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Remarks on relatively flat modules

By Kenji NISHIDA

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Introduction

Let R be a ring and I be an idempotent ideal of R. Put $\mathscr{T}=\{{}_RM; IM=M\}$, $\mathscr{F}=\{{}_RM; IM=0\}$, and $\mathscr{C}=\{{}_RM; \mathrm{Ann}_M I=0\}$. Then $(\mathscr{T}, \mathscr{F})$, $(\mathscr{F}, \mathscr{C})$ is a TTF-theory over the category of all left R-modules. Since I is an ideal, we can define a TTF-theory over the category of all right R-modules in the same way. We denote it $(\mathscr{T}', \mathscr{F}')$, $(\mathscr{F}', \mathscr{C}')$.

Bland calls a left R-module M relatively flat if, for any exact sequence $0\rightarrow A\rightarrow B\rightarrow C\rightarrow 0$ of right R-modules such that $C\in\mathscr{F}'$, a sequence $0\rightarrow A\otimes M\rightarrow B\otimes M\rightarrow C\otimes M\rightarrow 0$ is exact [1]. In this paper we call such a module I-flat.

It is well known that a flat module is characterized by the purity in the sense of Cohn. We shall define the I-purity and give the similar characterization of the I-flatness.

In section 2, we investigate the I-flatness of R/I-modules. We give the characterization of a ring which has the property that each I-flat module is codivisible with respect to $(\mathscr{T}, \mathscr{F})$.

Throughout this paper, all rings are associative with unit and all modules are unital.

The reader is referred to [5] about the torsion theories.

1. On I-purity

It is well known that the purity in the sense of Cohn and the flatness are closely related. We shall show that the similar relation holds between the I-purity and the I-flatness.

DEFINITION 1–1. *We call an exact sequence* $0\rightarrow L\rightarrow X\rightarrow M\rightarrow 0$ *of left R-modules I-pure if, for each* $A\in\mathscr{F}'$, *a sequence* $0\rightarrow A\otimes L\rightarrow A\otimes X\rightarrow A\otimes M\rightarrow 0$ *is exact.*

We call a submodule U of V I-pure if the induced sequence $0\rightarrow U\rightarrow V\rightarrow V/U\rightarrow 0$ *is I-pure.*

THEOREM 1–2. *The following conditions are equivalent for a left R-module M.*

(1) *M is I-flat.*

(2) *Every exact sequence* $0 \to L \to X \to M \to 0$ *is I-pure.*

(3) *There exists an I-pure exact sequence* $0 \to L \to X \to M \to 0$ *such that* X *is I-flat.*

PROOF. (1) implies (2); Let $A \in \mathscr{F}'$. We take an exact sequence $0 \to C \to B \to A \to 0$, where B is free. We get the following commutative diagram;

$$\begin{array}{ccccccccc}
 & & C\otimes L & \longrightarrow & C\otimes X & \longrightarrow & C\otimes M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow f & & \\
0 & \longrightarrow & B\otimes L & \longrightarrow & B\otimes X & \longrightarrow & B\otimes M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A\otimes L & \overset{g}{\longrightarrow} & A\otimes X & \longrightarrow & A\otimes M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \qquad (*)$$

where the rows and the columns are exact. By (1) and $A \in \mathscr{F}'$, f is a monomorphism. Then we can show that g is a monomorphism by a diagram chase. Hence $0 \to L \to X \to M \to 0$ is I-pure.

(2) implies (3); This is clear, since we can take X free.

(3) implies (1); Let $0 \to C \to B \to A \to 0$ be an exact sequence of right R-modules such that $A \in \mathscr{F}'$. We get a commutative diagram $(*)$, where now the middle column instead of the middle row is a short exact sequence. By (3) g is a monomorphism. Then we can show that f is a monomorphism by a diagram chase. Hence M is I-flat.

We shall give the elementwise characterization of I-purity.

