\,;\, 1) \cup F(3\,;\, 0, 0)$$

....

$$S_\Omega \cap \Big\{R_{n-1} < |w| \leq R_n\Big\} = \bigcup_{\substack{i_1+\cdots+i_{k-1}+k=n \\ 2\leq k\leq n}} F(k\,;\, i_1, \cdots, i_{k-1}).$$

Then $S_\Omega = \cup F(k\,;\, i_1, \cdots, i_k)$. Hence, by (8), we have $S_\Omega = H$.

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