Hyperinvariant subspaces for contractions of class $C_0$

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1. Introduction

Let $T$ be a bounded operator on a separable Hilbert space $\mathfrak{H}$. A subspace $\mathfrak{G}$ of $\mathfrak{H}$ is said to be hyperinvariant for $T$ if $\mathfrak{G}$ is invariant for every operator that commutes with $T$. In [2] the hyperinvariant subspaces for a unilateral shift were determined, and those for an isometry in [1]. Recall that $T$ is said to be of class $C_0$ if $T$ is a contraction (i.e., $\|T\| \leq 1$) and $T^n \to 0$ (strongly) as $n \to \infty$. Hence a unilateral shift is of class $C_0$. Let $T$ be of class $C_0$. Then it necessarily follows that

$$\delta^* \equiv \dim (1-TT^*) \mathfrak{G} \geq \dim (1-T^*T) \mathfrak{G} \equiv \delta$$

(see [6]). In the case of $\delta^* = \delta < \infty$, in an earlier paper [8] we established a canonical isomorphism between the lattice of hyperinvariant subspaces for $T$ and that for the Jordan model of $T$. In this paper we extend this result to the case of $\delta < \delta^* < \infty$. For an operator $T$ of this class we shall present complete description of the hyperinvariant subspaces $\mathfrak{R}$ with the property that every subspace of $\mathfrak{R}$ hyperinvariant for $T$ is hyperinvariant for the restricted operator $T|\mathfrak{R}$. The author wishes to express his gratitude to Prof. T. Ando for his constant encouragement.

2. Preliminaries

Let $\theta$ be an $n \times m$ $(\infty > n \geq m)$ matrix over $H^\infty$ on the unit circle. Such a matrix $\theta$ is called inner if $\theta(z)$ is isometry a.e. on the unit circle. For such an inner function $\theta$ a Hilbert space $\mathfrak{G}(\theta)$ and an operator $S(\theta)$ are defined by

$$\mathfrak{G}(\theta) = H^*_n \ominus \theta H^*_m \quad \text{and} \quad S(\theta)h = P_\theta(Sh) \quad \text{for} \ h \ in \ \mathfrak{G}(\theta),$$

where $H^*_n$ is the Hardy space of $n$-dimensional (column) vector valued functions, $P_\theta$ is the projection from $H^*_n$ onto $\mathfrak{G}(\theta)$, and $S$ is the simple unilateral shift, that is, $(Sh)(z) = zh(z)$. A contraction $T$ of class $C_0$ with $\delta = n$ and $\delta = m$ is unitarily equivalent to an $S(\theta)$ of this type [7]. Thus in the sequel we may discuss $S(\theta)$ in place of $T$.

For a completely non unitary contraction $T$, it is possible to define
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\[ \phi(T) \text{ for every function } \phi \text{ in } H^\infty. \] In particular, for $S(\theta)$ given above $\phi(S(\theta))$ can be equivalently defined by the following:

\[ \phi(S(\theta)) h = P_\theta \phi h \text{ for } h \text{ in } \mathfrak{H}(\theta) \text{ (see [5], [7]).} \]

If there is a function $\phi$ such that $\phi(T) = 0$, then $T$ is said to be of class $C_{0}$. $T$ of class $C_{0}$ with $\delta \leq \delta_* < \infty$ is of class $C_{0}$ if and only if $\delta = \delta_*$ [7].

Suppose $T_1$ is a bounded operator on $\mathfrak{H}_1$ and $T_2$ a bounded operator on $\mathfrak{H}_2$. If there exists a complete injective family $\{X_\alpha\}$ from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ (i.e., for each $\alpha$, $X_\alpha$ is an one to one bounded operator from $\mathfrak{H}_1$ to $\mathfrak{H}_2$ and $\cap X_\alpha \mathfrak{H}_1 = \mathfrak{H}_2$) such that for each $\alpha$ $X_\alpha T_1 = T_2 X_\alpha$, then we write $T_1 \preceq T_2$. If $T_1 \preceq T_2$ and $T_2 \preceq T_1$, then $T_1$ and $T_2$ are said to be completely injection-similar, and denote by $T_1 \approx^{ci} T_2$ [6].

An $n \times m \ (n \geq m)$ normal inner matrix $N'$ over $H^\infty$ is, by definition, of the form:

\[
N' = \begin{bmatrix}
\phi_1 & 0 & \cdots & 0 \\
0 & \phi_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \phi_m \\
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 \\
\end{bmatrix}
\]

where, for each $i$, $\phi_i$ is a scalar inner function and a divisor of its succesor. Then

\[
S(N') = S(\phi_1) \oplus \cdots \oplus S(\phi_m) \oplus S \cdots \oplus S \nabla_{n-m}
\]

is called a Jordan operator.

Let $\theta$ be an $n \times m \ (\infty > n \geq m)$ inner matrix over $H^\infty$ and $N$ a corresponding normal matrix, i.e., $N$ is the $n \times m$ normal inner matrix of the form (2), where $\phi_1, \phi_2, \ldots, \phi_m$ are the "invariant factors" of $\theta$, that is,

\[
\phi_k = \frac{d_k}{d_{k-1}} \text{ for } k = 1, 2, \ldots, m,
\]

where $d_0 = 1$ and $d_k$ is the largest common inner divisor of all the minors of order $k$. In this case, Nordgren [4] has shown that there exist pairs of matrices $\Delta_i, \Lambda_i$ and $\Delta'_i, \Lambda'_i \ (i = 1, 2)$ satisfying

\[
\Delta_i \theta = N \Lambda_i, \\
\theta \Lambda'_i = \Delta'_i N, \\
(\det \Lambda_i) (\det \Lambda'_i) \wedge d_m = 1,
\]
(5) \((\det \Delta_1) (\det \Delta'_1) \cap (\det \Delta_2) (\det \Delta'_2) = 1\),
(5') \((\det \Delta_1) (\det \Delta'_1) \cap (\det \Delta_2) (\det \Delta'_2) = 1\),

where \(x \cap y\) denotes the largest common inner divisor of scalar function \(x\) and \(y\) in \(H^\infty\). Setting

\((6)\) \(X_i = P_{N} \Delta_i |H(\theta)\) and
\((6')\) \(Y_i = P_{\theta} \Delta'_i |H(N)\) for \(i = 1, 2\),

\(\{X_1, X_2\}\) and \(\{Y_1, Y_2\}\) are complete injective families satisfying the following relations:

\((7)\) \(X_i S(\theta) = S(N) X_i\) and
\((8)\) \(S(\theta) Y_i = Y_i S(N)\) for \(i = 1, 2\).