In the following Theorem 1–3, Corollary 1–5, and 1–6, we assume that I is a finitely generated right ideal.

THEOREM 1–3. *Let* U *be a submodule of* V. *Then the following conditions are equivalent.*

(1) U *is an I-pure submodule of* V.

(2) *Let* $v_1, \cdots, v_m \in V$, $u_1, \cdots, u_n \in U$, *and* $d_{ij} \in R (i=1, \cdots, n; j=1, \cdots, m)$, *where* $\{d_{ij}\}$ *satisfies the following condition* (I).

Condition (I); *For any* $x_1, \cdots, x_n \in I$, *there exist* $r_1, \cdots, r_m \in R$ *such that* $x_i = \sum_j d_{ij} r_j \quad (i=1, \cdots, n)$.

If $u_i = \sum_j d_{ij} v_j$, *then there exist* $u'_1, \cdots, u'_m \in U$ *such that* $u_i = \sum_j d_{ij} u'_j \ (i=1, \cdots, n)$.

PROOF. (1) implies (2); Let $v_j, u_i, d_{ij} \ (i=1, \cdots, n; j=1, \cdots, m)$ be given as in (2). Define $\alpha : R^m \to R^n$ be $\alpha = (d_{ij})$. Put $M = \mathrm{Coker}\ \alpha$. By (I), for every $(x_1, \cdots, x_n) \in I^n$, there exists $(r_1, \cdots, r_m) \in R^m$ such that $(x_1, \cdots, x_n) = \alpha(r_1, \cdots, r_m)$. Thus we have $I^n \subset \mathrm{Im}\ \alpha$, that is, $MI = 0$. Hence $M \in \mathscr{F}'$. Let ω_1,

$\cdots, \omega_n \in M$ be the generators of M. Then we have $\sum_i \omega_i d_{ij}=0$ for all $j=1, \cdots, m$. By assumption $M\otimes U \rightarrow M\otimes V$ is a monomorphism. Thus we have $\sum \omega_i \otimes u_i = \sum \omega_i \otimes \sum d_{ij} v_j = \sum \omega_i d_{ij} \otimes v_j = 0$. Therefore, we have $\sum \omega_i \otimes u_i = 0$ in $M\otimes U$. By Lemma 2.3 of [3], there exist $u'_1, \cdots, u'_m \in U$ such that $u_i = \sum d_{ij} u'_j$ for all $i=1, \cdots, n$.

(2) implies (1); We need to show that $0 \rightarrow M\otimes U \rightarrow M\otimes V$ is exact for every $M\in \mathscr{F}'$. We may assume that M is finitely generated, and since every finitely generated module is a direct limit of finitely presented modules, we may even assume that M is finitely presented. Let $R^m \xrightarrow{\alpha} R^n \rightarrow M \rightarrow 0$ be exact. We represent $\alpha=(d_{ij})_{1\leq i\leq n, 1\leq j\leq m}$, $d_{ij}\in R$. Since $MI=0$, we have $I^n \subset \operatorname{Im} \alpha$. Thus $\{d_{ij}\}$ satisfies the condition (I). Consider the following commutative diagram;

$$\begin{array}{ccccccc} R^m\otimes U & \xrightarrow{f} & R^n\otimes U & \longrightarrow & M\otimes U & \longrightarrow & 0 \\ \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \rho} & & \\ R^m\otimes V & \xrightarrow{g} & R^n\otimes V & \longrightarrow & M\otimes V & \longrightarrow & 0, \end{array}$$

