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A relation between the F. and M. Riesz theorem and the structure of LCA groups

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§ 1. Introduction

Let G be a locally compact Abelian group and $M(G)$ the usual Banach algebra of all complex bounded regular measures on G.

Let $L^1(G)$ be a set of all functions intergrable on G with respect to the Haar measure dx.

Suppose Γ is a LCA group. We shall call Γ is a topological ordered group or simply ordered group if there exists a closed semigroup P of Γ such that (i) $P\cup(-P)=\Gamma$ and (ii) $P\cap(-P)=\{0\}$. Next we shall say that Γ is an algebraically ordered group if there exists a semigroup P such that (i) $P\cup(-P)=\Gamma$ and (ii) $P\cap(-P)=\{0\}$. [(1)].

Suppose $\Gamma=\hat{G}$ (the dual group of G) is an algebraically ordered group. A measure $\mu\in M(G)$ is said to be of analytic type if $\hat{\mu}(\gamma)=\int_G(-x,\gamma)\,d\mu(x)=0$ for all $\gamma<0$.

We put $M^a(G)=\{\mu\in M(G)\,;\ \mu$ is of analytic type$\}$.

Call a semigroup S satisfies the condition (*) if $S\cup(-S)=\Gamma$ and $S\cap(-S)=\{0\}$.

Our purpose is to prove the following theorem.

THEOREM 1. *Let G be a non-compact LCA group with its dual $\Gamma=\hat{G}$ is algebraically ordered.*

If (I) $M^a(G)\subset L^1(G)$

(II) *for any closed subgroup H of G (but $H\neq G$) $M(G/H)$ has a non-zero analytic measure ($H^\perp$ (annihilator of H) becomes naturally an algebraically ordered group.)*

then, $G=R$. And moreover $P=(0,\infty)$ or $(-\infty,0]$, where P is a semigroup which induces algebraically order into Γ.

REMARK 1. In the above theorem the condition (II) cannot be weakened. Indeed, let $F\neq\{0\}$ be a compact torsion-free group and D its dual. We put $G=T\oplus D$, then G is a non-compact LCA group, where T is a circle group.

Since $\hat{D}=F$ is an algebraically ordered group, we can construct a semigroup P of $\hat{G}$ which induces algebraically order in $\hat{G}$ as follows.

Put $$P=\left\{(n,f)\in Z\oplus F\,;\ n>0,\ \text{or}\ n=0\ \text{and}\ f\geqq 0\right\}.$$

Since P is not dense in $\hat{G}$, $M(G)$ has non-zero analytic measure. Moreover the following proposition A is established.

PROPOSITION A.

$$M^a(G)\subset L^1(G)$$

[Proof] Let μ be any measure of analytic type. Put $d\mu=d\mu_a+d\mu_s$, where $d\mu_a$ is absolutely continuous and $d\mu_s$ is singular. By [Lemma 1 of (1)], both μ_a and μ_s are of analytic type. Suppose $\mu_s\neq 0$, then there exists a non-negative integer n and $f\in F$ such that

$\hat{\mu}_s(n,f)\neq 0$. By Lemma 1 in § 2, we have $n\geqq 1$.

Put $n_0=\inf\{n\in Z;\ n\geqq 1,\ \hat{\mu}_s(n,f)\neq 0$ for some $f\in F\}$.

We define measures $\lambda_s\in M(G)$ and $\sigma\in L^1(G)$ as follows

$$\hat{\lambda}_s(n,f)=\hat{\mu}_s(n+n_0,f)\quad \text{for}\quad n\in Z,\ f\in F$$
$$d\sigma=dx\times d\delta_0,$$

where dx is a Haar measure on T and $d\delta_0$ is the unit point mass at 0 in D. Put $\nu=\lambda_s-\lambda_s*\sigma$.

By the definition of n_0, ν is of analytic type.

Therefore, by [Lemma 1 of (1)], λ_s is of analytic type, since $\lambda_s*\sigma$ belongs to $L^1(G)$. Hence, by Lemma 1 in § 2, $\hat{\lambda}_s(0,f)=0$ for all $f\in F$.

Since $\hat{\mu}_s(n_0,f)=\hat{\lambda}_s(0,f)$, we have a contradiction by the definition of n_0. Hence $\mu_s=0$ and μ belongs to $L^1(G)$.

§ 2. We state some propositions and lemmas before we prove the theorem

PROPOSITION 1. *Let $P\subset\hat{G}$ be a semigroup satisfying the condition* (*). *Then, the following* (a), (b) *and* (c) *are equivalent.*

(a) *$M(G)$ has no non-zero analytic measure*

(b) *P is dense in $\hat{G}$*

(c) *$-P$ is dense in $\hat{G}$.*

We omit the proof, since these facts are easily followed.

