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A Lindelöf type theorem on a Riemann surface

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1. Introduction and definitions

Let $f(z)$ be a bounded analytic function in $|z|<1$. If $f(z)$ has an asymptotic value α along some path in $|z|<1$ terminating at $e^{i\theta_0}$, then $f(z)$ has necessarily the angular limit α at $e^{i\theta_0}$ (Lindelöf's theorem). In this paper we study Lindelöf type theorem for an analytic mapping from a hyperbolic Riemann surface into another Riemann surface.

Let R be a hyperbolic Riemann surface. For a positive superharmonic function s on R and a closed set F in R, we denote by $s_F^R=s_F$ the lower envelope of the family of all positive superharmonic functions s' on R with $s'(z)\geq s(z)$ quasi-everywhere on F. Then s_F is superharmonic on R. Let γ: $z=z(t)$, $0\leq t<1$, be an arc in R such that γ tends to the ideal boundary of R as $t\rightarrow 1$. This means that for every compact set K in R there exists $t_0=t_0(K)$, $0<t_0<1$, with $\{z(t)|t_0\leq t<1\}\subset R-K$. Let $\{R_n\}_{n=1}^{\infty}$ be an exhaustion of R. For $0<\delta<1$, we set

$$\Omega_n(\gamma\,;\,\delta)=\left\{z\in R\,\middle|\,1_{\gamma\cap(R-R_n)}(z)>\delta\right\},$$

$$\Omega^*(\gamma\,;\,\delta)=\left\{z\in R\,\middle|\,1_{\gamma}(z)>\delta\right\},\ \Omega_n^*(\gamma\,;\,\delta)=\Omega^*(\gamma\,;\,\delta)\cap(R-\bar{R}_n).$$

Then $\bigcap_{n=1}^{\infty}\Omega_n^*(\gamma\,;\,\delta)=\phi$. If $\lim_{n\to\infty}1_{\gamma\cap(R-R_n)}(z)\not\equiv 0$, $\bigcap_{n=1}^{\infty}\Omega_n(\gamma\,;\,\delta)\neq\phi$. Let $\phi: R\rightarrow X$ be an arbitrary mapping from R into a compact metric space X. We define the following cluster sets:

$$\phi(\gamma\,;\,\delta)=\bigcap_{n=1}^{\infty}\overline{\phi(\Omega_n(\gamma\,;\,\delta))},\ \phi_{\Delta}(\gamma)=\bigcup_{0<\delta<1}\phi(\gamma\,;\,\delta),$$

$$\phi^*(\gamma\,;\,\delta)=\bigcap_{n=1}^{\infty}\overline{\phi(\Omega_n^*(\gamma\,;\,\delta))},\ \phi_{\Delta}^*(\gamma)=\bigcup_{0<\delta<1}\phi^*(\gamma\,;\,\delta).$$

In § 2, we show a relation between $\phi_{\Delta}(\gamma)$ and $\phi_{\Delta}^*(\gamma)$. If R is an open unit disk $\{|z|<1\}$ and $\gamma_\theta: z=z_\theta(t)=te^{i\theta}$, $0\leq t<1$, then $\phi^*(\gamma_\theta\,;\,\delta)$ coincides with an angular cluster set at $e^{i\theta}$ and $\phi_{\Delta}^*(\gamma_\theta)$ coincides with the outer angular cluster set at $e^{i\theta}$. Let $\phi(z)$ be an analytic mapping from R into another hyperbolic Riemann surface R', let R'^* be a metrizable compactification of R' and let $\lim_{t\to 1}\phi(z(t))=b\in R'^*$. We set

$$E(b)=E(b\,;\,R'^*)=\left\{a\in R'^*\,\middle|\,\lim_{n\to\infty}\overline{\lim_{w\to a}}\,1_{F_n(b)}(w)>0\right\},$$

where

$$F_n(b)=\left\{w\in R'\,\middle|\,d(w,b)\leq\frac{1}{n}\right\}\quad (d \text{ is a metric on } R'^*).$$

In § 3, we have $\phi_\Delta(\gamma)\subset E(b)$ (Proposition 2). By this proposition we investigate Lindelöf type theorem and Koebe type theorem. And we refer to Green lines and Kuramochi boundary.

