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On Harnack's pseudo-distance

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1. Introduction and terminology

In this paper we shall give a sufficient condition for which the Harnack's pseudo-distance h_R (denoted by d_R in [5]) on an arbitrary Riemann surface R is a real distance and we shall investigate a relation among the Harnack's pseudo-distance h_R, the Kobayashi's distance d_R ([4]) and the Carathéodory's distance c_R (cf. [4]).

For an arbitrary (open or closed) Riemann surface R, we denote by $HP=HP(R)$ the family of all positive harmonic functions on R. For any $a, b\in R$, we set

$$k_R(a, b) = \inf\left\{c\,;\ c^{-1}u(a)\leqq u(b)\leqq cu(a) \quad \text{for any} \quad u\in HP(R)\right\}$$

(Harnack's constant).

It is easy to see that $1\leqq k_R(a, b)<\infty$ and $(a, b)\rightarrow k_R(a, b)$ is continuous. Furthermore the following properties are easy to see:

$$k_R(a, b)=k_R(b, a) \quad \text{and} \quad k_R(a, b)\leqq k_R(a, c)\,k_R(c, b)\,.$$

The following definition is due to J. Köhn (cf. [2]).

DEFINITION 1. For any $a, b\in R$, we set $h_R(a, b)=\log\ k_R(a, b)$.

By definition, we see that $(a, b)\rightarrow h_R(a, b)$ is continuous and $R\in O_{HP}$ if and only if $h_R=0$. Furthermore h_R is a (real) distance if and only if $HP(R)$ separates points of R, i. e., for any $a, b\in R$ $(a\neq b)$, we can find $u\in HP(R)$ with $u(a)\neq u(b)$.

The following theorem is due to J. Köhn [5].

THEOREM 1 (An extension of Schwartz-Pick's theorem). *Let R, R' be two open Riemann surfaces. If f is an analytic mapping of R into R', then*

$$h_R(a, b)\geqq h_{R'}\big(f(a), f(b)\big) \quad \textit{for any } a, b\in R\,.$$

In particular, if $HP(R)$ separates points of R and R' is hyperbolic, then the equality holds if and only if f is an onto conformal mapping.

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LEMMA 1 ([5]). *In the case of* $R=\{|z|<1\}$, $h_R(a, b)$ *equals the Poincaré-Bergman metric on* R, *i. e.,*

$$h_R(a, b) = \log \frac{1+r}{1-r},$$

where $r=\left|\frac{b-a}{1-\bar{a}b}\right|$ $(a, b\in R)$.

2. Class $\mathscr{A}$

In the following we denote by ∂E the relative boundary of a subset E of a Riemann surface R. Furthermore, for a closed set X, we denote by $C(X)$ the family of all real-valued continuous functions on X. For an open set G in R and $f\in C(\partial G)$, we refer to [3] for the definition and properties of Dirichlet solution H_f^G.

DEFINITION 2. We denote by $\mathscr{A}$ the family of all hyperbolic Riemann surfaces R which satisfy the following condition:

For any closed disk K in R, $L_{0R-K}\, f$[1] $=H_f^{R-K}$ in $R-K$ $(f\in C(\partial K))$ implies $f=0$.

LEMMA 2. *Let* R' *be an arbitrary Riemann surface. If* K' *is a closed disk in* R', *then* $R=R'-K'$ *belongs to* $\mathscr{A}$.

PROOF. Let K be any closed disk in R. Let f be any function in $C(\partial K)$ with $L_{0R-K}\, f=H_f^{R-K}$ in $R-K$. We denote by u the common function. Let D be an open disk in R with $D\cap K=\emptyset$ and $K'\subset D$. By the aid of consistencies of operators L_0 and H, we can show that

$$u = L_{0R-(D\cup\partial D)}\, u = H_u^{R-(D\cup\partial D)} \quad \text{in} \quad D-K.$$

Thus we see that $\partial u/\partial\nu=0$ and $u=0$ on ∂K. This implies that $u=0$ in $D-K$. Hence $u=0$ in $R-K$ and $f=0$.

Let R be a hyperbolic Riemann surface. Let R_D^* be the Royden compactification of R and let $\Delta_D=R_D^*-R$ (cf. [3]). For $a\in R$, we denote by μ_a the harmonic measure on Δ_D with respect to a. Let $a_0\in R$ be fixed once for all and let $\mu\equiv\mu_{a_0}$. It is known that there exists a uniquely determined normal derivative, say $\phi[g_a]$, of g_a with respect to μ in the sense of F-Y. Maeda [6]. The existence and uniqueness of $\phi[g_a]$ are $\mu-$ a. e. For $f\in C(\Delta_D)$, we denote by $H_f^{R^*}$ the Dirichlet solution of f on R_D^*.

LEMMA 3 (cf. [6]). *Let* f *be a continuous fuction on* Δ_D. *Then*

$$H_f^{R^*}(a) = \frac{1}{2\pi}\int_{\Delta_D} f(\xi)\, \phi[g_a](\xi)\, d\mu(\xi) \qquad (a\in R).$$

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1) See [1] for the definition and properties of the operator L_0.

THEOREM 2. *If a Riemann surface R belongs to the class $\mathscr{A}$, then h_R is a distance.*

PROOF. It is sufficient to prove that $HB(R)$ separates points of R. Let a and b be any points in R such that $H_f^{R^*}(a)=H_f^{R^*}(b)$ for any $f\in C(\Delta_D)$. It follows from Lemma 3 that $\psi[g_a]=\psi[g_b]$ or $\psi[g_a-g_b]=0$ on Δ_D $\mu-$ a. e. By the aid of Theorem 8 in [6], we see that there is a closed disk K in R such that $K-\partial K\ni a$, b and $L_{oR-K}(g_a-g_b)$[2] $=g_a-g_b=H_{(g_a-g_b)}^{R-K}$ in $R-K$. By the assumption on R, we see that $g_a-g_b=0$ in $R-K$. Thus we obtain that $a=b$. This completes the proof.

COROLLARY. *If a Riemann surface R belongs to $\mathscr{A}$, then $HBD(R)$ separates points of R.*

PROOF. Let a, b be points in R with $a\neq b$. Then there is an $f\in C(\Delta_D)$ with $H_f^{R^*}(a)\neq H_f^{R^*}(b)$. Since $C_D(\Delta_D)=\{f\in C(\Delta_D);\ H_f^{R^*}\in HD(R)\}$ is dense in $C(\Delta_D)$ with respect to the uniform convergence topology on Δ_D (cf. [6]), we have the Corollary.

3. Invariant distances

Let R be an arbitrary Riemann surface. We denote by c_R (resp. d_R) the Carathéodory's distance (resp. the Kobayashi's distance) on R (cf. [4]).

THEOREM 3. *For an arbitrary Riemann surface R, we have the following inequalities:*

$$d_R(a, b)\geqq h_R(a, b)\geqq c_R(a, b) \qquad (a, b\in R).$$

Furthermore these invariant pseudo-distances do not identically equal one another.

PROOF. The inequalities follow from Proposition 1. 4 and Proposition 2. 5 of IV in [4]. Let R be a closed Riemann surface with genus $\geqq 2$. Then d_R is a distance (cf. Corollary 4. 13, IV, in [4]) but $h_R=0$. Hence $d_R\neq h_R$ for such an R. On the other hand, let $D=\{|z|<1\}$ and $D_0=\{0<|z|<1\}$. Then it is easy to see that $h_D\neq h_{D_0}$ and $c_D=c_{D_0}$. This completes the proof.

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