LEMMA 1. *Let G be a discrete Abelian group such that $\hat{G}$ is torsion-free (hence $\hat{G}$ is an algebraically ordered group).*

If P is any semigroup of $\hat{G}$ satisfying the condition (*), *then P is dense in $\hat{G}$.*

[Proof]

Let $m_{\hat{G}}$ be a normalized Haar measure on $\hat{G}$. If P is not dense in $\hat{G}$, there exists an element $\gamma_0 \in \hat{G} \backslash \bar{P}$ and a symmetric neighborhood U of 0 such that $\gamma_0 + U + U \subset \hat{G} \backslash \bar{P} \subset (-P) \backslash \{0\}$.

Then, $\{n\gamma_0 + U\}$ $(n=1, 2, 3, \cdots)$ are pairewise disjoint. Hence, $1 \geqq m_{\hat{G}} (\bigcup_{n=1}^{\infty} (n\gamma_0 + U)) = \infty$ and we have a contradiction.

We state a definition for lemma 3.

DEFINITION 1. *In this paper, non-empty subset* E^n *of* R^n $(n \geqq 2)$ *is called a RC-set if* E^n *contains some* $U^n(S_n)$. *Where* U^n *is an unitary transformation in* R^n *and* $S_n = \{(x, y) \in R^n;\ x \in R,\ y \in V_{n-1}\}$ $(V_{n-1}$ *is a non-empty open set in* $R^{n-1})$.

PROPOTISION 2. *Let* $P \subset R$ *be a semigroup satisfying the condition* (*). *Then,* P *is* (i) *closed or* (ii) *dense in* R.

Moreover, if P *is closed,* P *is* $[0, \infty)$ *or* $(-\infty, 0]$.

[Proof]

Case (i). We suppose that P is closed.

It is sufficient to consider the following two cases.

Case 1. there exists an element $x_0 \in -P$ with $x_0 < 0$

Case 2. there exists an element $x_0 \in -P$ with $x_0 > 0$

Firstly we consider the case 1. Since P^c is open and x_0 belongs to P^c, there exists an open interval $(-\delta + x_0, \delta + x_0) \subset P^c$ with $\delta + x_0 < 0$ for some $\delta > 0$.

Put $x_1 = \sup \{x < 0;\ (-\delta + x_0, x) \subset P^c\}$.

Then, $x_1 = 0$. Hence $[x_0, 0) \subset P^c \backslash \{0\} \subset -P$.

Since $-P$ is closed, $[x_0, 0] \subset -P$.

Therefore $-P = (-\infty, 0]$, because $-P$ is a semigroup.

That is $P = [0, \infty)$.

We can conclude $P = (-\infty, 0]$ by the same discussion if case 2 happens.

Case (ii). We suppose that P is not closed.

From case (i) $P \cap [0, \infty) \neq \phi$ and $P \cap (-\infty, 0] \neq \phi$.

Put
$$a_+ = \inf \{a \in P;\ a > 0\} \quad \text{and}$$
$$a_- = \sup \{a \in P;\ a < 0\}\ .$$

Then, $a_+ = a_- = 0$. Hence, P is dense in R.

REMARK 2. *Professor S. Koshi pointed out to the author that there exists a semigroup* S *of* R *such that it is dense in* R *and satisfies the condition* (*).

LEMMA 2. *Let P be a semigroup of R^2 satisfying the condition* (*). *Then, we have*

(a) *P contains a set which is transformed $S_2=\{(x, y)\in R^2\,;\ y\in R,\ x\geqq a$ for some $a\in R\}$ by some unitary transformation in R^2,*

or (b) *P is dense in R^2.*

[Proof] Suppose P is not dense in R^2.

Put $F_1=\{(x, 0)\in R^2\,;\ x\in R\}$ and $F_2=\{(0, y)\in R^2\,;\ y\in R\}$. We define semigroups P_1 and P_2 of R^2 by $P_1=F_1\cap R$ $(=F_1\cap R\oplus\{0\})$ and $P_2=F_2\cap R$ $(=F_2\cap R\oplus\{0\})$.

From proposition 2, it is sufficient to consider the following four case.

[case I] $P_1\cong[0, \infty)$ and P_2 is dense in R

[case II] P_1 is dense in R and P_2 is dense in R

[case III] $P_1\cong[0, \infty)$ and $P_2\cong[0, \infty)$

[case IV] $P_1\cong[0, \infty)$ and $P_2\cong(-\infty, 0]$.

But since [case IV] can be proved as [case III], We consider only [case I], [case II] and [case III].

Step 1. We shall begin with [case II].

But it is easy to check that P is dense in R^2, because P is a semigroup. Hence [case II] cannot be happened.

Step 2. We suppose that [case I] happens.