This implies \(S(\theta) \cap S(N)\) (cf. [6]).

To every subspace \(\mathfrak{L}\) of \(\mathfrak{S} (\theta)\), invariant for \(S(\theta)\), there corresponds an unique factorization \(\theta = \theta_2 \theta_1\) of \(\theta\) such that \(\theta_1\) is an \(k \times m\) inner matrix and \(\theta_2\) is an \(n \times k\) inner matrix \((n \geq k \geq m)\) satisfying

\(\mathfrak{L} = \theta_2 \{H_k^2 \ominus \theta_1 H_m^2\} = \theta_2 H_k^2 \ominus \theta H_m^2\).

In this case \(S(\theta) \parallel \mathfrak{L}\) and \(P_{\theta} S(\theta) \parallel \mathfrak{L}\) are unitarily equivalent to \(S(\theta_1)\) and \(S(\theta_2)\), respectively. For this discussion see [7].

Let \(M\) be an \(m \times m\) normal inner matrix over \(H^\infty\). Then, in [8], we showed that, in order that a factorization \(M = M_2 M_1\) corresponds to a subspace hyperinvariant for \(S(M)\), it is necessary and sufficient that both \(M_1\) and \(M_2\) are \(m \times m\) normal inner matrices.

3. Jordan operator

Let \(N = \begin{bmatrix} M \\ 0 \end{bmatrix}\) be an \(n \times m\) normal inner matrix over \(H^\infty\), that is, \(M\) is an \(m \times m\) normal inner matrix over \(H^\infty\). Then \(S(N)\) on \(\mathfrak{S} (N)\) are identified with

\(S(M) \oplus S_{n-m}\) on \(\mathfrak{S} (M) \oplus H_m^2\),

where \((S_{n-m} h)(z) = z h(z)\) for \(h\) in \(H_m^2\).

Let \(\mathfrak{R}\) be a hyperinvariant subspace for \(S(N)\). Then it is clear that \(\mathfrak{R}\) is decomposed to the direct sum,

\(\mathfrak{R} = \mathfrak{R}_1 \oplus \mathfrak{R}_2\),

where \(\mathfrak{R}_1\) is a subspace of \(\mathfrak{S} (M)\), hyperinvariant for \(S(M)\), and \(\mathfrak{R}_2\) is a subspace of \(H_m^2\), hyperinvariant for \(S_{n-m}\). In this case we have the fol-
Lema 1. For \( \mathfrak{R}_1 \) and \( \mathfrak{R}_2 \) which are hyperinvariant for \( S(M) \) and \( S_{n-m} \), respectively, in order that the direct sum \( \mathfrak{R} = \mathfrak{R}_1 \oplus \mathfrak{R}_2 \) is hyperinvariant for \( S(N) \), it is necessary and sufficient that \( \mathfrak{R}_2 = \{0\} \) or there exists an inner function \( \phi \) such that \( \mathfrak{R}_2 = \phi H_{n-m}^2 \) and \( \mathfrak{R}_1 \supseteq \phi (S(M)) \mathfrak{H}(M) \).

Proof. An operator \( X = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \) commutes with \( S(N) \), if and only if \( Y_{ij} \) satisfy the following conditions:

\[
Y_{11} S(M) = S(M) Y_{11}, \quad Y_{12} S_{n-m} = S(M) Y_{12},
\]
\[
Y_{21} S(M) = S_{n-m} Y_{21}, \quad Y_{22} S_{n-m} = S_{n-m} Y_{22}.
\]

Since \( S(M)^n \rightarrow 0 \) as \( n \rightarrow \infty \) and \( S_{n-m} \) is isometry, we have \( Y_{21} = 0 \). Thus if \( \mathfrak{R}_2 = \{0\} \), then it follows that \( X \mathfrak{R} \subseteq \mathfrak{R} \) for every \( X \) commuting \( S(N) \). By the lifting theorem (cf. [5], [7]), a bounded operator \( Y_{12} \) from \( H_{n-m}^2 \) to \( H(M) \) intertwines \( S_{n-m} \) and \( S(M) \), if and only if there is an \( m \times (n-m) \) matrix \( \Omega \) over \( H^\infty \) such that \( Y_{12} = P_M \Omega \). Thus, if \( \mathfrak{R}_2 = \phi H_{n-m}^2 \) and \( \mathfrak{R}_1 \supseteq \phi (S(M)) \mathfrak{H}(M) \) for some inner function \( \phi \), then we have

\[
X \mathfrak{R} = (Y_{11} \mathfrak{R}_1 + Y_{12} \phi H_{n-m}^2) \oplus Y_{22} \phi H_{n-m}^2
\]
\[
\subseteq (\mathfrak{R}_1 + P_M \phi H_{n-m}^2) \oplus \phi H_{n-m}^2
\]
\[
\subseteq (\mathfrak{R}_1 + P_M \phi H_{m}^2) \oplus \phi H_{n-m}^2
\]
\[
= (\mathfrak{R}_1 + \phi (S(M)) \mathfrak{H}(M)) \oplus \phi H_{n-m}^2
\]
\[
\subseteq \mathfrak{R}_1 \oplus \phi H_{n-m}^2 = \mathfrak{R}
\]

for every \( X \) commuting with \( S(N) \).

