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Fusion and groups admitting an automorphism of prime order fixing a solvable subgroup

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1. Introduction.

In this paper, we prove the following theorem:

THEOREM 1. *Let G be a finite group. Assume that G admits an automorphism α of order s, s a prime. Assume further that $C_G(\alpha)$ is a (solvable) $\{2, 3, s\}'$-group. Then G is solvable.*

This is a generalization of a theorem of B. Rickman [7], where he took up the case that $C_G(\alpha)$ is a cyclic q-group for some prime $q \geq 5$ distinct from s.

Since $C_G(\alpha)$ is a s'-group, G is a s'-group. Hence, it is well known that $\langle\alpha\rangle$ leaves invariant a Sylow q-subgroup for each prime divisor q of the order of G. Since $C_G(\alpha)$ is of odd order, $C_G(\alpha)$ is solvable by the well-known result of Feit-Thompson [3]. The proof of Theorem 1 depends on the anaysis of the fusion in G. Therefore, we need the following theorems: (For the definitions and notations, see § 2.)

THEOREM 2. *Let G be a finite group and $W_1, \cdots, W_n$ be conjugacy functors for a prime p. Assume that $\{W_1, \cdots, W_n\}$ controls p-fusion in every p-local of G. Then $\{W_1, \cdots, W_n\}$ controls p-fusion in G.*

THEOREM 3. *Let G be a finite group and $W_1, \cdots, W_n$ be conjugacy functors for a prime p such that $W_i(P) \supseteq Z(P)$ for each i and each p-group P. Assume that $\{W_1, \cdots, W_n\}$ controls p-fusion in every p-constrained p-local of G, then $\{W_1, \cdots, W_n, Z\}$ controls p-fusion in G.*

These theorems are generalizations of a well-known theorem of Alperin-Gorenstein [2].

COROLLARY 1. *Let W_1 and W_2 be conjugacy functors satisfying $W_i \supseteq Z$ $(i=1, 2)$. Assume that $N=N_N(W_1(P_0))\, N_N(W_2(P_0))\, O_{p'}(N)$ for every p-constrained p-local N of a finite group G and a Sylow p-subgroup P_0 of N. Then $\{W_1, W_2, Z\}$ controls p-fusion in G.*

COROLLARY 2. *Assume that G is S_4-free and $Z(S) \trianglelefteq N_G(J(S))$ for a Sylow 2-subgroup S of G. Then $N_G(Z(S))$ controls 2-fusion in G. In par-*

ticular, $Z(S)$ is a strrongly closed abelian 2-subgroup in S with respect to G.

REMARK. Theorem 1 follows from this corollary 2.

2. Notation and Preliminary results.

All groups considered in this paper are assumed to be finite. For a prime q, let $\mathrm{Syl}_q(G)$ denote the set of Sylow q-subgroups of the group G and $\mathrm{Syl}_{q,\langle\alpha\rangle}(G)$ denote the set of $\langle\alpha\rangle$-invariant Sylow q-subgroups of G. A conjugacy functor on G is a mapping W which satisfies the following three conditions for every p-subgroup T:

i) $W(T)\subseteq T$; ii) $W(T)\neq 1$ if $T\neq 1$; and
iii) $W(T^g)=W(T)^g$ for every element g in G

Let P be a Sylow p-subgroup of the group G, p a prime, and $W_1, \cdots, W_n$ be conjugacy functors on G. We say that $\{N_G(W_1(P)), \cdots, N_G(W_n(P))$ (or $C_G(W_i(P)))\}$ connects A with B, for two subsets A and B in P, if there are subsets $A_0, \cdots, A_m$ in P such that $A_0=A$, $A_m=B$, and for each $i=0, \cdots, m-1$, A_i is conjugate to A_{i+1} in $N_G(W_j(P))$ (or $C_G(W_j(P)))$ for some j in $\{1, \cdots, n\}$. We say that $\{W_1, \cdots, W_n\}$ controls p-fusion in G if there is a Sylow p-subgroup P of G satisfying the following property; whenever A and B are subsets of P and A is conjugate to B in G, then $\{N_G(W_1(P)), \cdots, N_G(W_n(P))\}$ connects A with B. The notation Z denotes the conjugacy functor which maps each p-group to its center. $W_1\supseteq W_2$ means that $W_1(T)\supseteq W_2(T)$ for every p-subgroup T of G. Suppose T is a p-group for some prime p. Let $d(T)$ be the maximum of the orders of the Abelian subgroups of T. Let $A(T)$ be the set of all Abelian subgroups of T of order $d(T)$. Let $J(T)=\langle A(T)\rangle$. Thus, $J(T)$ is the Thompson subgroup of T.

