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# On measurable norms and abstract Wiener spaces

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## § 1. Introduction

In [4], H. Kuo has shown that the following:

THEOREM A. *Let $H$ be a real separable Hilbert space with norm $\|\cdot\|_H$, and $\|\cdot\|$ be a continuous Hilbertian norm on $H$. Then the following conditions are equivalent.*

(1) $\|\cdot\|$ *is measurable.*

(2) *There exists a one-to-one Hilbert-Schmidt operator $T$ of $H$ such that $\|x\|=\|Tx\|_H$ for $x\in H$.*

In general, if $\|\cdot\|$ be a measurable norm (not necessarily Hilbertian one), there is a compact operator $K$ of $H$ such that $\|x\|\leqq\|Kx\|_H$ for $x\in H$. The above theorem shows that if a measurable norm $\|\cdot\|$ be a Hilbertian one, then the operator $K$ can be taken to be a Hilbert-Schmidt operator. However, if a measurable norm $\|\cdot\|$ be not a Hilbertian one, then this is not necessarily true. The counterexample can be found in [4].

The purpose of the present paper is to show that under the suitable conditions of the norm $\|\cdot\|$, Theorem $A$ can be extended to a non-Hilbertian case. Throughout the paper, we assume that linear spaces are separable with real coefficients.

## § 2. Basic definitions and well known results

### 1°. $p$-absolutely summing operators and $(*)_p$-conditions ($1\leqq p<\infty$)

Let $E$ and $F$ be Banach spaces.

A sequence $\{x_i\}$ with values in $E$ is called weakly $p$-summable if for all $x^*\in E^*$, the sequence $\{x^*(x_i)\}\in l_p$.

A sequence $\{x_i\}$ with values in $E$ is called absolutely $p$-summable if the sequence $\{\|x_i\|\}\in l_p$.

DEFINITION 2.1.1. *A linear operator $T$ from $E$ into $F$ is called $p$-absolutely summing if for each $\{x_i\}\subset E$ which is weakly $p$-summable, $\{T(x_i)\}\subset F$ is absolutely $p$-summable.*

We shall say "absolutely summing" instead of "1-absolutely summing".

THEOREM 2.1.1. (c.f. [7])

*Let $H$ be a Hilbert space, $E$ be a Banach space and $T$ be a continuous linear operator from $H$ into $E$. Then the following conditions are equivalent.*

(1) *$T$ is $p$-absolutely summing ($1\leqq p\leqq 2$).*

(2) *There exists a Hilbert space $G$ such that*

$$H \underset{U}{\longrightarrow} G \underset{V}{\longrightarrow} E$$

*$T=V\circ U$ where $U$ is a Hilbert-Schmidt operator and $V$ is a continuous linear operator, respectively.*

Recently, the author [9] has introduced the class of Banach spaces which satisfy the $(*)_p$-conditions. That is the following:

DEFINITION 2.1.2. *Let $E$ be a Banach space and $1\leqq p<\infty$. If the following condition $(*)_p$ is satisfied, then we shall say that a Banach space $E$ satisfies the $(*)_p$-conditions. The condition is as follows;*

$(*)_p$: *For any $\{x_n^*\}\subset E^*$ with $\|x_n^*\|=1$ $(n=1, 2, \cdots\cdots)$,*

$$\bigcap_{T\in L(F,E)} l_p(\|T^* x_n^*\|^p) = l_p$$

*where the totality of continuous linear operators from $F$ into $E$ is denoted by $L(F, E)$, and $F$ denoted by the following,*

$$F=\begin{cases} l_{p^*} & \text{if } p>1 \\ c_0 & \text{if } p=1 \end{cases} \quad (1/p+1/p^*=1).$$

Here, we have some examples of Banach spaces which satisfy the $(*)_p$-conditions, and those are as follows;

From the above definition, it is easily seen that if $E^*$ is isomorphic to a subspace of $l_p$, then $E$ satisfies the $(*)_p$-conditions. And also, by Theorem 2.1.1., if $E$ is isomorphic to a Hilbert space $H$, then $E$ satisfies the $(*)_p$-conditions ($1\leqq p\leqq 2$). More generally, $\mathscr{L}_{p^*,\lambda}$-space (c.f. [5]) satisfies the $(*)_p$-conditions, and especially, every $L_{p^*}(\mu)$-space satisfies the $(*)_p$-conditions (for more details, see [9]).