Conversely suppose \( \mathfrak{R} = \mathfrak{R}_1 \oplus \mathfrak{R}_2 \) is hyperinvariant for \( S(N) \), and \( \mathfrak{R}_2 \neq \{0\} \). Then by [2] there exists an inner function \( \phi \) such that \( \mathfrak{R}_2 = \phi H_{n-m}^2 \). Let \( \Omega_i \) (\( i = 1, 2, \ldots, m \)) be the \( m \times (n-m) \) matrix such that the \((j, k)\)-th entry of \( \Omega_i \) is 1 for \((j, k) = (i, 1)\) and 0 for \((j, k) \neq (i, 1)\). Setting

\[
X_i = \begin{bmatrix} 0 & Y_i \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Y_i = P_M \Omega_i,
\]
each \( X_i \) commutes with \( S(N) \), hence we have \( \mathfrak{R}_1 \supseteq \sum_{i=1}^{m} Y_i \phi H_{n-m}^2 = P_M \phi H_{m}^2 = \phi (S(M)) \mathfrak{H}(M) \). This completes the proof.

Theorem 1. In order that a factorization \( N = N_2 N_1 \) of \( N \) into the product of an \( n \times k \) inner matrix \( N_2 \) and an \( k \times m \) inner matrix \( N_1 \) (\( n \geq k \geq m \)) corresponds to a hyperinvariant subspace \( \mathfrak{R} \) for \( S(N) \), it is necessary and sufficient that \( N_1 \) and \( N_2 \) are normal matrices satisfying (i) or (ii):

(i) \( k = m \),
(ii) \( k=n \) and \( N_2 \) has the form \[
\begin{bmatrix}
M_2 & 0 \\
0 & \phi_{1_{n-m}}
\end{bmatrix}
\]

PROOF. First, assume that \( k=m \), and both \( N_1 \) and \( N_2 \) are normal inner matrices. Then, setting \( N_2 = \begin{bmatrix} M_2 \\ 0 \end{bmatrix} \), it follows that \( N_2 \{ H_m^2 \oplus N_1 H_m^2 \} = M_2 \{ H_m^2 \oplus N_1 H_m^2 \} \) is hyperinvariant for \( S(M) \) (see [8]). Therefore, by Lemma 1, it is hyperinvariant for \( S(N) \). Next, assume that \( N_1 \) and \( N_2 \) are normal matrices satisfying (ii). Set \( N_1 = \begin{bmatrix} M_1 \\ 0 \end{bmatrix} \). Then we have

\[
\mathcal{R} = N_2 \{ H_m^2 \oplus N_1 H_m^2 \} = M_2 \{ H_m^2 \oplus M_1 H_m^2 \} \oplus \phi H_{n-m}^2 .
\]

Normality of \( M_1 \) and \( M_2 \) implies that \( M_2 \{ H_m^2 \oplus M_1 H_m^2 \} \) is hyperinvariant for \( S(M) \). On the other hand, normality of \( N_2 \) implies \( M_2 H_m^2 \supseteq \phi H_{n-m}^2 \), and hence we have

\[
M_2 H_m^2 \oplus M H_{n-m}^2 \supseteq \phi \{ S(M) \} \phi (M) .
\]

Thus from Lemma 1 we deduce that \( \mathcal{R} \) is hyperinvariant for \( S(N) \).

Conversely, first, assume that \( \mathcal{R} = \mathcal{R}_1 \oplus \{0\} \) is hyperinvariant for \( S(N) \), and \( N = N_2 N_1 \) is the factorization corresponding to \( \mathcal{R} \). Since \( S(N)|\mathcal{R} = S(M)|\mathcal{R}_1 \) is of class \( C_0 \), \( S(N_1) \) is of class \( C_0 \) (cf. 2). This implies that \( N_1 \) is an \( m \times m \) inner matrix, that is, \( k=m \). Setting \( N_2 = \begin{bmatrix} M_2 \\ \Gamma \end{bmatrix} \), where \( M_2 \) is an \( m \times m \) matrix and \( \Gamma \) an \((n-m)\times m\) matrix, we have

\[
M = M_2 N_1, \quad \mathcal{R}_1 = M_2 \{ H_m^2 \oplus N_1 H_m^2 \} \quad \text{and} \quad \Gamma H_m^2 = \{0\} .
\]

Since \( \Gamma = 0 \) and \( N_2 \) is inner, it follows that \( M_2 \) is inner. Thus the hyperinvariance of \( \mathcal{R}_1 \) corresponding to \( M = M_2 N_1 \) implies that \( M_2 \) and \( N_1 \) are \( m \times m \) normal inner matrices. Next assume that \( \mathcal{R} = \mathcal{R}_1 \oplus \phi H_{n-m}^2 \) and \( \mathcal{R}_1 \supseteq \phi \{ S(M) \} \phi (M) \). Clearly we have

\[
P_{\mathcal{R}_1} S(N) |\mathcal{R}^1 = P_{\mathcal{R}_1} S(M) |\mathcal{R}_1 \supseteq \phi (1_{n-m}) .
\]

where \( \mathcal{R}_1^1 \) denotes the orthogonal complement of \( \mathcal{R}_1 \) in \( \phi (M) \). Since the right-hand operator is of class \( C_0 \) (page 129 of [7]), \( S(N_2) \) is of class \( C_0 \). This implies that \( N_2 \) is an \( n \times n \) matrix; i.e., \( k=n \). To the hyperinvariant subspace \( \mathcal{R}_1 \) for \( S(M) \) there corresponds a factorization \( M = M_2 M_1 \), where \( M_1 \) and \( M_2 \) are \( m \times m \) normal inner matrices. Thus setting \( N_2' = \begin{bmatrix} M_2 \\ 0 \\
0 & \phi_{1_{n-m}} \end{bmatrix} \) and \( N_1' = \begin{bmatrix} M_1 \\ 0 \end{bmatrix} \), it is clear that \( N = N_2' N_1' \) and \( \mathcal{R} = N_2' \{ H_m^2 \oplus N_1' H_m^2 \} \). From
the uniqueness of the factorization of $N$ into product of two inner matrices corresponding to (hyper) invariant subspace $\mathfrak{R}$, only this factorization $N = N_2'N_1'$ corresponds to $\mathfrak{R}$, that is, $N_2 = N_2'$ and $N_1 = N_1'$. Since

$$M_2(H_m \oplus M_1 H_m^*) = \mathfrak{L} \supseteq \phi S(M) \mathfrak{R} = M_2 \phi H_m^*,$$

we have $M_2 H_m^* \supseteq \phi H_m^*$; this implies that every entry of $M_2$ is a divisor of $\phi$. Therefore $N_2$ is an $n \times n$ normal inner matrix. Hence $N_1$ and $N_2$ are normal inner matrices satisfying (ii).

