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The unitary part of paranormal operators

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Let T be a *contraction* (i.e. $\|T\|\leq 1$) on a complex Hilbert space $\mathfrak{H}$. It is known ([3] Theorem 3.2) that there is a unique direct sum decomposition $T=T^{(u)}\oplus T^{(o)}$ on $\mathfrak{H}^{(u)}\oplus\mathfrak{H}^{(o)}$ such that $T^{(u)}=T|_{\mathfrak{H}^{(u)}}$ is unitary while $T^{(o)}=T|_{\mathfrak{H}^{(o)}}$ is *completely non-unitary*, that is, $T^{(o)}$ has no non-trivial reducing subspace on which $T^{(o)}$ is unitary. Actually $\mathfrak{H}^{(u)}$ is characterized as follows:

$$\mathfrak{H}^{(u)}=\left\{x\in\mathfrak{H}:\|T^{n}x\|=\|T^{*^{n}}x\|=\|x\|\quad n=1,2,3,\cdots\right\}.$$

Since the sequence $\{T^{*^{n}}T^{n}\}$ and $\{T^{n}T^{*^{n}}\}$ are non-negative, monotone decreasing, there exist their strong limits. Then by using the notations $A:=(\lim T^{*^{n}}T^{n})^{\frac{1}{2}}$ and $A_{*}:=(\lim T^{n}T^{*^{n}})^{\frac{1}{2}}$ the subspace $\mathfrak{H}^{(u)}$ is written in the following way:

$$\mathfrak{H}^{(u)}=\{x\in\mathfrak{H}:Ax=A_{*}x=x\}.$$

Recently Putnam ([1], Corollary 1 of Theorem 3) showed that if T is a *hyponormal* (i.e. $\|Tx\|\geq\|T^{*}x\|$) contraction A_{*} becomes the projection onto $\mathfrak{H}^{(u)}$. This result was derived from a rather deep property of a hyponormal operator. The purpose of this paper is to prove the same conclusion for a *paranormal* (i.e. $\|Tx\|^{2}\leq\|T^{2}x\|\,\|x\|$) contraction, with a very simple proof. Every hyperonormal operator is paranormal. In contrast to the case of hyponormality the sum of a paranormal operator and a scalar is not necessarily paranormal. This discrepancy makes it inevitable for us to take an approach different from that of Putnam as well as of Stampfli and Wadhwa [2].

THEOREM. *Let T be a paranormal contraction. Then A_{*} is the projection onto the subspace $\mathfrak{H}^{(u)}$.*

Proof. Define $\mathfrak{M}:=\overline{A_{*}(\mathfrak{H})}$. From the definition of A_{*}, $\|A_{*}T^{*}x\|=\lim_{n\to\infty}\|T^{*^{n+1}}x\|=\|A_{*}x\|$ for all $x\in\mathfrak{H}$. So there exists a partial isometry W such that $A_{*}T^{*}=WA_{*}$ and $W|_{\mathfrak{M}^{\perp}}=0$. Since W is isometric on $\mathfrak{M}$ and $TA_{*}=A_{*}W^{*}$ we have $TA_{*}WA_{*}=A_{*}W^{*}WA^{*}=A_{*}^{2}$, hence $\overline{T\mathfrak{M}}\supset\overline{A_{*}^{2}\mathfrak{H}}=\overline{A_{*}\mathfrak{H}}=\mathfrak{M}$, that is, $\overline{T\mathfrak{M}}=\mathfrak{M}$. Let $x\in\mathfrak{M}$, and define $y_{n}:=A^{*}W^{n}x$ $(n=0,1,2,\cdots)$. Then we have $Ty_{n+1}=TA_{*}W^{n+1}x=A_{*}W^{*}W^{n+1}x=A_{*}W^{n}x=y_{n}$.