THEOREM 2.1.2. (c.f. [9])

*Let $E$ be a Banach space, and $1\leqq p<\infty$. Then the following conditions are equivalent.*

(1) *$E$ satisfies the $(*)_p$-conditions.*

(2) *For any Banach space $F$, if $T$ is a $p$-absolutely summing operator from $E$ into $F$, then $T^*$ (adjoint of $T$) is a $p$-absolutely summing*

*operator from* $F^*$ *into* $E^*$.

### 2°. measurable norms and abstract Wiener spaces

Let $H$ be a real separable Hilbert space with norm $||\cdot||_H$. $F(H)$ will denote the partially ordered set of finite dimensional orthogonal projections $P$ of $H$. ($P>Q$ means $P(H)\supset Q(H)$ for $P, Q\in F(H)$).

DEFINITION 2.2.1. *The standard Gaussian measure in* $H$ *is the cylinder set measure* $\mu_H$ *defined as follows*:

$$\hat{\mu}_H(x)=\exp\left(-\frac{1}{2}||x||_H^2\right) \quad \text{for} \quad x\in H,$$

*where* $\hat{\mu}_H$ denote the Fourier-transform of $\mu_H$.

REMARK 2.2.1. *The standard Gaussian measure* $\mu_H$ *is finitely additive, but* $\mu_H$ *is not* $\sigma$*-additive.*

DEFINITION 2.2.2. *A norm* $||\cdot||$ *in* $H$ *is called measurable if for any* $\varepsilon>0$, *there exists* $P_0\in F(H)$ *such that if* $P\in F(H)$ *and* $P\perp P_0$ *then* $\mu_H\{||Px||>\varepsilon\}<\varepsilon$.

REMARK 2.2.2. (c.f. [4])

(1) *Let* $||\cdot||$ *be a measurable norm in* $H$. *Then* $||\cdot||$ *is continuous.*

(2) *Let* $T$ *be a one-to-one Hilbert-Schmidt operator of* $H$, *and define* $||x||=||Tx||_H$ *for* $x\in H$. *Then* $||\cdot||$ *is a measurable norm.*

(3) *Let* $||\cdot||$ *be a norm in* $H$. *If there exists a measurable norm which is stronger than* $||\cdot||$, *then* $||\cdot||$ *is a measurable norm.*

**Notation.** Let $||\cdot||$ be a measurable norm in $H$, and $B$ denote the completion of $H$ with respect to $||\cdot||$. And also $i$ denote the inclusion map from $H$ into $B$. The triple $(i, H, B)$ is called an abstract Wiener space. Theorem $A$ shows that if $B$ is a Hilbert space, then $(i, H, B)$ is an abstract Wiener space iff $i$ is a Hilbert-Schmidt operator.

THEOREM 2.2.1. (c.f. [1])

*Let* $||\cdot||$ *be a continuous norm in Hilbert space* $H$, *and* $\mu_H$ *the standard Gaussian measure in* $H$. *Let* $B$ *denote the completion of* $H$ *with respect to* $||\cdot||$. *Then the following conditions are equivalent.*

(1) $||\cdot||$ *is a measurable norm.*

(2) $\mu_H$ *can be extended to a* $\sigma$*-additive measure in* $B$.

## § 3. Main theorem and other results

### 1°. Main theorem

In this subsection, we shall prove the following main theorem which is a generalization of Theorem $A$ for non-Hilbertian cases.