2. A relation between $\phi_\Delta(\gamma)$ and $\phi^*_\Delta(\gamma)$

Let R be a hyperbolic Riemann surface, $\gamma: z=z(t)$, $0\leq t<1$ be an arc in R such that γ tends to the ideal boundary of R as $t\to 1$ and $\phi: R\to X$ be an arbitrary mapping from R into a compact metric space X.

LEMMA 1. (i) *If* $\lim_{n\to\infty}1_{\gamma\cap(R-R_n)}(z)\not\equiv 0$ *on* R, *then* $\phi(R)\subset\phi_\Delta(\gamma)$.

(ii) *If* $\lim_{n\to\infty}1_{\gamma\cap(R-R_n)}(z)\equiv 0$ *on* R, *then* $\phi_\Delta(\gamma)\subset\phi^*_\Delta(\gamma)$.

Proof. (i) We set $u(z)=\lim_{n\to\infty}1_{\gamma\cap(R-R_n)}(z)$. Then $u(z)$ is a positive harmonic function and $0<u(z)\leq 1$ on R. Suppose $0<u(z)<1$ on R. (If $u(z)\equiv 1$ on R, then $R\subset\Omega_n(\gamma\,;\,\delta)$ for any n and δ.) Since $1_{\gamma\cap(R-R_n)}(z)>u(z)$ for every $z\in R$ and every n, $z\in\Omega_n(\gamma\,;\,u(z))$ for every $z\in R$ and every n. Then $\phi(z)\in\phi(\gamma\,;\,u(z))\subset\phi_\Delta(\gamma)$ for every $z\in R$. Therefore $\phi(R)\subset\phi_\Delta(\gamma)$.

(ii) Since $1_{\gamma\cap(R-R_n)}(z)\leq 1_\gamma(z)$ for every $z\in R$, we have $\Omega_n(\gamma\,;\,\delta)\subset\Omega^*(\gamma\,;\,\delta)$ for every n and every δ, $0<\delta<1$. Now we fix δ, $0<\delta<1$ and m. By $\lim_{n\to\infty}1_{\gamma\cap(R-R_n)}(z)\equiv 0$ on R, there exists n such that $1_{\gamma\cap(R-R_n)}(z)\leq\delta$ for every $z\in\bar{R}_m$. Hence $\Omega_n(\gamma;\delta)\cap\bar{R}_m=\phi$. Then $\Omega_n(\gamma;\delta)=\Omega_n(\gamma;\delta)\cap(R-\bar{R}_m)\subset\Omega^*_m(\gamma;\delta)$. Hence we have $\phi(\gamma\,;\,\delta)\subset\phi^*(\gamma\,;\,\delta)$ and $\phi_\Delta(\gamma)\subset\phi^*_\Delta(\gamma)$.

PROPOSITION 1. *Let* $g(z,z_0)$ *be a Green function on* R *with pole at* z_0. *If* $\lim_{n\to\infty}1_{\gamma\cap(R-R_n)}(z)\equiv 0$ *on* R *and* $\lim_{\substack{z\in\Omega^*(\gamma;\delta)\\ z\to idbdy\ of\ R}}g(z,z_0)=0$ *for any* δ, $0<\delta<1$, *then* $\phi_\Delta(\gamma)=\phi^*_\Delta(\gamma)$.