4. Lattice isomorphism

Let $\theta$ be an $n \times m$ inner matrix and $N$ be the corresponding normal inner matrix. Set

$$\alpha(\mathfrak{I}) = \bigvee \mathfrak{I} \{ Z \mathfrak{R} : ZS(\theta) = S(N)Z \}$$

and

$$\beta(\mathfrak{I}) = \bigvee \mathfrak{I} \{ W \mathfrak{R} : WS(N) = S(\theta)W \}$$

for each subspace $\mathfrak{I}$ and $\mathfrak{R}$ hyperinvariant for $S(\theta)$ and $S(N)$, respectively, where $\bigvee \mathfrak{I}_i$ denotes the minimum subspace including all $\mathfrak{I}_i$. Since $S(\theta) \subseteq S(N)$, it is clear that $\alpha(\mathfrak{I})$ is the non trivial hyperinvariant subspace for $S(N)$, if $\mathfrak{I}$ is non trivial.

**Lemma 2.** If $\theta = \theta_2 \theta_1$ is the factorization corresponding to a non trivial hyperinvariant subspace $\mathfrak{I}$ for $S(\theta)$, then $\theta_1$ is an $m \times m$ inner matrix, or $\theta_2$ is an $n \times n$ inner matrix.

**Proof.** Let $S(\theta) = \begin{bmatrix} T_1 & * \\ 0 & T_2 \end{bmatrix}$ and $S(N) = \begin{bmatrix} S_1 & * \\ 0 & S_2 \end{bmatrix}$ be the triangulations corresponding to $\phi(\theta) = \mathfrak{I} \oplus \mathfrak{I}^\perp$ and $\phi(N) = \alpha(\mathfrak{I}) \oplus \alpha(\mathfrak{I})^\perp$, respectively. Theorem 1 implies that $S_1$ or $S_2$ is of class $C_0$. First, suppose $u(S_1) = 0$ for some $u$ in $H^\infty$. For the bounded operator $X_1$ given by (6) and every $f$ in $\mathfrak{I}$, in virtue of (3), it follows that

$$X_1 u(T_1)f = X_1 u\Bigl(S(\theta)\Bigr) f = P_N A_1 P_N u f = P_N A_1 u f,$$

$$= P_N u A_1 f = u \Bigl(S(N)\Bigr) X_1 f = 0.$$

Since $X_1$ is an injection, we have $u(T_1) f = 0$, which implies that $T_1$ is of class $C_0$, that is, $\theta_1$ is an $m \times m$ inner matrix. Next suppose $S_2$ is of class $C_0$, hence so is $S_2^*$. For $Y_i$ given by (6)' and every $Z$ such that $ZS(\theta) = S(N)Z$, in virtue of (8), $Y_iZ$ commutes with $S(\theta)$; this implies $Y_i Z \mathfrak{I} \subseteq \mathfrak{I}$ and hence $Y_i \alpha(\mathfrak{I}) \subseteq \mathfrak{I}$. Thus we have $Y_i \mathfrak{I}^\perp \subseteq \alpha(\mathfrak{I})^\perp$. From this and (8), for each
If \( h \in \mathfrak{L}^\perp \), it follows that
\[
Y_i^* T_i^* h = S_i^* Y_i^* h \quad \text{for} \quad i = 1, 2.
\]
From this we can deduce that
\[
Y_i^* u (T_i^*) h = u (S_i^*) Y_i^* h \quad \text{for every} \quad u \in H^\infty,
\]
(see (7) chap 3). Since \( Y_1 \mathfrak{H}(N) \lor Y_2 \mathfrak{H}(N) = \mathfrak{H}(\theta) \), we have \( u (T_i^*) = 0 \) for \( u \) satisfying \( u (S_i^*) = 0 \). Therefore \( \theta_2 \) is an \( n \times n \) inner matrix. This completes the proof.

The following theorem implies that the mapping \( \alpha : \mathfrak{L} \longrightarrow \alpha (\mathfrak{L}) \) is isomorphism from the lattice of hyperinvariant subspaces for \( S(\theta) \) onto that for \( S(N) \), and its inverse is given by \( \beta : \mathfrak{R} \longrightarrow \beta (\mathfrak{R}) \).

**Theorem 2.** For \( X_i \) and \( Y_i \) \((i = 1, 2)\) given by (6) and (6)',
\[
\alpha (\mathfrak{L}) = X_1 \mathfrak{L} \lor X_2 \mathfrak{L}, \quad \text{and} \quad \beta \ast \alpha (\mathfrak{L}) = \mathfrak{L},
\]
\[
\beta (\mathfrak{R}) = Y_1 \mathfrak{R} \lor Y_2 \mathfrak{R} \quad \text{and} \quad \alpha \ast \beta (\mathfrak{R}) = \mathfrak{R},
\]
where \( \mathfrak{L} \) and \( \mathfrak{R} \) are arbitrary hyperinvariant subspaces for \( S(\theta) \) and \( S(N) \), respectively.