where the rows are exact. Since ϕ is a monomorphism, ρ is a monomorphism if and only if $\operatorname{Im} g \cap \operatorname{Im} \phi = \operatorname{Im} \phi f$ by Lemma 11.3 of [5]. Take $(v_1, \cdots, v_m) \in V^m \cong R^m \otimes V$. Put $v=g(v_1, \cdots, v_m)=(\sum d_{ij} v_j)_i \in V^n$. If $v\in \operatorname{Im} \phi$, then there exists $(u_1, \cdots, u_n)\in U^n$ such that $u_i=\sum d_{ij} v_j$ for all $i=1, \cdots, n$. By assumption there exist $u'_1, \cdots, u'_m \in U$ such that $u_i = \sum d_{ij} u'_j$ for all $i=1, \cdots, n$, that is, $v\in \operatorname{Im} \phi f$. Thus we have $\operatorname{Im} g \cap \operatorname{Im} \phi \subset \operatorname{Im} \phi f$. But the converse inclusion always holds. Hence we have $\operatorname{Im} g \cap \operatorname{Im} \phi = \operatorname{Im} \phi f$, so that, ρ is a monomorphism.

COROLLARY 1–4. *Let M be a finitely presented right R-module such that $M\in \mathscr{F}'$. Then $M\cong N/K$, where N and K are isomorphic to some finite direct sums of R/I.*

PROOF. Let $R^m \xrightarrow{\alpha} R^n \rightarrow M \rightarrow 0$ be exact. We represent $\alpha=(d_{ij})_{1\leq i\leq n, 1\leq j\leq m}$. Since $MI=0$, we have $I^n \subset \operatorname{Im} \alpha$. Thus, for any $x_1, \cdots, x_n \in I$, there exist $r_1, \cdots, r_m \in R$ such that $x_i = \sum d_{ij} r_j$. Hence $\{d_{ij}\}$ satisfies (I). Now, for each $(y_1, \cdots, y_n) \in I^n$, there exists $(s_1, \cdots, s_m) \in R^m$ such that $y_i=\sum d_{ij} s_j$ for all $i=1, \cdots, n$. It is easily shown that I is an I-pure left ideal of R. Thus there exist $z_1, \cdots, z_m \in I$ such that $y_i = \sum d_{ij} z_j$ by Theorem 1–3. Therefore, $\alpha(I^m)=I^n$. This implies that $\bar{\alpha}: (R/I)^m \rightarrow (R/I)^n$, which is induced from α, is a monomorphism. Thus we have an exact sequence $0\rightarrow (R/I)^m \rightarrow (R/I)^n \rightarrow M \rightarrow 0$. Hence $M\cong N/K$, where $N\cong (R/I)^n$, $K\cong (R/I)^m$.

COROLLARY 1–5. *Let M, M', and M'' be R-modules and $M'' \subset M' \subset M$.*

(1) *If M' is an I-pure submodule of M and M'' is an I-pure submodule of M', then M'' is an I-pure submodule of M and M'/M'' is an I-pure submodule of M/M''.*

(2) *If M'' is an I-pure submodule of M, then M'' is an I-pure submodule of M'.*

COROLLARY 1–6. *Let N and P be submodules of M such that $N\cap P$ and $N+P$ are I-pure submodules of M. Then N and P are I-pure in M.*

PROOF. We shall prove the corollary for N. Take any $z_j=x_j+y_j\in N+P$, $x_j\in N$, $y_j\in P\,(1\leq j\leq m)$, $a_i\in N\,(1\leq i\leq n)$, and $d_{ij}\in R\,(1\leq i\leq n, 1\leq j\leq m)$, where $\{d_{ij}\}$ satisfies (I). If $a_i=\sum d_{ij}z_j=\sum d_{ij}x_j+\sum d_{ij}y_j$, then we put $b_i=a_i-\sum d_{ij}x_j$. Since $b_i\in P\cap N$ and $y_j\in P$, there exist $y'_j\in P\cap N$ such that $b_i=\sum d_{ij}y'_j$ by assumption. Thus $a_i=\sum d_{ij}(x_j+y'_j)$ and $x_j+y'_j\in N$. Therefore, N is I-pure in $N+P$ by Theorem 1–3. Hence N is I-pure in M by Corollary 1–5 (1).