THEOREM 3.1.1. *Let $H$ be a Hilbert space with norm $||\cdot||_H$, and $1 \leqq p \leqq 2$. Let $||\cdot||$ be a continuous norm in $H$ and $B$ the completion of $H$ with respect to $||\cdot||$. Then, if a Banach space $B^*$ (dual of $B$) satisfies the $(*)_p$-conditions, the following conditions are equivalent.*

(1) *$||\cdot||$ is a measurable norm.*

(2) *There exists a one-to-one Hilbert-Schmidt operator $T$ of $H$ such that $||x|| \leqq ||Tx||_H$, $x \in H$.*

To prove this theorem, the following lemma is very useful.

LEMMA 3.1.1. *Let $B$ be a Banach space, and $\mu$ be a cylinder set measure in $B$. Then, if $\mu$ is $\sigma$-additive, $\hat{\mu}$ (Fourier-transform of $\mu$) is continuous relative to the absolutely summing topology.*

*The continuity of $\hat{\mu}$ means the following: There exists the sequence of continuous seminorms $\{p_n\}$ in $B^*$ (dual of $B$) such that the natural injection from $B^*$ into $(B^*)_{p_n}$ is absolutely summing, and $\hat{\mu}$ is continuous relative to the seminorms $\{p_n\}$; namely, for any $\varepsilon > 0$ there exists $n$ and $\delta > 0$, such that the inequality $p_n(x^*) \leqq \delta$, $x^* \in B^*$ implies that $|1 - \hat{\mu}(x^*)| \leqq \varepsilon$.*

The proof can be done by the same way as lemma 3.1.1. in [10], and so we omit it.

LEMMA 3.1.2. *Let $H$ be a Hilbert space with norm $||\cdot||_H$, and $||\cdot||$ be a measurable norm in $H$. Let $B$ denote the completion of $H$ with respect to $||\cdot||$ and $i$ the inclusion map from $H$ into $B$. Then, we have that the adjoint map $i^*$ from $B^*$* into *$H^*$ is adsolutely summing.*

PROOF. Since a norm $||\cdot||$ is measurable, by Theorem 2.2.1., a standdard Gaussian measure $\mu_H$ in $H$ can be extended to a $\sigma$-additive one in $B$. Hence, by Lemma 3.1.1., $\hat{\mu}(x^*)$, $x^* \in B^*$ is continuous relative to the absolutely summing topology. Since

$$\hat{\mu}_H(x^*) = \exp\left(-\frac{1}{2}||i^* x^*||_H^2\right), \quad x^* \in B^*,$$

it is easily seen that there exists a positive constant $C$ and $n$ such that

$$||i^* x^*||_H \leqq C p_n(x^*), \quad x^* \in B^*.$$

From this, we have easily the assertion.

Next, using the above lemma, we shall prove the main theorem.

PROOF of THEOREM 3.1.1.

(1) $\Rightarrow$ (2): let $||\cdot||$ be a measurable norm. Then, by Lemma 3.1.2., the natural map from $B^*$ into $H^*$ is absolutely summing. Since a Banach

space $B^*$ satisfies the $(*)_p$-conditions, and the natural map is also $p$-absolutely summing (c.f. [7]), by Theorem 2.1.2., the natural map from $H^{**}$ into $B^{**}$ is $p$-absolutely summing. Here, $H=H^{**}$, hence the natural map from $H$ into $B$ is $p$-absolutely summing.

Thus, by Theorem 2.1.1., there exists a Hilbert space $G$ with norm $\|\cdot\|_G$ such that

$$H\subset G\subset B$$

where the natural map from $H$ into $G$ is a Hilbert-Schmidt operator, and the map from $G$ into $B$ is a continuous linear operator, respectively. Since a norm $\|\cdot\|_G$ be Hilbertian, it is easily seen that there exists a one-to-one continuous linear operator $T$ of $H$ such that $\|x\|_G=\|Tx\|_H$, $x\in H$.