**Proof.** Let \( \theta = \theta_2 \theta_1 \) and \( N = N_2 N_1 \) be the factorizations of \( \theta \) and \( N \) corresponding to \( \mathfrak{L} \) and \( \alpha (\mathfrak{L}) \), respectively. Then the proof of Lemma 2 implies that both \( \theta_1 \) and \( N_1 \) are \( k \times m \) matrices and both \( \theta_2 \) and \( N_2 \) are \( n \times k \) matrices, where \( k = n \) or \( k = m \). Since \( X_i \mathfrak{L} \subseteq \alpha (\mathfrak{L}) \) and \( Y_i \alpha (\mathfrak{L}) \subseteq \mathfrak{L} \), it clearly follows that
\[
\Delta_i \theta_2 H_k^2 \subseteq N_2 H_k^2 \quad \text{and} \quad \Delta_i' N_2 H_k^2 \subseteq \theta_2 H_k^2,
\]
which guarantee the existence of \( k \times k \) matrices \( A_i \) and \( B_i \) over \( H^\infty \) satisfying
\[
\Delta_i \theta_2 = N_2 A_i \quad \text{and} \quad \Delta_i' N_2 = \theta_2 B_i.
\]
This and (3) implies that
\[
A_i \theta_1 = N_1 A_i \quad \text{and} \quad B_i N_1 = \theta_1 A_i'.
\]
By (13) we have
\[
\Delta_i' \Delta_i \theta_2 = \theta_2 B_i A_i,
\]
and by (13')
\[
B_i A_i \theta_1 = \theta_1 A_i' A_i.
\]
Thus, if \( k = n \), then \( \det A_i \) is a divisor of \( (\det \Delta_i) (\det \Delta_i') \), and if \( k = m \) then \( \det A_i \) is a divisor of \( (\det \Delta_i) (\det \Delta_i') \). To prove the first relation of (11), suppose that
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$$f \in \alpha(\mathfrak{L}) \ominus \{X_1 \mathfrak{L} \vee X_2 \mathfrak{L}\}.$$ 

Then $f$ is orthogonal to $\Delta_1 \theta_2 H^2 \ominus \Delta_2 \theta_2 H^2$. On the other hand, $f \in \alpha(\mathfrak{L})$ implies the existence of $g$ belonging to $H^2 \ominus N_1 H^2_m$ such that $f = N_d g$. Thus for every $h$ in $H^2$, we have for $i=1,2$

$$0 = (f, \Delta_i \theta_2 h) = (N_d g, N_2 A_i h) = (g, A_i h).$$

If $k=n$, then, by (5) and Beurling's theorem

$$A_i H^2_n \supseteq (\det A_i) H^2_m \supseteq (\det A_i) \ (\det A_i') H^2_n$$

induce $A_1 H^2_n \vee A_2 H^2_n = H^2_n$ and hence $g=0$. If $k=m$, then it follows that from (13) and (4) $\det N_i$ is a divisor of $d_m$, and that $A_i H^2_m \supseteq (\det A_i) \ (\det A_i') H^2_m$; this implies, by (4), $N_1 H^2_m \vee A_i H^2_m = H^2_m$. Consequently we have $g=0$. Thus we showed that if $k=n$, then $\alpha(\mathfrak{L}) = X_1 \mathfrak{L} \vee X_2 \mathfrak{L}$, and if $k=m$, then $\alpha(\mathfrak{L}) = X_1 \mathfrak{L} = X_2 \mathfrak{L}$. The rest is proved in a similar way. Thus we can conclude the proof.

**Corollary 1.** Let $\theta$ be an $n \times m$ $(n>m)$ inner matrix over $H^\infty$. Then for any non constant scalar inner function $\phi$, $\overline{\phi(S(\theta)) \mathfrak{H}(\theta)}$ is a non trivial hyperinvariant subspace for $S(\theta)$.

**Proof.** Since $\{X_1, X_2\}$ is a complete injective family, it is clear that

$$\overline{\alpha(\phi(S(\theta)) \mathfrak{H}(\theta))} = \phi(\overline{S(N)}) \mathfrak{H}(N).$$

The following relation:

$$\mathfrak{H}(M) \oplus \phi H^2_{n-m} \supseteq \phi(S(N)) \mathfrak{H}(N) \supseteq \{0\} \oplus \phi H^2_{n-m}$$

implies that $\overline{\phi(S(N)) \mathfrak{H}(N)}$ is trivial and hence so $\phi(\overline{S(\theta)}) \mathfrak{H}(\theta)$ is by Theorem 2.

**Corollary 2.** $K\phi(S(\theta)) = \{h \in \mathfrak{H}(\theta) : \phi(S(\theta)) h = 0\}$ is a non trivial hyperinvariant subspace for $S(\theta)$ if and only if $\phi \wedge d_m \neq 1$.

**Proof.** It is clear that $K\phi(S(\theta))$ is hyperinvariant for $S(\theta)$ and

$$\alpha(K\phi(S(\theta))) = K\phi(S(N)) = K\phi(S(M) \oplus \{0\}).$$

Since, by the definition, we have $d_m = \det M$, we must show that $K\phi(S(M)) = \{0\}$ if and only if $\phi \wedge (\det M) = 1$.

But this results have already been proved in [3].

5. **Restricted operators**

For an arbitrary subspace $\mathfrak{L}$ of $\mathfrak{H}(\theta)$ we define the subspace $\alpha'(\mathfrak{L})$ of
\[ \mathfrak{H}(N) \]

Similarly define the subspace \( \mathfrak{R} \) of \( \mathfrak{H}(\theta) \) by

\[ \mathfrak{R} = Y_1 \mathfrak{R} \vee Y_2 \mathfrak{R} \quad \text{for a subspace} \quad \mathfrak{R} \text{ of } \mathfrak{H}(N). \]

Then by Theorem 2 \( \alpha'(\mathfrak{L}) = \alpha(\mathfrak{L}) \) if \( \mathfrak{L} \) is hyperinvariant for \( S(\theta) \).

**Theorem 3.** Let \( \mathfrak{L} \) be a hyperinvariant subspace for \( S(\theta) \). If \( \mathfrak{L}' \) is a subspace of \( \mathfrak{L} \), hyperinvariant for \( S(\theta)|\mathfrak{L} \), then \( \alpha'(\mathfrak{L}') \) is a subspace of \( \alpha'(\mathfrak{L}) \), hyperinvariant for \( S(N)|\alpha'(\mathfrak{L}) \) and \( \beta'(\alpha'(\mathfrak{L}')) = \mathfrak{L}' \).