Proof. We have only to prove $\phi^*_\Delta(\gamma)\subset\phi_\Delta(\gamma)$ by Lemma 1 (ii). Suppose $\alpha\in\phi^*(\gamma\,;\,\delta)$ for some δ, $0<\delta<1$. Then there exists a sequence $\{z_k\}_{k=1}^\infty$ such that $1_\gamma(z_k)>\delta$, $z_k\to$ *the ideal boundary of* R as $k\to\infty$ and $\lim_{k\to\infty}\phi(z_k)=\alpha$. For any m there is a constant C_m such that $1_{\gamma\cap\bar{R}_m}(z)\leq C_m\,g(z,z_0)$ for every $z\in R$. By $\lim_{k\to\infty}g(z_k,z_0)=0$, $\lim_{k\to\infty}1_{\gamma\cap\bar{R}_m}(z_k)=0$. Since $1_\gamma(z)\leq 1_{\gamma\cap(R-R_m)}(z)+1_{\gamma\cap\bar{R}_m}(z)$ on R, we have $\lim_{k\to\infty}1_{\gamma\cap(R-R_m)}(z_k)\geq\delta$. Hence for any δ', $0<\delta'<\delta$, there is some $n=n(m,\delta')$ such that $\{z_k\}_{k=n}^\infty\subset\Omega_m(\gamma\,;\,\delta')$. Then $\alpha\in\phi(\gamma\,;\,\delta')\subset\phi_\Delta(\gamma)$. Thus we have $\phi^*_\Delta(\gamma)\subset\phi_\Delta(\gamma)$.

3. A relation between $\phi_\Delta(\gamma)$ and E(b)

Let R be a hyperbolic Riemann surface and R^* be a metrizable compactification of R and $\Delta=R^*-R$. For every $b\in R^*$, we set

$$F_n(b)=\left\{z\in R \,\middle|\, d(z, b)\leq\frac{1}{n}\right\} \qquad (n=1, 2, \cdots),$$

where d is a metric on R^*. We consider the following subsets of R^*:

$E(b)=E(b\,;R^*)=\left\{a\in R^* \,\middle|\, \lim_{n\to\infty}\overline{\lim_{z\to a}}\, 1_{F_n(b)}(z)>0\right\}$,

$E_1(b)=E_1(b\,;R^*)=\{a\in R^* \mid$ Any positive superharmonic function $s(z)$ on R with $\lim_{z\to b}s(z)=+\infty$ has always the property $\lim_{z\to a}s(z)=+\infty\}$,

$E=E(R^*)=\{a\in\Delta \mid$ There is not a barrier function at $a\}$,

$\Delta_s=\Delta_s(R^*)=\left\{b\in\Delta \,\middle|\, \lim_{n\to\infty}1_{F_n(b)}(z)>0 \text{ for every } z\in R\right\}$.

LEMMA 2. (i) *$b\in E(b)$ for every $b\in R^*$.*

(ii) *If $b\in R^*-\Delta_s$, then $E(b)\subset E_1(b)$.*

(iii) *If $b\in R$, then $E_1(b)=\{b\}$. If $b\in\Delta-\Delta_s$, then $E_1(b)\subset\Delta$. If $b\in\Delta_s$, then $R\subset E(b)$.*

(iv) *If $b\in\Delta$, then $E(b)\cap\Delta\subset E\cup\{b\}$.*

Proof. (i) By $\lim_{z\to b}1_{F_n(b)}(z)=1$ for any n, we have $b\in E(b)$.

(ii) We note that for every $b\in R^*-\Delta_s$ there exists a positive superharmonic function $s_b(z)$ on R with $\lim_{z\to b}s_b(z)=+\infty$. Let $b\in R^*-\Delta_s$ and $a\notin E_1(b)$. Then there exists a positive superharmonic function s on R such that $\lim_{z\to b}s(z)=+\infty$ and $\overline{\lim_{z\to a}}\,s(z)=\lambda<+\infty$. We set $\inf_{z\in F_n(b)}s(z)=\alpha_n$. Then $\lim_{n\to\infty}\alpha_n=+\infty$ and $1_{F_n(b)}(z)\leq\frac{1}{\alpha_n}s(z)$ on R. Hence

$$\lim_{n\to\infty}\overline{\lim_{z\to a}}\,1_{F_n(b)}(z)\leq\lim_{n\to\infty}\overline{\lim_{z\to a}}\,\frac{1}{\alpha_n}s(z)=\lambda\lim_{n\to\infty}\frac{1}{\alpha_n}=0\,.$$

Then $a\notin E(b)$. Hence we have $E(b)\subset E_1(b)$ for every $b\in R^*-\Delta_s$.