Since T is paranormal, we have $||y_n||^2=||Ty_{n+1}||^2\leq||T^2y_{n+1}||\cdot||y_{n+1}||=||y_{n-1}||\cdot||y_{n+1}||$ $(n=1, 2, \cdots)$, hence $\{||y_n||^2\}$ is convex with respect to n, and *bounded*: $||y_n||^2=||A_*W^nx||^2\leq||x||^2$ $(n=0, 1, 2, \cdots)$, therefore $\{||y_n||\}$ is non-incerasing. In particular $||y_0||\geq||y_1||$, that is, $||A_*x||\geq||A_*Wx||$. On the other hand, we have $||A_*x||=||A_*W^*Wx||=||TA_*Wx||\leq||A_*Wx||$, so $||A_*x||=||A_*Wx||=||TA_*Wx||$. Since $A_*Wx=T^*T(A_*Wx)=T^*A_*x$, it follows $T^*\mathfrak{M}\subset\mathfrak{M}$ and $||T^*A_*x||=||A_*x||$. Hence we showed $\mathfrak{M}$ reduces T and $T^*|_{\mathfrak{M}}$ is an isometry. Then $A_*^2=\lim_{n\to\infty}(TP_{\mathfrak{M}})^n(T^*P_{\mathfrak{M}})^n=P_{\mathfrak{M}}$ where $P_{\mathfrak{M}}$ is the projection onto $\mathfrak{M}$. Therefore $A_*=P_{\mathfrak{M}}$. To prove $T^*\mathfrak{M}=\mathfrak{M}$, take arbitrary $x\in\mathfrak{M}\ominus T^*\mathfrak{M}$. We can easily show that $TT^*x=x$ and $T^2T^*x=0$. Since T is paranormal we have $||x||^2=||TT^*x||\leq||T^2T^*x||\cdot||T^*x||=0$, hence $x=0$. Consequently $\mathfrak{M}=T^*\mathfrak{M}$, and $T^*|_{\mathfrak{M}}$ (and $T|_{\mathfrak{M}}$) is unitary. Therefore $\mathfrak{M}\subset\mathfrak{H}^{(u)}$. The reverse inclusion is trivial. Q. E. D.

COROLLARY 1. *Let T be a paranormal completely non-unitary contraction. Then $T\in C_{\cdot 0}$, i.e. $\lim_{n\to\infty} T^{*^n}=0$.*

Proof. By Theorem completely non-unitarity is equivalent to $A_*=0$.

Q. E. D.

COROLLARY 2. *Let T be a paranormal contraction. Then* $\lim_{n\to\infty}||T^nx||\geq\lim_{n\to\infty}||T^{*^n}x||$ *for all $x\in\mathfrak{H}$.*

Proof. Let $x\in\mathfrak{H}$. Then we devide x into $x=A_*x+(I-A^*)x$. By the Theorem A_* is the projection onto the subspace $\mathfrak{H}^{(u)}$ hence we have $||T^nx||^2=||T^nA_*x||^2+||T^n(I-A^*)x||^2\geq||T^nA_*x||^2=||A_*x||^2$ for all non-negative interger n. Consequently we have $\lim_{n\to\infty}||T^nx||^2\geq||A_*x||^2=\lim_{n=\infty}||T^{*^n}x||^2$.

Q. E. D.

By the almost same arguement as in the proof of the Theorem, we can obtain the following proposition;

PROPOSITION. *Let T be a paranormal contraction. Let U be unitary. If $TW=WU$ where W has dense range, then T is unitary.*

In contrast to the Theorem, it is not always true that A is a projection if T is a paranormal contraction. This can be seen in the following example. Let $\{e_n\}_{n=0}^{n=\infty}$ be an orthonormal basis of $\mathfrak{H}$. Let $Te_n=\frac{1}{2}e_{n+1}$ or $=e_{n+1}$ according as $n=0$ or $n\geq1$. Then T is a paranormal contraction, and by simple computation we have $Ae_0=\frac{1}{2}e_0$ and $A^2e_0=\frac{1}{4}e_0$. Hence A is not a projection.

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