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On radicals of principal blocks

By Kaoru MOTOSE

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§ 1. Introduction

Let K be an algebraically closed field of characteristic p, G a finite group with a p-Sylow subgroup $P\neq 1$, KG the group algebra of G over K and B_1 the principal block of KG with Cartan matrix $C_1=(c_{st})$. Further, we shall represent $[J(KG)\,;\,K]$ the K-dimension of the radical $J(KG)$ of KG, and u_s, f_s $(s=1, 2, \cdots, r)$ the degrees of all principal indecomposable left ideals U_s of KG and all irreducible modules $F_s=U_s/J(U_s)$, respectively, where F_1 is the trivial module.

R. Brauer and C. Nesbitt [1, p. 580] assert $u_1 f_s \geq u_s$ for all s and so $[J(KG):K]\leq|G|(1-1/u_1)$. From this estimation, it is easily seen that $[J(KG:K]=|G|(1-1/u_1)$ is equivalent to $u_1 f_s=u_s$ for all s. In this paper, we shall call the following question Wallace's problem.

If $[J(KG):K]=|G|(1-1/u_1)$, then is P normal?

As was pointed out by D. A. R. Wallace in Math. Reviews 22 (1961), # 12146, the solution of this problem [8, Theorem] contains an error but holds good for p-solvable groups. Recently, some studies on Wallace's theorem [8, Theorem] are given by Y. Tsushima [7] and the author [5]. The result of R. Brauer and C. Nesbitt [1, p. 580] assert also $[J(B_1):K]\leq[B_1:K]$ $(1-1/u_1)$. And so $[J(B_1):K]=[B_1:K]$ $(1-1/u_1)$ if and only if $u_1 f_s=u_s$ for all $F_s\in B_1$.

Using P. Fong's theorem [3, Lemma (3A)], Wallace's theorem [8, Theorem] is slightly modified as the following:

THEOREM A (D. A. R. Wallace). *Let G be a p-solvable group.*

$[J(B_1):K]=[B_1:K](1-1/u_1)$ *if and only if G is a p-solvable group with p-length* 1.

In the present paper, we shall show that if P is cyclic, then $[J(B_1):K]=[B_1:K]$ $(1-1/u_1)$ if and only if G is a p-solvable group with p-length 1. As an immediate consequence of this and Wallace's theorem [8, Theorem], we can see that Wallace's problem is valid for a group with a cyclic p-Sylow subgroup.

§ 2. Groups with $[J(B_1):K]=[B_1:K]\,(1-1/u_1)$

In the next, E. C. Dade's theorem [2, Theorem 68.1] will play an important role.

THEOREM. *Suppose that P is cyclic. Then,* $[J(B_1):K]=[B_1:K]\,(1-1/u_1)$ *if and only if G is a p-solvable group with p-length* 1.

PROOF. Since P is cyclic, by Dade's theorem [2, Theorem 68.1] and rearrangement of F_s, Cartan matrix (c_{st}) of B_1 is

$$\begin{pmatrix}
2 &&&&&&&&& \\
& 2 && * &&&&&& \\
&& \cdot &&& * &&&& \\
& * && \cdot &&&&&& \\
&&&& \cdot &&&&& \\
&&&&& 2 &&&& \\
&&&&&& h+1 & h & \cdots & h \\
&&&&&& h & h+1 & \cdots & h \\
&& * &&&&&&& \\
&&&&&& h & h & \cdots h & h+1
\end{pmatrix}$$

, where h is the number of exceptional irreducible characters in B_1, the degree $e=(|P|-1)/h$ is the number of non-exceptional irreducible characters in B_1, and elements of $*$-parts are 0 or 1. From the condition $u_1 f_s=u_s$ for all $F_s \in B_1$ and the form of above matrix, we obtain the following inequality:

$$\begin{aligned}
|P|\,(\textstyle\sum_s f_s) &\leq u_1(\textstyle\sum_s f_s) \\
&= \textstyle\sum_s \sum_t c_{st} f_t \\
&\leq \left(\mathrm{Max}_t\,(\textstyle\sum_s c_{st})\right)(\textstyle\sum_t f_t) \\
&\leq (eh+1)\,(\textstyle\sum_t f_t) \\
&= |P|\,(\textstyle\sum_t f_t)
\end{aligned}$$

Whence it follows, $|P|=u_1$ and $\sum_t |P| f_t=\sum_t(\sum_s c_{st}) f_t$. Noting that $f_s>0$ for all s and $\sum_s c_{st} \leq |P|$ (see the above inequality), we obtain the following:

$$\textstyle\sum_{s=1}^{e} c_{st}=|P|=eh+1 \text{ for all } t \qquad (\#)$$

Since $h+1 \geq c_{ss}$ for all s and $h \geq c_{st}$ for all $s \neq t$, by the equation (#), Cartan matrix of B_1 is

$$\begin{pmatrix} h+1 & h & \cdots & h \\ h & h+1 & \cdots & h \\ \\ h & h & \cdots h & h+1 \end{pmatrix}$$

and hence $eh+1=|P|=u_1=(h+1)f_1+hf_2+\cdots+hf_e$, which implies $f_1=f_2=\cdots=f_e=1$. Thus, by [2, Theorem 65.2], $O_{p',p}(G)=\cap_{F_s\in B_1} Ker F_s$ contains the commutator subgroup of G. This means G is a p-solvable group with p-length 1. The converse is valid by Theorem A.

