Hokkaido Mathematical Journal Vol. 6 (1977) p. 243-248

On the existence of p-blocks with given defect groups

By Tomoyuki WADA

(Received November 16, 1976: Revised December 21, 1976)

1. Introduction.

In [2] Brauer and Fowler proved the following theorem;

THEOREM 1. (Brauer-Fowler (5E), [2]). *Let G be a finite group of even order. Suppose that there exists a conjugate class K of involutions of G and a p-subgroup P such that $z^{x}\neq z^{-1}$ for every $x\in K$, $z\in P^{\#}$, then there exists an irreducible complex character χ such that $p^{c-d}|\chi(1)$, where $p^{c}=|P|$, $p^{d}\top|C_{G}(x)|$.*

Theorem 1 gives us a sufficient condition for the existence of a p-block of defect 0. We can also find in Theorem 1 a sufficient condition for the existence of a p-block with defect group D. Indeed in 2. we shall generalize Theorem 1, and improve the results of N. Ito (Lemma 1, [5]), P. Fong (Theorem (1 G), [4]) and Y. Tsushima (Theorem 1, [7]). In 3. we shall apply our results to a (p, q)-group which shall be defined bellow.

Notations. G denotes always a finite group in this paper. Let us set $|G|_{p}=|P|$ for some $P\in \mathrm{Syl}_{p}(G)$. By Irr (G) we shall mean the set of all irreducible complex characters of G. We denote $p^{d}\top n$, when $p^{d}|n$ and $p^{d+1}\nmid n$ for a prime number p and integers d, n. Let K be a conjugate class of G. We denote by $D(K)$ a p-defect group of K, i.e. a Sylow p-subgroup of $C_{G}(x)$ for some $x\in K$. Let B be a p-block of G. We also denote by $D(B)$ a defect group of B. For distinct prime numbers p, q we call G a (p, q)-group when G satisfies the following condition: p and q are in $\pi(G)$, and $C_{G}(x)$ is a q'-group for every p-element x of G. By a p-element we shall mean a non-trivial p-element. For a fixed prime number p $\mathfrak{p}$ is a prime ideal divisor of p in the field $Q(\varepsilon)$ where ε is a primitive $|G|$-th root of unity. Other notations are standard.

2. A generalization of Theorem 1.

THEOREM 2. *Let D be a p-subgroup of G. Suppose there exist conjugate classes $K_{1}, \cdots, K_{r}$ of G such that $D(K_{i})=D$, $i=1, \cdots, r$, and xy^{-1} is not a p-element for every $x, y\in K_{1}\cup\cdots\cup K_{r}$, then there exist p-blocks $B_{1}, \cdots, B_{r}$ of G which satisfy the following conditions;*

a) $D(B_i)=D,\ i=1,\cdots,r$,

b) $\chi_1,\cdots,\chi_r$ *are linearly independent mod* $\mathfrak{p}$ *on* $K_1,\cdots,K_r$ *for every* $\chi_i\in B_i$ *whose height is* 0 *in* B_i.

PROOF. Set $\{\chi_1,\cdots,\chi_s\}=\mathrm{Irr}(G)$ and $\psi_j=\sum_{i=1}^{s}(\chi_i(x_j)\overline{\chi_i(x_j)}/\chi_i(1))\cdot\bar{\chi}_i$ for $x_j\in K_j,\ j=1,\cdots,r$, then our assumption implies $\psi_{j|P^\#}=0$ for $P\in\mathrm{Syl}_p(G)$, $j=1,\cdots,r$. Since $\psi_j(1)=|C_G(x_j)|$, $\psi_{j|P}=(|C_G(x_j)|/|P|)\cdot\rho_P$, where ρ_P is the regular character of P.
Therefore

$$|C_G(x_j)|/|P|=(\psi_{j|P},1_P)=\sum_{i=1}^{s}\left(\chi_i(x_j)\overline{\chi_i(x_j)}/\chi_i(1)\right)\cdot(\bar{\chi}_{i|P},1_P)\,.$$

This yields

$$|G:P|=\sum_{i=1}^{s}\left(|G:C_G(x_j)|\chi_i(x_j)/\chi_i(1)\right)\cdot\overline{\chi_i(x_j)}\cdot(\bar{\chi}_{i|P},1_P)$$

for $j=1,\cdots,r$.
If $j\neq k$, then our assumption also implies that

$$\sum_{i=1}^{s}\left(|G:C_G(x_j)|\chi_i(x_j)/\chi_i(1)\right)\cdot(\bar{\chi}_{i|P},1_P)\cdot\overline{\chi_i(x_k)}$$
$$=\sum_{z\in P}\left(|G:C_G(x_j)|/|P|\right)\cdot\left(\sum_{i=1}^{s}\chi_i(x_j)\overline{\chi_i(x_k)}\,\overline{\chi_i(z)}/\chi_i(1)\right)=0$$

Now we shall consider the following matrices:

$A=|G:P|\cdot I$, where I is the identity matrix of $r\times r$ type,

$W=\left(\omega_i(x_j)\cdot(\bar{\chi}_{i|P},1_P)\right)$, where $\omega_i(x_j)=|G:C_G(x_j)|\chi_i(x_j)/\chi_i(1)$,

$X=\left(\chi_i(x_j)\right)$.

Then above argument shows that

$$A={}^tW\cdot\bar{X}\,,$$

where ${}^tW$ is the transposed matrix of W and $\bar{X}=\overline{(\chi_i(x_j))}$. Since det $A=|G:P|^r\not\equiv 0\ (\mathrm{mod}\ p)$, the rank mod $\mathfrak{p}$ of W and X is r. This yields the following lemma.