Obviously, $T$ is a Hilbert-Schmidt operator. Thus, we have easily the assertion.

(2) $\Rightarrow$ (1): By Remark 2.2.2., it is obvious.

REMARK 3.1.1. *In Theorem* 3.1.1., *let $i$ denote the inclusion map from $H$ into $B$. Then, we can say that if a Banach space $B^*$ satisfies the $(*)_p$-conditions* ($1\leqq p\leqq 2$), *$(i, H, B)$ is an abstract Wiener space iff $i$ is a Hilbert-Schmidt operator. However, if $p>2$, then the above result is not necessarily true. The counterexample can be found in the next subsection.*

COROLLARY 3.1.1. *Let $H$ be a Hilbert space with inner product $(\cdot, \cdot)_H$, and $\{e_n\}$ be a complete orthonormal system in $H$. We define a continuous norm $\|\cdot\|$ in $H$ by*

$$\|x\|=\left(\sum_{n=1}^{\infty}\lambda_n|(x, e_n)|^p\right)^{1/p}, \qquad x\in H$$

*where $0<\lambda_n<\infty$, and $1\leqq p\leqq 2$. Let $B$ denote the completion of $H$ with respect to $\|\cdot\|$, and $i$ the inclusion map from $H$ into $B$.*

*Then, we have that $(i, H, B)$ is an abstract Wiener space iff $i$ is a Hilbert-Schmidt operator from $H$ into $B$.*

PROOF. Since a Banach space $B$ is isomorphic to $l_p$, therefore $B^*$ (dual of $B$) satisfies the $(*)_p$-conditions. Thus, by Theorem 3.1.1., we have the assertion.

REMARK 3.1.2. *In the above corollary, if $p>2$, then the above result is not necessarily true. That case is discussed in the next subsection* (*see*; *Proposition* 3.2.1.).

## 2°. Other results

In this subsection, we shall discuss the cases of $l_p(\lambda_n)$ and $L_p(X, \mu)$.

PROPOSITION 3.2.1. *Let* $||\cdot||$ *be a continuous norm in Hilbert space* $H$ *defined by the same way as Corollary* 3.1.1., *namely*;

$$||x|| = \left(\sum_{n=1}^{\infty} \lambda_n |(x, e_n)|^p\right)^{1/p}, \qquad x \in H$$

*where* $0<\lambda_n<\infty$, *and* $1 \leqq p<\infty$. *Let* $B$ *denote the completion of* $H$ *with respect to* $||\cdot||$, *and* $i$ *the inclusion map from* $H$ *into* $B$.

*Then, the following conditions are equivalent.*

(1) $(i, H, B)$ *is an abstract Wiener space.*

(2) *The adjoint map* $i^*$ *from* $B^*$ *into* $H^*$ *is absolutely summing.*

(3) $\sum_{n=1}^{\infty} \lambda_n < \infty$.

PROOF.

(1) $\Rightarrow$ (2): By Lemma 3.1.2., it is obvious.

(2) $\Rightarrow$ (3): Since $H$ is linearly isometric to $l_2$, and $B$ is linearly isometric to $l_p(\lambda_n)$, respectively, therefore, this is the particular case of Proposition 4.2.1. in [8].

(3) $\Rightarrow$ (1): It is sufficient to show that if the condition (3) be satisfied, $(i, l_2, l_p(\lambda_n))$ is an abstract Wiener space. However, by Lemma 3.2.1. in [10], the condition (3) implies that a standard Gaussian measure in $l_2$ can be extended to a $\sigma$-additive one in $l_p(\lambda_n)$.

Thus, by Theorem 2.2.1., we have the assertion.