(iii) Let $b\in R$ and let $g(z, b)$ be the Green function of R with pole at b. Since $\sup_{z\in R-V(b)}g(z, b)<+\infty$ for any neighbourhood $V(b)$ of b, we have $E_1(b)=\{b\}$ for every $b\in R$. Let $b\in\Delta-\Delta_s$ and $a\in R$. Then $\lim_{n\to\infty}\overline{\lim_{z\to a}}\,1_{F_n(b)}(z)=\lim_{n\to\infty}1_{F_n(b)}(a)=0$. Hence we have $E(b)\cap R=\phi$ for every $b\in\Delta-\Delta_s$. If $b\in\Delta_s$, then $\lim_{n\to\infty}\overline{\lim_{z\to a}}\, 1_{F_n(b)}(z)=\lim_{n\to\infty} 1_{F_n(b)}(a)>0$ for every $a\in R$. Hence we have $R\subset E(b)$ for every $b\in\Delta_s$.

(iv) Let $c\in\Delta-E\cup\{b\}$. Then there exists a positive superharmonic function $s_c(z)$ such that $\lim_{z\to c} s_c(z)=0$ and $\inf_{z\in R-V(c)} s_c(z)>0$ for any neighbourhood $V(c)$ of c. Since $b\neq c$, there is some n_0 and some neighbourhood $U(c)$ of c such that $\overline{F_{n_0}(b)}\cap\overline{U(c)}=\phi$. We set $\inf_{z\in F_{n_0}(b)} s_c(z)=\alpha_{n_0}$. Then $\alpha_{n_0}>0$ and $1_{F_{n_0}(b)}(z)\leq\dfrac{1}{\alpha_{n_0}}s_c(z)$ on R. Hence $\overline{\lim_{z\to c}}\, 1_{F_{n_0}(b)}(z)\leq\dfrac{1}{\alpha_{n_0}}\overline{\lim_{z\to c}}\, s_c(z)=0$. Thus $c\notin E(b)$. Therefore we have $E(b)\cap\Delta\subset E\cup\{b\}$.

PROPOSITION 2. *Let ϕ be an analytic mapping from R into another hyperbolic Riemann surface R' and R'^* be a metrizable compactification of R'. Let $\gamma: z=z(t)$, $0\leq t<1$ be an arc such that $z(t)$ tends to the ideal boundary of R' as $t\to 1$. If $\lim_{t\to 1}\phi(z(t))=b\in R'^*$, then $\phi_\Delta(\gamma)\subset E(b\,;R'^*)$.*

Proof. We fix δ $(0<\delta<1)$. We have only to show that for every $a\in R'^*-E(b)$ there exists $m=m(a)$ such that $\overline{\phi(\Omega_m(\gamma\,;\delta))}\not\ni a$. Let $a\in R'^*-E(b)$. Then there exists some n_0 and some neighbourhood $V(a)$ of a such that $1_{F_{n_0}(b)}(w)<\dfrac{\delta}{2}$ on $V(a)\cap R'$. Since $\phi(z(t))$ tends to b as $t\to 1$, there exists some $\gamma_m=\gamma\cap(R-R_m)$ such that $\phi(\gamma_m)\subset F_{n_0}(b)$. Then by $1_{\phi(\gamma_m)}(w)\leq 1_{F_{n_0}(b)}(w)$ on R', we have $1_{\phi(\gamma_m)}(w)<\dfrac{\delta}{2}$ on $V(a)\cap R'$. Next we note $1_{\gamma_m}(z)\leq 1_{\phi(\gamma_m)}\circ\phi(z)$ on R. Hence $1_{\phi(\gamma_m)}\circ\phi(z)>\delta$ for every $z\in\Omega_m(\gamma\,;\delta)$, i. e. $1_{\phi(\gamma_m)}(w)>\delta$ on $\phi(\Omega_m(\gamma\,;\delta))$. Therefore we have $(V(a)\cap R')\cap\phi(\Omega_m(\gamma\,;\delta))=\phi$ and $a\notin\overline{\phi(\Omega_m(\gamma\,;\delta))}$. Hence we have $\bigcap_{n=1}^{\infty}\overline{\phi(\Omega_n(\gamma\,;\delta))}\subset E(b)$ for every $0<\delta<1$. Thus $\phi_\Delta(\gamma)\subset E(b)$.