We need the following lemmas:

LEMMA 1. (Shult, [8], Theorem 3.1)

Let V be a group of order p (p a prime) of operators acting on a group G and $(|G|, 2p)=1$. Let A be a faithful KGV-module where the characteristic of the field K does not divide $|GV|$. If $C_A(V)=0$, then V centralizes G.

LEMMA 2. (Thompson [9])

Let V be a group of order p (p a prime) of operators acting on a group G. If $C_G(V)=1$, then G is nilpotent.

LEMMA 3. (Glauberman [4], Corollary 3)

Let G be a group, p a prime, P a Sylow p-subgroup of G, and Q a subgroup of $Z(P)$. If $Q\trianglelefteq N_G(J(P))$ and if p is odd and $p-1$ does not divide the index $|N_G(Q):C_G(Q)|$ then Q is weakly closed in P with respect to G.

LEMMA 4. (Glauberman [5], Corollary 10)

Let S be a Sylow 2-subgroup of the group G. Suppose that $C_G(O_2(G)) \subseteq O_2(G)$ and G is S_4-free. Then

$$G=\left\langle C_G\left(Z(S)\right),\ N_G\left(J(S)\right)\right\rangle.$$

LEMMA 5. (Goldschmidt [6])

Let G be a finite non-abelian simple group. Assume that G has a strongly closed abelian 2-subgroup. Then G is isomorphic to one of the following groups:

a) $L_2(2^n)$ $n\geq 3$, $Sz(2^{2n+1})$ $n\geq 1$, $U_3(2^n)$ $n\geq 2$,

b) $L_2(q)$ $q\equiv 3, 5 \pmod 8$, and

c) the groups of type Janko-Ree.

LEMMA 6.

Suppose that a solvable group G admits an automorphism V of order s, s a prime. Assume that $C_G(V)$ is a $\{2, 3, s\}'$-group. Then $G=0_{q',q}(G)\,C_G(V)$ for each prime $q\in\pi(G)-\pi(C_G(V))$.

PROOF. Let G be a minimal counterexample to Lemma 6. Then we may assume that $O_{q'}(G)=1$, so that $O_q(G)=F(G)\supseteq C_G(F(G))$. Let T be a V-invariant Hall q'-subgroup of G. We will show that V centralizes T. Let $T_2\in Syl_{2,V}(T)$. If $T_2\neq 1$, then $q\neq 2$. Since $C_G(V)$ is odd order, V acts on $T_2F(G)$ as a fixed point free automorphism group. Thus, $T_2F(G)$ is nilpotent by Thompson's theorem [9], a contradiction. Thus T is of odd order. Applying Lemma 1 to $GF(q)\,TV$ acting on $O_q(G)$, we have $[T, V] \subseteq C_G(F(G))\cap T\subseteq F(G)\cap T=1$, so $T\subseteq C_G(V)$. Let $Q\in Syl_{q,V}(G)$, then we have $G=TQ$ and $G\trianglerighteq [G, V]=[Q, V]=Q$. Therefore, $G=O_q(G)\,C_G(V)$, a contradiction.

3. Proof of Theorem 1.

In this section, we assume that Corollary 2 is true. Let G be a minimal counterexample to the Theorem 1.

(I) G is a simple group.

Proof. By minimality of G, $G=G_1\times G_1^{\alpha}\times\cdots\times G_1^{\alpha^{s-1}}$ or G is simple. If $G=G_1\times\cdots\times G_1^{\alpha^{s-1}}$, then $C_G(\alpha)\cong G_1$ is a non-abelian simple group, a contradiction.

(II) For each prime $q\in\pi(G)-\{\pi(C_G(\alpha)), 2\}$ and $Q\in Syl_{q,\langle\alpha\rangle}(G)$, $Z(Q)$ is weakly closed in Q with respect to G.

Proof. By minimality and simplicity of G, $N_G(J(Q))$ and $N_G(Z(Q))$ are solvable. Thus by Lemma 6, $N_G(J(Q))=O_{q'}(N_G(J(Q)))\,N_G(Q)$, so we have $Z(Q) \trianglelefteq N_G(J(Q))$. Set $N=N_G(Z(Q))$, then $N=O_{q'}(N)\,N_N(Q)=C_G(Z(Q))\,C_N(\alpha)$. Thus we have that $N_G(Z(Q))/C_G(Z(Q))$ is $\pi(C_G(\alpha))$-group, in particular, $q-1$ does not divide the index $|N_G(Z(Q)):C_G(Z(Q))|$. Hence, by Lemma 3, we have that $Z(Q)$ is weakly closed in Q with respect to G.

(III) G is S_3-free. In particular, G is S_4-free.

Proof. Let $Q\in Syl_{3,\langle\alpha\rangle}(G)$. By (II), $N_G(Z(Q))$ controls 3-fusion in G. Since $N_G(Z(Q))/O_{3'}(N_G(Z(Q)))$ is of odd order by Lemma 6, G is S_3-free.

