Hokkaido Mathematical Journal Vol. 6 (1977) p. 249-254

On a certain change of affine connections on an almost quaternion manifold

By Shigeyoshi FUJIMURA

(Received November 13, 1976)

§ 1. Introduction

Let (M, V) be an almost quaternion manifold[1] of dimension $4m$ $(\geqq 8)$, i.e., a manifold M which admits a 3-dimensional vector bundle V consisting of tensors of type $(1,1)$ over M satisfying the following condition: In any coordinate neighborhood U of M, there is a local base $\{F, G, H\}$ of V such that

$$\begin{cases} F^2 = G^2 = H^2 = -I, \\ FG = -GF = H, \quad GH = -HG = F, \quad HF = -FH = G, \end{cases}$$

where I is an identity tensor field of type $(1,1)$ on M. Such a local base $\{F, G, H\}$ of V is called a canonical local base of V in U (cf. [2][2]). We shall discuss in the local and use this canonical local base of V. For convenience sake, we put $J_1=I$, $J_2=F$, $J_3=G$ and $J_4=H$.

We now consider an affine connection Γ and a curve $C=x(t)$ on an almost quaternion manifold (M, V) satisfying

$$\nabla_{\dot{x}(t)} \dot{x}(t) = \sum_{i=1}^{4} \phi_i(t) J_i \dot{x}(t)$$

where $\dot{x}(t)$ is the vector tangent to C at the point $x(t)$, $\phi_i(t)$ $(i=1, \cdots, 4)$ are certain functions of the parameter t and ∇ is an operator of covariant differentiation with respect to Γ. Such a curve will be called a Q-planar curve. The purpose of this paper is to prove the following theorem conjectured in the previous paper ([1, p. 242]):

THEOREM. *In an almost quaternion manifold* (M, V) *of dimension* $4m$ $(\geqq 8)$, *affine connections* Γ *and* $\bar{\Gamma}$ *have all Q-planar curves in common if and only if there exist local* 1-*forms* ψ_i $(i=1, \cdots, 4)$ *on* M *satisfying*

$$(1) \qquad S(X, Y)+S(Y, X) = \sum_{i=1}^{4} \left\{\psi_i(X) J_i Y + \psi_i(Y) J_i X\right\},$$

1) Throughout this paper, we assume that manifolds, tensor fields, curves and affine connections are differentiable and of class C^∞.

2) Numbers in brackets refer to the references at the end of the paper.

where ∇ *and* $\bar{\nabla}$ *are operators of covariant differentiation with respect to* Γ *and* $\bar{\Gamma}$ *respectively, and* $S(X, Y)=\bar{\nabla}_X Y-\nabla_X Y$.

§ 2. Proof of Theorem

Let W be an n-dimensional real vector space which admits linear transformations L_2, L_3 and L_4 of W satisfying

$$L_i^2=-L_1\,,\qquad L_i L_j=\operatorname{sgn}\begin{pmatrix}2 & 3 & 4\\ i & j & k\end{pmatrix} L_k$$

for $k\neq i,\ j$ and $i\neq j$ $(i, j, k=2, 3, 4)$, where L_1 and $\operatorname{sgn}\begin{pmatrix}2 & 3 & 4\\ i & j & k\end{pmatrix}$ denote the identity transformation of W and the sign of the permutation $\begin{pmatrix}2 & 3 & 4\\ i & j & k\end{pmatrix}$ respectively. Such a vector space will be called to have a quaternion structure $\{L_i\}$ and we denote it by $(W, \{L_i\})$. The following can be obtained easily.

LEMMA 1. *For a nonzero vector* X, $L_i X (i=1, \cdots, 4)$ *are linearly independent.*

LEMMA 2. *If vectors* $L_1 X, \cdots, L_4 X$ *and* Y *are linearly independent, then* $L_1 X, \cdots, L_4 X, L_1 Y, \cdots, L_4 Y$ *are also linearly independent.*

COROLLARY. *The dimension* n *of* $(W, \{L_i\})$ *is* $4m$ *and there exist vectors* $e_1, \cdots, e_m$ *of* W *such that* $\{L_1 e_1, \cdots, L_4 e_1, \cdots, L_1 e_m, \cdots, L_4 e_m\}$ *is a base of* W.