Then, $-P_2$ is dense in F_2, because P_2 is dense in F_2.

Hence $-P$ is dense in $\{(x, y)\,;\ x\leqq 0,\ y\in R\}$ and P is dense in $\{(x, y)\,;\ x\geqq 0,\ y\in R\}$.

Since P is not dense in R^2, $-P$ is so. Therefore, $-P$ is not dense in $\{(x, y)\,;\ x\geqq 0,\ y\in R\}$.

Hence there exists a non-empty open set $U\subset\{(x, y)\,;\ x\geqq 0,\ y\in R\}$ such that $(-P)\cap U=\phi$. Hence $U\subset P$ and P contains a set $\{(x, y)\,;\ x\geqq a,\ y\in R\}$ for some $a\in R$.

Step 3. Since P contains $\{(x, y)\,;\ x\geqq 0,\ y\geqq 0\}$, it is sufficient to consider the case that P contains an element $z_0=(x_0, y_0)$ $(x_0>0$ and $y_0<0)$.

Put $a=\dfrac{y_0}{x_0}$, $F_a=\{(x, ax)\,;\ x\in R\}$ and $P_{F_a}=P\cap F_a$.

Then, P_{F_a} is dense in $F_a\cdots(3)_1$ or not dense in $F_x\cdots(3)_2$. If the case $(3)_1$ happens, P contains $\{(x, y)\,;\ y>ax\}$. Hence (a) is established.

If the case $(3)_2$ happens, P contains $\{(x, y)\,;\ x\geqq 0,\ y>ax\}$.

Now we put $a^{\#}=\inf\{a<0\,;\ (x, ax)\in P,\ x>0\}$, then P contains $\{(x, y)\,;\ y>a^{\#}x,\ x\in R\}$.

Where if $a^{\#}=-\infty$, P contains $\{(x, y)\,;\ x>0,\ y\in R\}$.

Hence (a) is established.

LEMMA 3. *Let P be a closed semi-group of R^n ($n\geqq 2$) satisfying the condition (*). Then, we have*

(a) *P contains a RC-set E_n in R^n*

or (b) *P is dense in R^n.*

[Proof] From proposition 2 and lemma 2, we can prove the Lemma 3 by using the induction.

We drop the detail.

§ 3. Proof of Theorem

Finally, we prove the theorem. Put $\Gamma=\hat{G}$, then by the structure theorem [(4); p 40], Γ contains an open subgroup $R^n\oplus F$, where F is a compact subgroup and $n\geqq 0$.

Put $H=F^{\perp}$ (annihilator of F), then $\widehat{G/H}=F$. Hence, by lemma 1 and proposition 1, we have $F=\{0\}$.

Hence Γ contains R^n ($n\geqq 0$) as an open subgroup.

Since G is non-compact, we have $n\geqq 1$.

Let i be an identity map from R^n into itself. Then, there exists a homomorphisim ϕ from Γ into R^n such that $\phi|R^n=i$, because R^n is divisible.

Since R^n is an open subgroup, ϕ is continuous. Hence by [(2); p 59], we have

$$\Gamma\cong R^n\oplus\Gamma/R^n\,.$$

We put $D=\Gamma/R^n$, then $\Gamma=R^n\oplus D$, where D is discrete.

Step 1. Firstly, we shall prove that $n=1$.

Suppose $n\geqq 2$. Put $P_n=P\cap P^n\,(=P\cap R^n\oplus\{0\})$, then by lemma 3, we have

[1] P_n contains some RC-set $E_n\subset R^n$

or [2] P_n is dense in R^n.

The case [2] cannot be happened because of the hypothysis (II) of the theorem.

If the case [1] happens, then there exists a unitary transformation T_n in R^n and a subset $S_n=\{(x, y)\,;\ x\in R,\ y\in V_{n-1}\}$ such that $T_n(E_n)=S_n$.

Where V_{n-1} is a non-empty open set in R^{n-1}.

Then there exists a non-zero integrable function $h\in L^1(R^{n-1})$ such that $supp\,(\hat{h})\subset V_{n-1}$. We define a measure $\lambda'\in M(R^n)$ by

$$d\lambda'(x, y)=d\delta_0(x)\times h(y)\,dy\,,$$

where δ_0 is a dirac measure at 0 in R.

Then we have $supp(\hat{\lambda}')\subset S_n$.