(IV) $Z(S)\trianglelefteq N_G(J(S))$ for some $S\in Syl_{2,\langle\alpha\rangle}(G)$.

Proof. By minimality of G, $N_G(J(S))$ is solvable, so we have that $N_G(J(S))=O_{2'}(N_G(J(S)))\,N_G(S)\trianglerighteq Z(S)$, by Lemma 6.

(V) A contradiction.

Proof. Since G is S_4-free and $Z(S)\trianglelefteq N_G(J(S))$ for some Sylow 2-subgroup S of G, we have that $Z(S)$ is a strongly closed Abelian 2-subgroup in S with respect to G, by Corollary 2. Since G is simple, we have that G is isomorphic to one of the following groups:

$L_2(2^n)\ n\geq 3$, $Sz(2^{2n+1})\ n\geq 1$, $U_3(2^n)\ n\geq 2$, $L_2(q)\ q\equiv 3, 5$ (mod 8), and the groups of type Janko-Ree,

by the result of Goldschmidt [6]. However, it is easily proved that none of these groups have automorphisms satisfying the conditions of Theorem 1, a contradiction.

4. Proof of Theorem 2 and Theorem 3.

In this section, we will only prove Theorem 3. But, by a slight change, we can get a proof of Theorem 2.

Suppose that Theorem 3 is false and let G be a counterexample to Theorem 3. Let $W_1, \cdots, W_n$ be the conjugacy functors with $W_i\supseteq Z$ for every i. Then there are two subsets A and B in P which are conjugate in G, but $\{N_G(W_1(P)), \cdots, N_G(W_n(P)), C_G(Z(P))\}$ cannot connect A with B, where P is a Sylow p-sugroup of G. By Alperin's theorem [1], we may assume that A and B are contained in a p-constrained p-local $N_G(H)$ with $1\neq H\leq P$ and A is conjugate to B in $N_G(H)$, namely, there is a p-constrined p-local $N_G(H)$ $(1\neq H\leq P)$ satisfying that $N_P(H)$ has two subsets A and B which a are conjugate in $N_G(H)$ but $\{N_G(W_1(P)), \cdots, N_G(W_n(P)), C_G(Z(P))\}$ cannot connect A with B. Moreover, we may assume that $C_G(H)=Z(H)\times O_{p'}(N_G(H))$. Choose such a subgroup H in P satisfying the following conditions:

i) $N_P(H)$ is maximal in such groups,

ii) H is of maximal order subject to i).

We will show that $N_P(H)\in Syl_p(N_G(H))$. Suppose false, then $H\neq P$ and P contains a conjugate L of H which satisfies $N_P(L)\in Syl_p(N_G(L))$. Then, by Alperin's theorem [1] there are an integer m and elements $x_1, \cdots, x_m$ in G and subgroups $K_1, \cdots, K_m$ in P such that $x_i\in N_G(K_i)$ $(i=1, \cdots, m)$, $C_G(K_i)=Z(K_i)\times O_{p'}(N_G(K_i))$ for each i, $N_p(H)\subseteq K_1$, $N_P(H)^{x_1\cdots x_i}\subseteq K_{i+1}$ $(i=1, \cdots, m-1)$, $N_P(H)^{x_1\cdots x_m}\subseteq N_P(L)$ and $H^{x_1\cdots x_m}=L$. Since $|N_P(K_i)|\gneqq|N_P(H)|$, $\{N_G(W_1(P)), \cdots, N_G(W_n(P)), C_G(Z(P))\}$ connects A with $A^{x_1\cdots x_m}$ and $B^{x_1\cdots x_m}$ with B, by maximality of H. But, since $|N_P(L)|\gneqq|N_P(H)|$, $\{N_G(W_1(P)), \cdots, N_g(W_n(P)), C_G(Z(P))\}$ connects $A^{x_1\cdots x_m}$ with $B^{x_1\cdots x_m}$, by maximality of H, so $\{N_G(W_1(P)), \cdots, N_G(W_n(P)), C_G(Z(P))\}$ connects A with B, a contradiction. So we have that $N_P(H)\in Syl_p(N_g(H))$. By the hypothesis of Theorem 3, $\{W_1, \cdots, W_n\}$ controls p-fusion in $N_G(W_1(P_0))$, where $P_0=N_P(H)$. If $C_p(W_1(P_0))\notin Syl_p(C_G(W_1(P_0)))$, then there is an element y in $C_G(Z(P))$ such that $N_P(W_1(P_0))^y\subseteq P$ and $N_P(W_1(P_0)^y)\in Syl_p(N_{C_G(Z(P))}(W_1(P_0)^y))$, in particular, $C_P(W_1(P_0)^y)\in Syl_p(C_G(W_1(P_0)^y))$. Then P contains two subsets A^y and B^y which are conjugate in $N_G(W_1(P_0)^y)$ and $C_P(W_1(P_0)^y)\in Syl_p(C_G(W_1(P_0)^y))$. Since $y\in C_G(Z(P))$, $\{N_G(W_1(P)), \cdots, N_G(W_n(P)), C_G(Z(P))\}$ connects A with A^y and B^y with B. So we may assume that $C_P(W_1(P_0))\in Syl_p(C_G(W_1(P_0)))$. Set $N=N_G(W_1(P_0))$, then $N=N_N(C_P(W_1(P_0))\, W_1(P_0)\cdot C_N(W_1(P_0))=N_N(P_1)\, C_N(Z(P))$, where $P_1=C_P(W_1(P_0))\, W_1(P_0)$, by the Frattini argument. Thus, there is a subset B_0 in P such that A is conjugate to B_0 in $N_N(P_1)$ and B_0 is conjugate to B in $C_N(Z(P))$. Clearly, $N_G(P_1)$ is p-constrained p-local and $C_G(P_1)=Z(P_1)=Z(P_1)\times O_{p'}(N_G(P_1))$. Therefore, by maximality of H, $|N_P(P_1)|\gneqq|N_P(H)|$ implies that $\{N_G(W_1(P)), \cdots, N_G(W_n(P)), C_B(Z(P))\}$ connects A with B_0. Since $C_G(Z(P))$ connects B_0 with B, we have a contradiction.