Let Q be a W-valued quadratic form on $(W, \{L_i\})$ which satisfies

$$(2)\qquad Q(X)=\sum_{i=1}^{4}\alpha_i(X)\, L_i X$$

for any vector X and certain functions $\alpha_1, \cdots, \alpha_4$ on W, and B the W-valued bilinear form associated with Q, i.e.,

$$(3)\qquad 2B(X, Y)=Q(X+Y)-Q(X)-Q(Y)$$

for any vectors X and Y. From (2) and (3), for any real number t, we have

$$\begin{aligned}2B(X, tY)&=Q(X+tY)-Q(X)-Q(tY)\\ &=Q(X+tY)-Q(X)-t^2 Q(Y)\\ &=\sum_{i=1}^{4}\left\{\alpha_i(X+tY)\, L_i(X+tY)-\alpha_i(X)\, L_i X-t^2\alpha_i(Y) L_i Y\right\}\end{aligned}$$

and

$$2B(X, tY)=2tB(X, Y)$$

$$= t\left\{Q(X+Y)-Q(X)-Q(Y)\right\}$$

$$= t\sum_{i=1}^{4}\left\{\alpha_i(X+Y)\,L_i(X+Y)-\alpha_i(X)\,L_iX-\alpha_i(Y)\,L_iY\right\}.$$

Thus, we have

$$\sum_{i=1}^{4}\left\{\alpha_i(X+tY)-\alpha_i(X)-t\alpha_i(Y)\right\}L_i(X+tY)$$

$$=\sum_{i=1}^{4}\left\{\alpha_i(X+tY)\,L_i(X+tY)-\alpha_i(X)\,L_iY-t^2\alpha_i(Y)\,L_iY\right\}$$

$$-t\sum_{i=1}^{4}\left\{\alpha_i(X)\,L_iY+\alpha_i(Y)\,L_iX\right\}$$

$$=t\sum_{i=1}^{4}\left\{\alpha_i(X+Y)-\alpha_i(X)-\alpha_i(Y)\right\}L_i(X+Y),$$

from which, if $L_1X, \cdots, L_4X, L_1Y, \cdots, L_4Y$ are linearly independent, we have

$$\alpha_i(X+tY)-\alpha_i(X)-t\alpha_i(Y)=t\left\{\alpha_i(X+Y)-\alpha_i(X)-\alpha_i(Y)\right\}$$

and

$$t\left\{\alpha_i(X+tY)-\alpha_i(X)-t\alpha_i(Y)\right\}=t\left\{\alpha_i(X+Y)-\alpha_i(X)-\alpha_i(Y)\right\}$$

for every i $(i=1, \cdots, 4)$. Therefore, we have

LEMMA 3. *If vectors $L_1X, \cdots, L_4X, L_1Y, \cdots, L_4Y$ are linearly independent,*

$$\alpha_i(X+Y)=\alpha_i(X)+\alpha_i(Y) \qquad (i=1, \cdots, 4).$$

Assume that vectors $L_1X, \cdots, L_4X, L_1Y, \cdots, L_4Y$ are linearly independent, and put $Z=\sum_{i=1}^{4} r_iL_iX$ $(\neq 0)$ for real numbers $r_1, \cdots, r_4$. Then, from Lemmas 2 and 3, we have

$$(4)\qquad \alpha_i(X+Y+Z)=\alpha_i(X)+\alpha_i(Y+Z)$$

$$=\alpha_i(X)+\alpha_i(Y)+\alpha_i(Z).$$

When $X+Z\neq 0$, from Lemmas 2 and 3 and (4), we have

$$\alpha_i(X+Z)=\alpha_i(X+Y+Z)-\alpha_i(Y)$$

$$=\alpha_i(X)+\alpha_i(Z) \qquad (i=1, \cdots, 4).$$

When $X+Z=0$, from (4), we have

$$(5)\qquad \alpha_i(X)+\alpha_i(Z)=0 \qquad (i=1, \cdots, 4).$$

Therefore, together with Lemma 3, we have

LEMMA 4. *If* dim $W \geqq 8$ *and* $X+Y\neq 0$ *for nonzero vectors* X *and* Y,

$$\alpha_i(X+Y)=\alpha_i(X)+\alpha_i(Y) \qquad (i=1, \cdots, 4).$$

Since $Q(tX)=t^2Q(X)$, from (2), we have

$$t\sum_{i=1}^{4}\left\{\alpha_i(tX)-t\alpha_i(X)\right\} L_iX=0 . \tag{6}$$

From (6) and Lemma 1, we can obtain

LEMMA 5. *For any nonzero real number* t *and any nonzero vector* X,

$$\alpha_i(tX)=t\alpha_i(X) \qquad (i=1, \cdots, 4).$$

PROPOSITION. *In a real vector space* W *with a quaternion structure* $\{L_i\}$ *of dimension* $4m\,(\geqq 8)$, *if a* W*-valued quadratic form* Q *on* W *satisfies* (2), *then there exist linear functions* $\beta_1, \cdots, \beta_4$ *on* W *such that* $\beta_i(X)=\alpha_i(X)$ $(i=1, \cdots, 4)$ *for any nonzero vector* X, *and the* W*-valued bilinear form* B *associated with* Q *is given by*

$$2B(X, Y)=\sum_{i=1}^{4}\left\{\beta_i(X)\, L_iY+\beta_i(Y)\, L_iX\right\}$$

for any vectors X *and* Y.