**Proof.** Let \( \theta = \theta_2 \theta_1 \) and \( N = N_2 N_1 \) be the factorization of \( \theta \) and \( N \) corresponding to \( \mathfrak{L} \) and \( \alpha'(\mathfrak{L}) = \alpha(\mathfrak{L}) \), respectively.

\[ \mathfrak{L} = \theta_2 \{ H_i^2 \theta - \theta_1 H_i^2 \} \]

implies that \( \theta_2 | \mathfrak{H}(\theta_1) \) is unitary from \( \mathfrak{H}(\theta_1) \) onto \( \mathfrak{L} \). Hence, in virtue of

\[ \left( (S(\theta)|\mathfrak{L}) \right) \left( \theta_2 \mid \mathfrak{H}(\theta_1) \right) = \left( \theta_2 \mid \mathfrak{H}(\theta_1) \right) \left( S(\theta_1) \right), \]

it follows that \( (\theta_2 | \mathfrak{H}(\theta_1))^{-1} \mathfrak{L}' \) is hyperinvariant for \( S(\theta_1) \). Now for \( A_i \) and \( B_i \) given by (13), from (14) or (14'). \( \det A_i \) \( \det B_i \) is a divisor of \( \det A_i' \) \( \det A_i' \) or \( \det A_i \) \( \det A_i \). Thus by (5) or (5)' we have

\[ (17) \quad \det A_i \wedge \det B_i = 1. \]

It is easy to show that for \( X_i = P_{N_i} A_i | \mathfrak{H}(\theta_1) \),

\[ X_i' \left( \theta_2 \mid \mathfrak{H}(\theta_1) \right)^{-1} \mathfrak{L}' \vee X_i' \left( \theta_2 \mid \mathfrak{H}(\theta_1) \right)^{-1} \mathfrak{L}' \]

is hyperinvariant for \( S(N_i) \), by making use of (13'), (4) and (17), as we have shown Theorem 2. By making use of (3), (4), (5) and (6). Since \( N_2 | \mathfrak{H}(N_1) \) is unitary from \( \mathfrak{H}(N_1) \) onto \( \alpha'(\mathfrak{L}) = \alpha(\mathfrak{L}) \),

\[ \left( (S(N)|\alpha(\mathfrak{L})) \right) \left( N_2 \mid \mathfrak{H}(N_1) \right) = \left( N_2 \mid \mathfrak{H}(N_1) \right) S(N_1) \]

implies that

\begin{align*}
N_2 \left( X_i' \left( \theta_2 \mid \mathfrak{H}(\theta_1) \right)^{-1} \mathfrak{L}' \vee X_i' \left( \theta_2 \mid \mathfrak{H}(\theta_1) \right)^{-1} \mathfrak{L}' \right) \\
= N_2 \left( P_{N_i} A_i \left( \theta_2 \mid \mathfrak{H}(\theta_1) \right)^{-1} \mathfrak{L}' \vee P_{N_i} A_i \left( \theta_2 \mid \mathfrak{H}(\theta_1) \right)^{-1} \mathfrak{L}' \right) \\
= P_{N_i} N_2 A_i \left( \theta_2 \mid \mathfrak{H}(\theta_1) \right)^{-1} \mathfrak{L}' \vee P_{N_i} N_2 A_i \left( \theta_2 \mid \mathfrak{H}(\theta_1) \right)^{-1} \mathfrak{L}' \\
= P_{N_i} A_i \left( \theta_2 \mid \mathfrak{H}(\theta_1) \right)^{-1} \mathfrak{L}' \vee P_{N_i} A_i \theta_2 \left( \theta_2 \mid \mathfrak{H}(\theta_1) \right)^{-1} \mathfrak{L}' \\
= P_{N_i} \mathfrak{L}' \vee P_{N_i} \mathfrak{L}' = X_i \mathfrak{L}' \vee X_i \mathfrak{L}' = \alpha'(\mathfrak{L}')
\end{align*}

is hyperinvariant for \( S(N)|\alpha'(\mathfrak{L}) \). \( \beta'(\alpha'(\mathfrak{L}')) = \mathfrak{L}' \) is proved by the same way.
as Theorem 2. Thus we complete the proof.

The same argument as the proof of Theorem 3 yields.

Theorem 3'. Let $\mathcal{N}$ be a hyperinvariant subspace for $S(N)$. If $\mathcal{N}$ is a subspace of $\mathcal{N}$, hyperinvariant for $S(N)|\mathcal{N}$, then $\beta'(\mathcal{N})$ is a subspace of $\beta'(\mathcal{N})$, hyperinvariant for $S(\theta)|\beta'(\mathcal{N})$, and $\alpha'(\beta'(\mathcal{N})) = \mathcal{N}$.

Theorem 4. Let $\mathcal{L}$ be a subspace hyperinvariant for $S(\theta)$. Then $\mathcal{L}$ is a subspace of $\mathcal{H}(\theta)$, hyperinvariant for $S(\theta)$, if it is a subspace of $\mathcal{L}$, hyperinvariant for $S(\theta)|\mathcal{L}$.

Proof. Set $\alpha'(\mathcal{L}) = \mathcal{N}$ and $\alpha(\mathcal{L}) = \alpha(\mathcal{L}) = \mathcal{N}$. Theorem 3 implies that $\mathcal{N}$ is hyperinvariant for $S(N)|\mathcal{N}$. Let $N = N_2N_1$ be the factorization of $N$ corresponding to $\mathcal{N}$. Then $(N_2N_1)^{-1}\mathcal{N}$ is a subspace of $\mathcal{H}(N_1)$, hyperinvariant for $S(N_1)$. Since $N_1$ is a $k \times m$ ($k = n$ or $k = m$) normal inner matrix over $H^\infty$, by Theorem 1 there is an $l \times m$ normal inner matrix $N'_1$ and an $k \times l$ normal inner matrix $N'_2$ such that

$$N_1 = N'_2N'_1 \text{ and } (N_2|\mathcal{H}(N_1))^{-1}\mathcal{N} = N'_2\{H^2_1 \ominus N'_1H^2_m\},$$

where $n \geq k \geq l \geq m$, and $l = m$ or $l = n$. It is easy to show that $N_2N'_2$ and $N'_1$ satisfy the condition (i) or the condition (ii) of Theorem 1; this implies that

$$\mathcal{N}' = N_2N'_2\{H^2_1 \ominus N'_1H^2_m\}$$

is hyperinvariant for $S(N)$. Thus

$$\beta(\mathcal{N}') = \beta'(\mathcal{N}') = \beta'(\alpha'(\mathcal{L}')) = \mathcal{L}'$$

is hyperinvariant for $S(\theta)$. Thus we conclude the proof.