2. On I-flat modules

Following Bland [2], we call a left R-module M codivisible if, for any exact sequence $0\to C\to B\to A\to 0$, where C is torsionfree, the induced map Hom $(M, B)\to$Hom (M, A) is onto. A left R-module C is a codivisible cover of M if C is codivisible and there exists an epimorphism $C\to M$ whose kernel is small in C [2].

In this section we investigate the I-flatness of left R/I-modules which are regarded as an R-module. Under a condition, we also give the characterization of a ring whose I-flat modules are codivisible with respect to $(\mathscr{T}, \mathscr{F})$.

LEMMA 2–1. *R/I is an I-flat left R-module.*

PROOF. Let $M\in\mathscr{T}'$. Then we have an exact sequence $0\to M\otimes I\to M\otimes R\to M\otimes R/I\to 0$, since $M\otimes I=0$. Hence R/I is I-flat by Theorem 1–2.

COROLLARY 2–2. *Every free R/I-module is I-flat as an R-module.*

LEMMA 2–3. *Let M be an R/I-module. M is an I-flat R-module if and only if M is a flat R/I-module.*

PROOF. Assume that M is I-flat. Let $0\to A\to B\to C\to 0$ be an exact sequence of right R/I-modules. Then $C\in\mathscr{T}'$. Thus by assumption $0\to A\otimes M\to B\otimes M\to C\otimes M\to 0$ is exact.

Conversely, let $0\to A\to B\to C\to 0$ be an exact sequence of right R-modules such that $C\in\mathscr{T}'$. Take an exact sequence $0\to K\to F\to M\to 0$, where F is a free R/I-module. By Corollary 2–2 F is an I-flat R-module. We get the following commutative diagram ;

$$\begin{array}{ccccccc}
A\otimes K & \longrightarrow & B\otimes K & \longrightarrow & C\otimes K & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle \phi} & & \\
A\otimes F & \xrightarrow{\psi} & B\otimes F & \longrightarrow & C\otimes F & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
A\otimes M & \xrightarrow{\gamma} & B\otimes M & \longrightarrow & C\otimes M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

with the exact rows and the columns. Since M is flat as an R/I-module, ϕ is a monomorphism. Since F is I-flat, ψ is a monomorphism. Thus γ is a monomorphism by a diagram chase. Hence M is I-flat as an R-module.

COROLLARY 2–4. *If M is an I-flat R-module, then M/IM is a flat R/I-module.*

REMARK. If $(\mathscr{T}, \mathscr{F})$ is hereditary, then the converse holds as well (cf. Theorem 2.4 of [1]).

Finally, we shall prove the following theorem.

THEOREM 2–5. *If $(\mathscr{T}, \mathscr{F})$ is hereditary, then the following conditions are equivalent.*

(1) *Every I-flat left R-module is codivisible with respect to $(\mathscr{T}, \mathscr{F})$.*

(2) *Every left R-module M has a codivisible cover with respect to $(\mathscr{T}, \mathscr{F})$.*

(3) *R/I is a left perfect ring.*

PROOF. The equivalence of (2) and (3) is stated in Theorem 11 of [4].

(1) implies (3); We need to show that each flat R/I-module is projective. Let M be a flat R/I-module. Then M is an I-flat R-module by Lemma 2–3. By (1) M is codivisible with respect to $(\mathscr{T}, \mathscr{F})$. Hence M is a projective R/I-module by Proposition 6 of [4].

(3) implies (1); Let M be an I-flat left R-module. By Corollary 2–4 M/IM is a flat R/I-module. Thus M/IM is a projective R/I-module by (3). Hence M is codivisible with respect to $(\mathscr{T}, \mathscr{F})$ by Theorem 8 of [4].

Addendum:

Recently the author has received a paper by H. Katayama entitled "Flat and projective properties in a torsion theory, Res. Rep. of Ube Tech. Coll., No. 15. (1972)" where we have found that our Theorem 1–2 is also obtained independently [cf. Proposition 2.4].

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Department of Mathematics
Hokkaido University
Sapporo Japan