The next example shows that the case $\phi_\Delta(\gamma)=E(b)$ happens and $E(b)$ is not always a single point.

Example. Let R be the upper half disk $\{|z|<1, \mathrm{Im}\, z>0\}$. Let $\omega(z)$ be the harmonic measure of the segment $[-1,0]$ with respect to R at $z\in R$. We take $\{\omega(z)\}$ for Q. Then Q-compactification R_Q^* of R (cf. 96 in [1]) is metrizable and resolutive. For every $0<\theta<\pi$, we set $\gamma_\theta: z=z_\theta(t)=\dfrac{1}{2}(1-t)e^{i\theta}$, $0\leq t<1$. We know that γ_θ defines an ideal boundary point b_θ in R_Q^* as $t\to 1$ and $b_{\theta_1}\neq b_{\theta_2}$, if $\theta_1\neq\theta_2$. Let $\phi: R\to R$ be the identity mapping i. e. $\phi(z)=z$. Then we have $\lim_{t\to 1}\phi(z_\theta(t))=b_\theta$ and $\phi_\Delta(\gamma_\theta)=E(b_\theta\,;R_Q^*)=\{b_\theta\in\Delta_Q=R_Q^*-R\,|\,0<\theta<\pi\}$.

4. Remarks on Kuramochi boundary

LEMMA 3. *Let R^* be a metrizable compactification of R and $\Delta=R^*-R$. Let K_0 be a closed disk and $R_0=R-K_0$.*

(i) *Let $b\in\Delta$. Then $\lim_{n\to\infty} 1_{F_n(b)}^{R_0}(z)\equiv 0$ on R_0 if and only if $\lim_{n\to\infty} 1_{F_n(b)}(z)\equiv 0$ on R.*

(ii) *Let $b\in\Delta-\Delta_s$ and $a\in\Delta$. Then $\lim_{n\to\infty}\lim_{z\to a} 1_{F_n(b)}^{R_0}(z)=0$ if and only if $\lim_{n\to\infty}\overline{\lim_{z\to a}}\, 1_{F_n(b)}(z)=0$.*

Proof. "if part" of (i) and (ii) are obvious. (i) We may assume $K_0\cap F_1(b)=\phi$. We set $\sup_{z\in K_0} 1_{F_n(b)}(z)=\alpha_n$. We note that $1_{F_n(b)}(z)\leq 1_{F_n(b)}^{R_0}(z)+\alpha_n 1_{K_0}(z)$ on R_0. Suppose $\lim_{n\to\infty} 1_{F_n(b)}^{R_0}(z)=0$. Then $\lim_{n\to\infty} 1_{F_n(b)}(z)\leq(\lim_{n\to\infty}\alpha_n)\, 1_{K_0}(z)\leq 1_{K_0}(z)$. Since $\sup_{z\in F_n(b)} 1_{K_0}(z)<1$ for any n, we have $\lim_{n\to\infty} 1_{F_n(b)}(z)\equiv 0$ on R. (We know that if $\lim_{n\to\infty} 1_{F_n(b)}(z)\not\equiv 0$ on R then $\sup_{z\in F_n(b)}(\lim_{k\to\infty} 1_{F_k(b)}(z))=1$ for any n.)

(ii) Suppose $b\in\Delta-\Delta_s$ and $\lim_{n\to\infty}\overline{\lim_{z\to a}}\, 1_{F_n(b)}^{R_0}(z)=0$. Then $\lim_{n\to\infty}\overline{\lim_{z\to a}}\, 1_{F_n(b)}(z)\leq(\lim_{n\to\infty}\alpha_n)\cdot 1=0$. Hence we have $\lim_{n\to\infty}\overline{\lim_{z\to a}}\, 1_{F_n(b)}(z)=0$.

We refer to Abschnitte 16, 17 in [1] for the definitions and properties of the Kuramochi compactification R_N^* of R. Let $\Delta_{N,1}$, $\Delta_{N,0}$ and $\Delta_{N,S}$ be the set of all minimal points of $\Delta_N=R_N^*-R$, the set of all non minimal points of Δ_N and the set of all singular points of Δ_N respectively.