Now, we determine a particular hyperinvariant subspace $\mathcal{L}_*$ for $S(\theta)$ by the following relation:

$$\mathcal{L}_* = \{h \in \mathcal{H}(\theta) : S(\theta)^n h \to 0 \text{ as } n \to \infty\} \quad \text{(7) P. 73}.$$

Then, from $\alpha(\mathcal{L}_*) \subseteq \mathcal{H}(M)$ and $\beta(\mathcal{H}(M)) \subseteq \mathcal{L}_*$, it follows that $\alpha(\mathcal{L}_*) = \mathcal{H}(M)$.

Theorem 5. Let $\mathcal{L}$ be a subspace hyperinvariant for $S(\theta)$. In order that if $\mathcal{L}'$ is a subspace of $\mathcal{L}$, hyperinvariant for $S(\theta)$, then $\mathcal{L}'$ is hyperinvariant for $S(\theta)|\mathcal{L}$, it is necessary and sufficient that there is a function $\phi$ in $H^\infty$ such that

$$\mathcal{L} = \phi(S(\theta)\mathcal{H}(\theta)) \text{ or } \mathcal{L} = \phi(S(\theta)\mathcal{H}(\theta)) \cap \mathcal{L}_*.$$

Proof. Sufficiency. Case $a$: suppose $\mathcal{L} = \phi(S(\theta)\mathcal{H}(\theta))$ and hence $\alpha(\mathcal{L}) = \phi(S(N))\mathcal{H}(N)$. Let $N = N_2N_1$ be the factorization corresponding to $\alpha(\mathcal{L})$. Then $N_2 = \text{diag}(\phi_1, \cdots, \phi_m, \phi, \cdots, \phi)$, where $\phi_i = \phi_i \wedge \psi_i$ for $i = 1, 2, \cdots, m$. Set $\phi = \phi_i u_i$ and $\phi_i = \phi_i v_i$ for $i = 1, 2, \cdots, m$. Then it follows that for $i =
1, 2, \cdots, m-1,
\phi_{i+1} = \phi \land \psi_{i+1} = \phi_t u_t \land \phi_t v_t \frac{\phi_{i+1}}{\phi_t} = \phi_t \left( u_t \land \psi_{i+1} \frac{\phi_{i+1}}{\phi_t} \right).

Since \( u_t \land \psi_{i+1} = 1 \), this implies that
\begin{equation}
\frac{\phi_{i+1}}{\phi_t} \land \psi_{i+1} = 1.
\end{equation}

Let \( \mathcal{L}' \) be a subspace of \( \mathcal{L} \), hyperinvariant for \( S(\theta) \). Then there is the factorization \( N_1 = N_2' N'_1 \), where \( N'_1 \) is a \( k \times m \) inner matrix and \( N'_2 \) is an \( n \times k \) inner matrix, such that \( \alpha(\mathcal{L}') = N_2' \{ H_2' \cup N'_1 H_1' \} \) (see [7] P. 291). The hyperinvariance of \( \alpha(\mathcal{L}') \) implies that \( N_2' \) and \( N'_1 \) are normal inner matrices satisfying (i) or (ii) of Theorem 1. First, assume (i). Then \( N'_1 \) is an \( m \times m \) normal inner matrix and hence \( N'_2 \) is an \( n \times m \) inner matrix. From the normalities of \( N_2 N'_2 \) and \( N_2 \), we can deduce that \( N'_2 \) has the form \[
\begin{bmatrix}
M' \\
0
\end{bmatrix},
\]
where \( M' = \text{diag} (t_1, t_2, \cdots, t_m) \). Since \( \phi_t t_i \) is a divisor of \( \phi_t \), it follows that \( t_i \) is a divisor of \( \psi_{i+1} \) and, by (18), \( \phi_{i+1} \frac{\phi_{i+1}}{\phi_t} t_i = 1 \). Then normality of \( N_2 N'_2 \) implies that there is an inner function \( \omega_t \) such that \( \omega_t = \frac{\phi_{i+1} t_{i+1}}{\phi_t t_i} \). From \( t_i \omega_t = \frac{\phi_{i+1} t_{i+1}}{\phi_t t_i} \), \( t_{i+1} \), it follows that \( t_i \) is a divisor of \( t_{i+1} \). Thus \( N'_2 \) is normal. Hence \( N_2^{-1} \alpha(\mathcal{L}') = N'_2 \{ H_2' \cup N'_1 H_1' \} \) is hyperinvariant for \( S(N_1) \). Therefore \( \alpha(\mathcal{L}') \) is hyperinvariant for \( S(N_1) \alpha(\mathcal{L}) \). Consequently \( \beta' (\alpha(\mathcal{L}')) = \beta (\alpha(\mathcal{L}')) = \mathcal{L}' \) is hyperinvariant for \( S(\theta) \alpha(\mathcal{L}) \). Next assume that \( N_2 N'_2 \) and \( N'_1 \) satisfy (ii). Then we have \( N'_2 = \text{diag} (t_1, t_2, \cdots, t) \), for inner functions \( t_1, t_2, \cdots, t_m \) and \( t \). It is proved as above that \( t_i \) is a divisor of \( t_{i+1} \) for \( i = 1, 2, \cdots, m-1 \). Since \( \phi_m t_m \) is a divisor of \( \phi_t \), \( t_m \) is a divisor of \( u_m t \). On the other hand since \( t_m \) is a divisor of \( v_m \) and \( v_m \land u_m = 1 \), \( t_m \) is a divisor of \( t \). Thus it follows that \( N'_2 \) is normal. Consequently in the same way as above we can deduce that \( \mathcal{L}' \) is hyperinvariant for \( S(\theta) \mathcal{L} \).