PROOF. Putting

$$\beta_i(X)=\begin{cases} \alpha_i(X) & \text{when } X\neq 0 \\ 0 & \text{when } X=0 \end{cases} \tag{7}$$

for every i $(i=1, \cdots, 4)$, from (5) and Lemmas 4 and 5, it follows that each of β_i $(i=1, \cdots, 4)$ is linear on W. Therefore, using (2), (3) and (7), this completes the proof.

PROOF of THEOREM. When we put

$$Q_p(X)=(\bar{\nabla}_XX-\nabla_XX)(p)$$

for any point p in M and any vector field X defined around p, and denote by $T_p(M)$ the tangent space of M at p, since $Q_p(X)$ depends upon the vector $X(p)$ at p but not the vector field X and there exists the unique Q-planar curve $x(t)$ such that $x(t_0)=p$ and $\dot{x}(t_0)=X_p$ for every $X_p\in T_p(M)$, we see that Q_p is the $T_p(M)$-valued quadratic form on $T_p(M)$. Thus, Theorem is a direct consequence of Proposition.

§ 3. Remarks

REMARK 1. Linear functions β_i $(i=1, \cdots, 4)$ on $(W, \{L_i\})$ defined in (7) are given by

$$\beta_1(X)=\frac{2}{n^2-4}\left\{\sum_{i=1}^{4}(\mathrm{Tr}_i B)(X)+(n-2)(\mathrm{Tr}_1 B)(X)\right\}$$

and

$$\beta_j(X)=\frac{2}{n^2-4}\left\{\sum_{i=1}^{4}(\mathrm{Tr}_i B)(L_j X)\right.$$
$$\left.-(n-2)\sum_{a=1}^{n} e^a\Big(B(L_j e_a, X)\Big)\right\} \qquad (j=2, 3, 4)$$

for any vector X, where $\{e_1, \cdots, e_n\}$ and $\{e^1, \cdots, e^n\}$ are any base of W and its dual base respectively, and $\mathrm{Tr}_i B$ $(i=1, \cdots, 4)$ are defined by

$$(\mathrm{Tr}_i B)(X)=\sum_{a=1}^{n} e^a\Big(B(L_i e_a, L_i X)\Big)$$

for any vector X.

REMARK 2. In an almost quaternion manifold (M, V), let $\{F, G, H\}$ and $\{F', G', H'\}$ be canonical local bases of V in the neighborhoods U and U' of M respectively. Then, if $U\cap U'\neq\phi$, we have

$$\begin{cases} F' = s_{11}F+s_{12}G+s_{13}H, \\ G' = s_{21}F+s_{22}G+s_{23}H, \\ H' = s_{31}F+s_{32}G+s_{33}H, \end{cases}$$

in $U\cap U'$ where $(s_{ij})\in SO(3)$ $(i, j=1, 2, 3)$ (cf. [2, p. 484]). Thus, using Remark 1, we see that the right hand side of (1) is independent of the choice of canonical local bases of V. And from Remark 1, it follows that each of local 1-forms ϕ_i $(i=1, \cdots, 4)$ on M in (1) is differentiable.

REMARK 3. Let (W', J) be a real vector space W' with a complex structure J of dimension $2m$ $(\geqq 4)$, i.e., a linear transformation J of W' such that $J^2=-I$, where I is an identity transformation of W'. By virtue of the same method as that of the proof of Proposition, which is different from that given in S. Tachibana and S. Ishihara ([3, p. 95]), we can prove that, if a W'-valued quadratic form Q' on (W', J) satisfies

$$Q'(X)=\lambda(X)X+\mu(X)JX$$

for any vector X and certain functions λ and μ on W', then there exist linear functions λ' and μ' on W' such that $\lambda'(X)=\lambda(X)$ and $\mu'(X)=\mu(X)$

for any nonzero vector X, and the W'-valued bilinear form B' associated with Q' is given by

$$2B'(X, Y) = \lambda'(X)Y + \lambda'(Y)X + \mu'(X)JY + \mu'(Y)JX$$

for any vectors X and Y.

References

[1] S. FUJIMURA: *Q*-connections and their changes on an almost quaternion manifold, Hokkaido Math. J., 5 (1976), 239–248.

[2] S. ISHIHARA: Quaternion Kählerian manifolds, J. Diff. Geo., 9 (1974), 483–500.

[3] S. TACHIBANA and S. ISHIHARA: On infinitesimal holomorphically projective transformations in Kählerian manifolds, Tôhoku Math. J., 12 (1960), 77–101.

Department of Mathematics
Ritsumeikan University