PROPOSITION 3. *If $b\in\Delta_{N,1}-\Delta_{N,S}$, then $E(b\,;R_N^*)\cap\Delta_{N,1}=\{b\}$.*

Proof. Let $a\in\Delta_1$, $a\neq b$. We shall show $a\notin E(b)$. Let $n\geq 2$. Suppose $n\tilde{g}_a(z)\geq\tilde{g}_b(z)$ for every $z\in R_0=R-K_0$. Then by minimum principle and $\int_{\partial K_0} {}^*d\tilde{g}_a=\int_{\partial K_0} {}^*d\tilde{g}_b=2\pi$, we may suppose $n\tilde{g}_a(z)>\tilde{g}_b(z)$ for every $z\in R_0$. Since $b\in\Delta_{N,1}-\Delta_{N,S}$, by Satz 17, 16 in [1] we see that $n\tilde{g}_a(z)-\tilde{g}_b(z)$ is a full-superharmonic function on R_0. Hence, by $a\in\Delta_1$, there exists a positive number c such that $c\tilde{g}_b(z)=n\tilde{g}_a(z)$ on R_0. Hence $a=b$. This is a contradiction. Then there exists a point $z_0\in R_0$ such that $n\tilde{g}_a(z_0)<\tilde{g}_b(z_0)$. We set $s_n(z)=\dfrac{\tilde{g}_{z_0}(z)}{\tilde{g}_{z_0}(b)}$ for every $z\in R_0$. We note $\tilde{g}_w(z)=\tilde{g}_z(w)$ for every $(z,w)\in R_N^*\times R_N^*$. Hence $s_n(z)$ is a full-superharmonic function on R_0 and $\lim_{z\to b} s_n(z)=1$ and $\lim_{z\to a} s_n(z)=\dfrac{\tilde{g}_{z_0}(a)}{\tilde{g}_{z_0}(b)}<\dfrac{1}{n}$. Thus we see that there exists a family of superharmonic functions $\{s_n\}$ such that $\lim_{z\to b} s_n(z)=1$ but $\lim_{z\to a} s_n(z)<\dfrac{1}{n}$. Hence we have $\lim_{n\to\infty}\overline{\lim_{z\to a}}\, 1^{R_0}_{F_n(b)}(z)=0$. Therefore we have $\lim_{n\to\infty}\overline{\lim_{z\to a}}\, 1_{F_n(b)}(z)=0$ by Lemma 3. Thus $a\notin E(b)$.

5. Theorems

THEOREM 1. (*Koebe type theorem*) *Let ϕ be an analytic mapping from R into another hypebolic Riemann surface R'. Let $\gamma: z=z(t)$, $0\leq t<1$ be an arc such that $z(t)$ tends to the ideal boundary of R as $t\to 1$ and $\lim_{n\to\infty} 1_{\gamma\cap(R-R_n)}(z)\neq 0$ on R. If $\lim_{t\to 1}\phi(z(t))=\alpha\in R'^*-\Delta_s$, then $\alpha\in R$ and $\phi(z)\equiv\alpha$ for every $z\in R$.*

Proof. By Lemm 1 (i) and Proposition 2, $\phi(R)\subset\phi_\Delta(\gamma)\subset E(\alpha\,;\,R'^*)$. If $\alpha\in\Delta-\Delta_s$, by Lemma 2 (iii), $E(\alpha)\subset\Delta$. Then $\phi(R)\subset\Delta$. This is a contradiction. Then $\alpha\in R$ and $E(\alpha)=\{\alpha\}$ by Lemma 2 (iii). Thus we have $\phi(R)=\{\alpha\}$.