This completes the proof of Theorem 3.

Next we shall prove the corollaries.

Proof of Corollary 1. We need only show that $\{W_1, W_2\}$ controls p-fusion in N. Set P is a Sylow p-subgroup of N. Suppose P contains two subsets A and B which are conjugate in N. Then, since $N=N_G(W_1(P))\, N_G(W_2(P))\, O_{p'}(N)$, there are elements a and b in $N_N(W_1(P))$ and $N_N(W_2(P))\, O_{p'}(N)$, respectively, such that $A^a=B^b$. Then $A^a=B^b$ is contained in some Sylow p-subgroup P_1 of $N_N(W_1(P))\cap N_N(W_2(P))\, O_{p'}(N)$. Since $N_N(W_1(P))\cap N_N(W_2(P))\, O_{p'}(N)$ contains P, there is an element c in $N_N(W_1(P))\cap N_N(W_2(P))\, O_{p'}(N)$ such that $P_1^c=P$. Then A is conjugate to $A^{ac}=B^{bc}$ in $N_N(W_1(P))$ and $A^{ac}=B^{bc}$ is conjugate to B in $N_N(W_2(P))\, O_{p'}(N)$. By the same way, we have that there are elements d and e in $N_N(W_2(P))$ and $O_{p'}(N)$, respectively, such that $bc=ed$ and P contains B, $E^e=B^{bcd^{-1}}$, and B^{bc}. Since $e\in O_{p'}(N)$, we have that $B=B^e$. Therefore, $N_N(W_1(P))$ connects A with $A^{ac}=B^{bc}$ and $N_N(W_2(P))$ connects B^{bc} with $B^{bcd^{-1}}=B^e=B$. Hence we have that $\{W_1, W_2\}$ controls p-fusion in N.

Proof of Corollary 2. Suppose false. Then $Z(S)$ is not weakly closed in S with respect to G. Therefore, there is an element g in G such that $Z(S)\neq Z(S)^g\subseteq S$. By Alperin's theorem [1], there is a subgroup H of S such that $H\supseteq Z(S)$ and $Z(S)\not\trianglelefteq N_G(H)$. Choose such a subgroup H in S satisfying the following conditions:

i) $N_S(H)$ is of maximal order in such groups,

ii) H is of maximal order subject to i).

Then we have that $N_S(H)$ is a Sylow 2-subgroup of $N_G(H)$ and $C_S(H)\subseteq H$, the proof of these results is similar to the proof of Theorem 3. Since $N_G(H)$ is 2-constrained, $N_G(H)=\langle N_{N_G(H)}(J(N_S(H))),\ C_{N_G(H)}(Z(N_S(H)))\rangle\, O(N_G(H))$, by Lemma 4. Since $\langle O(N_G(H)),\ C_G(Z(N_S(H)))\rangle\subseteq C_G(Z(S))$, we have that $Z(S)\subseteq J(N_S(H))$ and $Z(S)\not\trianglelefteq N_G(J(N_S(H)))$. By maximality of $N_S(H)$, $|N_S(J(N_S(H)))|\leq|N_S(H)|$. Thus we have that $N_S(H)=S$, but $Z(S)\not\trianglelefteq N_G(J(N_S(H)))=N_G(J(S))$, which contradicts the hypothesis of Corollary 2.

This completes the proof of Corollaries.

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