REMARK 3.2.1. *In the above porposition, if* $1 \leqq p \leqq 2$, *the conditions* (1), (2), (3) *and* (4) *are equivalent* (*c.f. Corollary* 3.1.1.); *where the condition* (4) *is the following*:

(4) *The map* $i$ *from* $H$ *into* $B$ *is a Hilbert-Schmidt operator.*

*However, if* $p>2$, *the condition* (4) *is not necessarily equivalent to the above equivalent conditions. Indeed, let the sequence* $\lambda_n$ *be taken as follows*;

$$\sum_{n=1}^{\infty} \lambda_n < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} (\lambda_n)^{\frac{2}{p}} = \infty,$$

*then, we have easily the counterexample.*

**Notation.** Let $(X, \mathfrak{B}, \mu)$ be a measure space. The $\mu$-measurable set $E$ of positive measure is called an atom whenever for any $\mu$-measurable subset $E_1$ of $E$ we have either $\mu(E_1)=0$ or $\mu(E-E_1)=0$.

If $(X, \mathfrak{B}, \mu)$ be a $\sigma$-finite measure space, then we may show that $X = X_1 + X_2$ uniquely, where neither $X_1$ nor any of its measurable subsets is an atom, and $X_2$ is a union of an at most countable number of atoms of finite measure. When this, we shall say $X_1$ non-atomic part of $\mu$.

PROPOSITION 3.2.2. *Let $(X, \mathfrak{B}, \mu)$ be a non-trivial finite measure space, and $1 \leqq p \leqq 2$. Let $i$ denote the natural injection from $L_2(X, \mu)$ into $L_p(X, \mu)$. Then the following conditions are equivalent.*

(1) *$(i, L_2(X, \mu), L_p(X, \mu))$ is an abstract Wiener space.*

(2) *The natural injection $i$ from $L_2$ into $L_p$ is a Hilbert-Schmidt operator.*

(3) *For any $\{X_n\} \subset X$ which is measurable and pairwise disjoint, we have*

$$\sum_{n=1}^{\infty} \mu(X_n)^{1-\frac{p}{2}} < \infty .$$

PROOF. Since a Banach space $(L_p)^*$ satisfies the $(*)_p$-conditions, and $1 \leqq p \leqq 2$, by Theorem 3.1.1., the equivalence of (1) and (2) be obvious. On the other hand, by Lemma 3.1.2. and Theorem 4.2.1. in [8], (1) implies (3). It suffices to show that (3) implies (2):

Suppose that the condition (3) be satisfied, then it is easily seen that the non-atomic part of $\mu$ has zero measure. Since $\mu(X) < \infty$, $\mu$ is concentrated on at most countable sets $\{x_n\}$ in $X$. When this, without loss of generality, we may assume that the sequence $\{x_n\}$ be an infinite one. Thus, $L_2(X, \mu)$ is identified to $l_2(\lambda_n)$, and $L_p(X, \mu)$ be identified to $l_p(\lambda_n)$; where

$$\lambda_n = \mu\{x_n\} , \quad \text{and} \quad \sum_{n=1}^{\infty} (\lambda_n)^{1-\frac{p}{2}} < \infty .$$

Hence, it sufficies to show that the natural injection from $l_2(\lambda_n)$ into $l_p(\lambda_n)$ is a Hilbert-Schmidt operator: but this can be proved by Proposition 4.1.1. in [8]. That completes the proof.

COROLLARY 3.2.1. *Let $(X, \mathfrak{B}, \mu)$ be a finite measure space, and $1 \leqq p \leqq 2$. Let $i$ denote the natural injection from $L_2(X, \mu)$ into $L_p(X, \mu)$. If the non-atomic part of $\mu$ has a positive measure, then $(i, L_2, L_p)$* is not an abstract Wiener space.

EXAMPLE. *Let $\mu$ be a Lebesgue measure on $([a, b], \mathfrak{B})$, and $1 \leqq p \leqq 2$. Let $i$ denote the natural injection from $L_2$ into $L_p$. Then, $(i, L_2, L_p)$ is not an abstract Wiener space.*

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