THEOREM 2. (*Lindelöf type theorem*) *Let f be an analytic function on R with $f(R)\notin O_g$ and let $\gamma: z=z(t)$, $0\leq t<1$ be an arc such that $z(t)$ tends to the ideal boundary of R as $t\to 1$ and $\lim_{n\to\infty} 1_{\gamma\cap(R-R_n)}(z)\equiv 0$ on R. If $\lim_{t\to 1}f(z(t))=\alpha$, then $f_\Delta(\gamma)=\{\alpha\}$.*

Proof. If $\alpha\in f(R)$, then $E(\alpha\,;\overline{f(R)})=\{\alpha\}$ by Lemma 2 (iii). Let $b\in\partial f(R)$. If $b\neq\infty$ (resp. $b=\infty$), then there exists $\varepsilon>0$ (resp. $n>0$) such that $\Omega_\alpha=f(R)\cup\{|w-\alpha|<\varepsilon\}$ (resp. $\Omega_\alpha=f(R)\cup\{n<|w|<+\infty\}$) is a hyperbolic Riemann surface. Then $f(R)\subset\Omega_\alpha$, $\alpha\in\Omega_\alpha$, $\Omega_\alpha\notin O_g$. Let $s_\alpha(w)=g(w,\alpha\,;\,\Omega_\alpha)|f(R)$, where $g(w,\alpha\,;\,\Omega_\alpha)$ is a Green function of Ω_α with pole at α. Then $\overline{\lim_{w\to\alpha}}\, s_\alpha(w)=+\infty$ but $\overline{\lim_{w\to\beta}}\, s_\alpha(w)<+\infty$ for any $\beta(\neq\alpha)\in\overline{f(R)}$. Hence $E(\alpha\,;\overline{f(R)})=E_1(\alpha\,;\overline{f(R)})=\{\alpha\}$ by Lemma 2 (ii). Thus we have $f_\Delta(\gamma)=E(\alpha\,;\overline{f(R)})=\{\alpha\}$ by Proposition 2.

We consider the Green lines issuing from a fixed point $z_0\in R$. The set L_r of all regular Green lines l admits the Green measure m. Godfroid proved that any AD-function f on R posseses a radial limit almost everywhere on L_r, i. e. $\lim_{\substack{z\in l\\ g(z,z_0)\to 0}} f(z)$ exists for every $l\in L_r$ except a set of Green measure zero (cf. P. 203 in [3]).

THEOREM 3. *If $R\notin O_{AD}$, then $\lim_{n\to\infty} 1_{l\cap(R-R_n)}(z)\equiv 0$ on R for almost every $l\in L_r$.*

Proof. Suppose that there exists a subset $B\subset L_r$ such that $m(B)>0$ and $\lim_{n\to\infty} 1_{l\cap(R-R_n)}(z)\neq 0$ on R for any $l\in B$. Let f be a non constant AD-

function. Since $m(B)>0$, there is some $l_0 \in E$ such that $\lim_{\substack{z\in l_0 \\ g(z,z_0)\to 0}} f(z)$ exists by Godfroid's theorem. Set $\lim_{\substack{z\in l_0 \\ g(z,z_0)\to 0}} f(z)=\alpha$. We know $f(R)\notin O_g$ for every AD-function f on R. Since $\lim_{n\to\infty} 1_{l_0\cap(R-R_n)}(z)\neq 0$ on R, by Theorem 1, $f(z)\equiv\alpha$ on R. This is contradiction.

By Godfroid's theorem and Theorem 1 we obtain the next theorem.

THEOREM 4. *Let f be an AD-function. Then $f_\Delta(l)$ is a single point for every $l\in L_r$ except for a set of Green measure zero.*

By Proposition 2 and Proposition 3 we have the next theorem.

THEOREM 5. *Let ϕ be an analytic mapping from R into another hyperbolic Riemann surface R' and $\gamma: z=z(t)$, $0\leq t<1$ be an arc such that $z(t)$ tends to the ideal boundary of R as $t\to 1$ and R'^*_N be the Kuramochi compactification of R'. We suppose $\lim_{t\to 1}\phi(z(t))=b\in R'^*_N$. If $b\in R'$, then $\phi_\Delta(\gamma)=\{b\}$. If $b\in\Delta_{N,1}-\Delta_{N,S}$, then $\phi_\Delta(\gamma)\cap(R'^*_N-\Delta_{N,0})=\{b\}$.*

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