We next define $\lambda'_{T_n}\in M(R^n)$ by

$$\lambda'_{T_u}(K)=\lambda'(T_n K) \qquad \text{for a Borel measurable set } K \text{ of } R^n.$$

Then, since T_n is a unitary transformation, we have

$$\begin{aligned}\hat{\lambda}'_{T_n}(s) &= \int_{R^n} e^{-i(s,t)}\, d\lambda'_{T_n}(t)\\ &= \int_{R^n} e^{-i(s,t)}\, d\lambda'(T_n t)\\ &= \hat{\lambda}'(T_n s) \qquad \text{for } s\in R^n.\end{aligned}$$

Hence we have $supp(\hat{\lambda}'_{T_n})\subset E_n$. And λ'_{T_n} is a singular measure, because λ' is so. We define a measure $\mu\in M(G)$ by

$$d\mu=d\lambda'_{T_n}\times dm_{\hat{D}},$$

where $m_{\hat{D}}$ is a Haar measure on $\hat{D}$. Easily, we can check that μ is a non-zero singular measure and

$$supp(\hat{\mu})\subset E_n+\{0\}\subset P.$$

In other words, $\mu\,(\neq 0)$ belongs to $M^a(G)\backslash L^1(G)$.

This contradicts to the hypothysis (I) of theorem.

Hence we have $\hat{G}=R\oplus D$.

Step 2. Finally, we shall prove that $G=R$ and that $P=[0,\infty)$ or $(-\infty,0]$.

Suppose P is closed. Then, by [(2); theorem 2], if $D\neq\{0\}$, there exists a non-zero measure $\mu\in M^a(G)\backslash L^1(G)$.

This contradicts to the hypothysis. Hence we have $D=\{0\}$.

We next consider the case that P is not closed.

Suppose $D\neq\{0\}$. By the hypothysis, $P_R=P\cap R\ (=R\oplus\{0\}\cap P)$ is closed. We consider only the case $P_R\cong[0,\infty)$. Moreover, we may assume that there exists no positive minimal element in D, because otherwise $\hat{G}$ is $R\oplus Z$ and we can prove as same way.

Then, there exists an element $(x_0,-d_0)\in P$ ($x_0\in R$, $d_0\in D$; $x_0>0$, $d_0>0$).

Since P is a semigroup satisfying the condition (*), there exists a non-negative real number $a_0\in R$ such that P cotains $\{(x,-nd_0);\ x\geqq na_0,\ n=0,1,2,\cdots\}\cup\{(x,-nd_0);\ x>na_0,\ n=-1,-2,\cdots\}$ or $\{(x,-nd_0);\ x>na_0,\ n=1,2,\cdots\}\cup\{(x,-nd_0);\ x\geqq na_0,\ n=0,-1,-2,\cdots\}$.

Put $\Lambda=\{(x,nd_0);\ x\in R,\ n\in Z\}$. Nextly, we define a homomorphism β of Λ into itself as follows

$$\beta(x,nd_0)=(na_0+x,\ nd_0).$$

Easily, we can check that Λ is an open subgroup of $\hat{G}$ and β is a continuous isomorphism with $\beta(\Lambda)=\Lambda$.

Let $h\in L^1(R)$ be a non-zero integrable function such that $supp(\hat{h})\subset(1, \infty)$. Let $\Lambda_{d_0}=\{nd_0;\ n\in Z\}$ be a subgroup of D and $F=\Lambda_{d_0}^{\perp}$ an annihilator of Λ_{d_0} in $\hat{D}$.

Since D has not minimal positive element, F is an infinite compact subgroup of $\hat{D}$.

We define a non-zero singular measure $\lambda\in M(G)=M(R\oplus\hat{D})$ by

$$d\lambda=h(x)\,dx\times dm_F,$$

where m_F is a Haar measure on F.

Since $\hat{m}_F(d)=\chi_{\Lambda_{d_0}}(d)$ (characteristic function of Λ_{d_0}), we have

$$supp(\hat{\lambda})\subset[1, \infty)\oplus\Lambda_{d_0}.$$

We next define a non-zero measure $\mu\in M(G)$ by

$$\hat{\mu}(t, d)=\hat{\lambda}o\beta(t, d) = \begin{cases}\hat{\lambda}(\beta(t, d)) & \text{if } (t, d)\in\Lambda \\ 0 & \text{otherwise}.\end{cases}$$

Then, we have

$supp(\hat{\mu})\subset\{(x, nd_0);\ n\in Z,\ x>-na_0+1\}\subset P.$

Hence, μ is of analytic type and, by Riemann-Lebesgue's lemma, μ does not belong to $L^1(G)$.

This contradicts to the condition (I) of theorem.

Hence we have $G=R$. Moreover, by the condition (II) of theorem, proposition 1 and proposition 2, P must be $[0, \infty)$ or $(-\infty, 0]$. q.e.d.

COROLLARY 1. *Let G be a non-compact LCA group with its dual $\hat{G}$ is algebraically ordered. If this order is archimedean, then the condition* (II) *of theorem can be weakened as follows.*

(II)' *$M(G)$ has a non-zero analytic measure.*

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