Case b: suppose \( \mathcal{L} = \overline{\phi (S(\theta)) \mathcal{L} (\theta)} \cap \mathcal{L} \). Then by Corollary 1 and \( \alpha(\mathcal{L}_\Phi) = \overline{\phi (M)} \) we have
\[
\alpha(\mathcal{L}) = \overline{\phi (S(N)) \mathcal{L} (N)} \cap \overline{\phi (M)} = \overline{\phi (S(M)) \mathcal{L} (M)},
\]
because \( \alpha \) is a lattice isomorphism. Let \( N = N_2 N_1 \) be the factorization corresponding to \( \alpha(\mathcal{L}) \). Then it follows that
\[
N_2 = \begin{bmatrix} M_2 \\ 0 \end{bmatrix} \quad \text{with} \quad M_2 = \text{diag} (\phi_1, \phi_2, \cdots, \phi_m),
\]
where \( \phi_i = \phi \land \psi_i \) for \( i = 1, 2, \cdots, m \). Let \( \mathcal{L} \) be a subspace of \( \mathcal{L} \), hyperinvariant for \( S(\theta) \), and \( N_1 = N_2' N'_1 \) be the factorization of \( N_1 \) such that \( N =
\( (N_2N_2')N_1' \) is the factorization of \( N \) corresponding to \( \alpha' (\mathfrak{V}) = \alpha (\mathfrak{V}) \). The hyperinvariance of \( \alpha (\mathfrak{V}) \) for \( S(N) \) implies that \( N_2N_2' \) and \( N_1' \) are normal inner matrices satisfying (i). In the same way as Case \( a \) it follows that \( N_2' \) is an \( m \times m \) normal inner matrix. Therefore it is simple to show that \( \mathfrak{V} \) is hyperinvariant for \( S(\theta)|\mathfrak{V} \).

**Necessity.** Let \( \mathfrak{V} \) be the hyperinvariant subspace for \( S(\theta) \) such that \( \mathfrak{V} \) is hyperinvariant for \( S(\theta)|\mathfrak{V} \), if \( \mathfrak{V} \) is a subspace of \( \mathfrak{L} \), hyperinvariant for \( S(\theta) \). Then, for every subspace \( \mathfrak{R}' \) of \( \alpha (\mathfrak{V}) \) such that \( \mathfrak{R}' \) is hyperinvariant for \( S(N) \), it follows from Theorem 3 that \( \beta (\mathfrak{V}') = \beta' (\mathfrak{V}') \) is hyperinvariant for \( S(\theta)|\mathfrak{V} \). Hence, by Theorem 3, \( \mathfrak{V}' = \alpha' (\beta' (\mathfrak{R}')) \) is hyperinvariant for \( S(N)|\alpha (\mathfrak{V}) \). Let \( N = N_2N_1' \) be the factorization corresponding to \( \alpha (\mathfrak{V}) \). Then \( N_2 \) and \( N_1 \) are normal inner matrices.

**Case \( a' \):** assume that \( N_1 \) and \( N_2 \) have the form:

\[
N_1 = \text{diag} (\xi_1, \xi_2, \ldots, \xi_m) \quad \text{and} \quad N_2 = \begin{bmatrix} M_2 \\ 0 \end{bmatrix}
\]

with \( M_2 = \text{diag} (\eta_1, \eta_2, \ldots, \eta_m) \).

Then it follows that \( \eta_i \) and \( \xi_i \) satisfy (18), that is \( \frac{\eta_i+1}{\eta_i} \) and \( \xi_i \) are relatively prime. In fact, if it were not true, then we have

\[
\omega \equiv \frac{\eta_i+1}{\eta_i} \wedge \frac{\xi_j}{\xi_{j-1}} \neq 1 \quad \text{for some} \quad j: 1 \leq j \leq i, \quad \xi_0 = 1 .
\]

Set

\[
M_2' = \text{diag} (\eta_1, \ldots, \eta_{j-1}, \eta_j \omega, \eta_j \omega, \ldots, \eta_i \omega, \eta_{i+1}, \ldots, \eta_m)
\]

\[
N_1' = \text{diag} (\xi_1, \ldots, \xi_{j-1}, \xi_j, \omega, \xi_{j+1}, \omega, \ldots, \xi_i, \omega, \xi_{i+1}, \ldots, \xi_m)
\]

and \( N_2' = \begin{bmatrix} M_2' \\ 0 \end{bmatrix} \). It is clear that \( \mathfrak{R}' \equiv N_2' \{ H_2 \oplus N_1' H_2 \} \) is a subspace of \( \alpha (\mathfrak{V}) \).

Since \( N_1' \) and \( N_2' \) are normal inner matrices, by Lemma \( \mathfrak{V} \) is hyperinvariant for \( S(N) \). However,

\[
(N_2 \mathfrak{S} (N_1))^{-1} N_2' \mathfrak{S} (N_1') = \text{diag} (1, \ldots, 1, \omega, \ldots, \omega, 1, \cdots 1) \mathfrak{S} (N_1')
\]

implies that \( \mathfrak{V}' \) is not hyperinvariant for \( S(N)|\alpha (\mathfrak{V}) \). Thus we have \( \frac{\eta_{i+1}}{\eta_i} < \xi_i = 1 \). Since \( \xi_i \) is a divisor of \( \xi_{i+1} \), it follows that

\[
\eta_m \wedge \psi_i = \eta_m \wedge (\eta_i \xi_i) = \eta_i \left( \frac{\eta_m}{\eta_i} \wedge \xi_i \right) = \eta_i .
\]

Thus we have

\[
\alpha (\mathfrak{V}) = \eta_m (S(M)) \mathfrak{S} (M) = \eta_m (S(N)) \mathfrak{S} (N) \cap \mathfrak{S} (M).
\]

Consequently \( \mathfrak{L} = \eta_m (S(\theta)) \mathfrak{S} (\theta) \cap \mathfrak{L}_* \).
Case $b'$: assume that $N_1$ and $N_2$ are normal inner matrices satisfying (ii). In this case, we can show
\[
\mathfrak{L} = \phi(S(\theta)) \Phi(\theta)
\]
for some $\phi$ in $H^\infty$ in the same way as Case $a'$. Thus we complete the proof of Theorem 5.

References