The following is the solution of Wallace's problem for a group with a cyclic p-Sylow subgroup P.

COROLLARY. *Suppose that P is cyclic. Then* $[J(KG):K]=|G|(1-1/u_1)$ *if and only if P is normal in G.*

PROOF. Assume that $[J(KG):K]=|G|(1-1/u_1)$. Then $u_1 f_s=u_s$ for all s, and hence $[J(B_1):K]=[B_1:K](1-1/u_1)$. Thus, G is a p-solvable group by Theorem and so P is normal by Wallace's theorem [8, Theorem]. The converse is given in [8, Theorem].

Let $\boldsymbol{f}$ and $\boldsymbol{u}$ be column vectors with componenets $f_1, f_2, \cdots, f_e$ and $u_1, u_2, \cdots, u_e$, respectively, where $f_1, f_2, \cdots, f_e$ and $u_1, u_2, \cdots, u_e$ are the sets of degrees of all irreducible modules and the principal indecomposable modules contained in B_1. In what follows, $(\boldsymbol{x}, \boldsymbol{y})$ means the inner product of real vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

The next shows that Wallace's problem is sharply related to Frobeniusean root (the largest characteristic root) of Cartan matrix C_1 of B_1.

PROPOSITION. The following are equivalent:

(1) $[J(B_1):K]=[B_1:K](1-1/u_1)$.
(2) u_1 is a characteristic root of C_1.
(3) u_1 is a Frobeniusean root of C_1.

PROOF. (1)$\Rightarrow$(2): If $[J(B_1):K]=[B_1:K](1-1/u_1)$, then $u_1\boldsymbol{f}=\boldsymbol{u}=C_1\boldsymbol{f}$.

(2)$\Rightarrow$(3): Since C_1 is a non-negative matrix, by Frobenius' theorem [4, pp. 404, 546 and 552], and the indecomposability of C_1 (see [2, Theorem 46.3]), there exist a positive number v and a positive vector $\boldsymbol{x}$ such that $C_1\boldsymbol{x}=v\boldsymbol{x}$ and every characteristic root of C_1 is not larger than v. Since $u_1\boldsymbol{f}\geq\boldsymbol{u}=C_1\boldsymbol{f}$ (see [1, p. 580]) and C_1 is symmetric, we obtain $u_1(\boldsymbol{f},\boldsymbol{x})\geq(\boldsymbol{u},\boldsymbol{x})=(C_1\boldsymbol{f},\boldsymbol{x})=(\boldsymbol{f},C_1\boldsymbol{x})=(\boldsymbol{f},v\boldsymbol{x})=v(\boldsymbol{f},\boldsymbol{x})$. Hence, $(u_1-v)(\boldsymbol{f},\boldsymbol{x})\geq 0$ and $(\boldsymbol{f},\boldsymbol{x})>0$ implies $u_1\geq v$. Thus, u_1 is a Frobeniusean root of C_1.

(3)$\Rightarrow$(1): By Frobenius' theorem and the indecomposability of C_1, there exists a positive vector $\boldsymbol{x}$ such that $C_1\boldsymbol{x}=u_1\boldsymbol{x}$. Since C_1 is symmetric, we find $(u_1 \boldsymbol{f}, \boldsymbol{x})=(\boldsymbol{f}, u_1 \boldsymbol{x})=(\boldsymbol{f}, C_1 \boldsymbol{x})=(C_1 \boldsymbol{f}, \boldsymbol{x})=(\boldsymbol{u}, \boldsymbol{x})$, and so $(u_1 \boldsymbol{f}-\boldsymbol{u}, \boldsymbol{x})=0$. Noting that $u_1 \boldsymbol{f}-\boldsymbol{u} \geq 0$ and $\boldsymbol{x}>0$, we obtain $u_1 \boldsymbol{f}=\boldsymbol{u}$, and hence $[J(B_1): K]=[B_1: K](1-1/u_1)$.

§ 3. Some remarks on nilpotency index of $J(KG)$

We shall denote the nilpotency index of $J(KG)$ by $t(G)$.

REMARK 1. From the proof of Theorem and [10, Lemma 4.2], we can see that if P is cyclic, then $t(G) \leq |P|$. More generally, if the defect group D of a block B is cyclic, then the nilpotency index of the radical of B is not larger than $|D|$.

REMARK 2. If $p=3$ and a 3-Sylow subgroup of G is of order 3, then $t(G)=3$. This is proved by Remark 1 and [9, Theorem].

REMARK 3. As the complete answer to the question posed in [6], the following theorem is obtained by Y. Tsushima. The result is informed to the author in a private communication. The author wishes to express his greatful thanks to Mr. Y. Tsushima, who kindly permit to cite it here.

THEOREM B (Y. Tsushima). *Let G be a p-solvable group. Then $t(G) = |P|$ if and only if P is cyclic.*

EXAMPLE. If G is not p-solvable, then the above theorem is not valid. Now, let G be the alternative group of degree 5, and $p=5$. Then Cartan matrix is $\begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and so $t(G) \leq 4$, specially shows $t(G) \neq |P|$. However, a p-Sylow subgroup of G is cyclic.

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Department of Mathematics
Shinshu University