LEMMA 3. *There exist* $\chi_1,\cdots,\chi_r\in \mathit{Irr}(G)$ *such that det* $(\chi_i(x_j))\not\equiv 0$, $\chi_i(x_i)\not\equiv 0$ *and* $\omega_i(x_i)\not\equiv 0$ *(mod* $\mathfrak{p}$*).*

PROOF. Since the rank mod $\mathfrak{p}$ of X is r, there exist $\chi_1,\cdots,\chi_r\in\mathrm{Irr}(G)$ such that det $(\chi_i(x_j))\not\equiv 0$ (mod $\mathfrak{p}$). By suitable permutation we may assume $\chi_i(x_i)\not\equiv 0$ (mod $\mathfrak{p}$), $i=1,\cdots,r$. Suppose there exists $\chi_i(1\leqq i\leqq r)$ such that

$\omega_i(x_i)\equiv 0 \pmod{\mathfrak{p}}$, then $\omega_i(x_k)\equiv 0 \pmod{\mathfrak{p}}$ for every $k=1,\cdots,r$. Therefore the i-th column of ${}^tW$ is 0 mod $\mathfrak{p}$, and

$$A\equiv {}^tW(i)\cdot \bar{X}(i) \pmod{\mathfrak{p}},$$

where ${}^tW(i)$ is a $r\times(s-1)$ matrix obtained by excluding the i-th column from ${}^tW$, $\bar{X}(i)$ is a $(s-1)\times r$ matrix obtained by excluding the i-th row from $\bar{X}$. Since always det $A\not\equiv 0 \pmod{p}$, the rank mod $\mathfrak{p}$ of ${}^tW(i)$ and $\bar{X}(i)$ remains to be r. By applying the above argument to $X(i)$ we can finally obtain our $\chi_1,\cdots,\chi_r$.

If $\omega_i(x_i)\not\equiv 0 \pmod{\mathfrak{p}}$, then χ_i belongs to a p-block B_i with $D(B_i)\subseteq D$. If $\chi_i(x_i)\not\equiv 0 \pmod{\mathfrak{p}}$, then χ_i belongs to a p-block B_i with defect of $B_i\geqq d$, where $|D|=p^d$. Therefore Lemma 3 implies that there exist $\chi_1,\cdots,\chi_r\in \mathrm{Irr}(G)$ such that χ_i belongs to a p-block B_i of G with $D(B_i)=D$ and χ_i is of height 0 in B_i, and they are linearly independent mod $\mathfrak{p}$ on $K_1,\cdots,K_r$. Finally suppose $\chi'_1,\cdots,\chi'_r$ are any characters such that $\chi'_i\in B_i$ and χ'_i is of height 0 in B_i, then $\omega_i(x_j)\equiv\omega'_i(x_j) \pmod{\mathfrak{p}}$, where ω_i and ω'_i are corresponding to χ_i and χ'_i. Since det $(\chi_i(x_j))\not\equiv 0 \pmod{\mathfrak{p}}$, we can easily obtain det $(\chi'_i(x_j))\not\equiv 0 \pmod{\mathfrak{p}}$. This completes the proof of Theorem 2.

COROLLARY 4. *Suppose* $O_{p'}(G)$ *contains conjugate classes* $K_1,\cdots,K_r$ of G with $D(K_i)=D$, $i=1,\cdots,r$, *then there exists p-blocks* $B_1,\cdots,B_r$ *of* G *which satisfy the following conditions*;

a) $D(B_i)=D$, $i=1,\cdots,r$,

b) $\chi_1,\cdots,\chi_r$ *are linearly independent mod* $\mathfrak{p}$ *on* $K_1,\cdots,K_r$ *for every* $\chi_i\in B_i$ *whose height is* 0 *in* B_i.

REMARK. Theorem 2 is a generalization of Theorem 1 of [7] and corollary 4 is a generalization of Lemma 1 of [5] and an improvement of Theorem (1 G) of [4].

3. (p,q)-groups.

First in this section we shall argue on the existence of a p-block of defect 0 of (p,q)-groups. These arguments are easy application of Theorem 2. Secondly we shall argue on $(p,2)$-groups.

THEOREM 2′. *Let* G *be a* (p,q)-*group. Then the following holds.*

1) *G possesses a p-block of defect* 0, *or*

2) *For every conjugate class* K *of q-elements of* G *there exist* $x,y\in K$ *such that* xy^{-1} *is a p-element.*

REMARK. Since a (p,q)-group is a (q,p)-group, the dual assertion for p and q in Theorem 2′ holds. There exist groups satisfying 1) and 2); for example, $G=A_5$, $p=5$, $q=2$.

COROLLARY 5. *Let G be a (p, q)-group.*

(1) *Suppose $|O_{p'}(G)|_q \neq 1$, then G possesses a p-block of defect* 0.

(2) *Suppose G is p-solvable or q-solvable, then G possesses a p-block of defect* 0 *or a q-block of defect* 0.

PROOF. (1) is clear by Theorem 2$'$. (2) Let G be p-solvable. If $O_p(G)\neq 1$, then G possesses a q-block of defect 0 by the duality of (1). We may assume $O_p(G)=1$ and $O_{p'}(G)\neq 1$. If $|O_{p'}(G)|_q\neq 1$, then G possesses a p-block of defect 0 by (1). If $|O_{p'}(G)|_q=1$, then $G/O_{p'}(G)$ is also a (p, q)-group and $O_p(G/O_{p'}(G))\neq 1$. This implies that $G/O_{p'}(G)$ possesses a q-block of defect 0 by the duality of (1), and this block is also a q-block of defect 0 of G. By the same way it is proved when G is q-solvable.

THEOREM 6. *Let G be a $(p, 2)$-group, then*

1) *G possesses a p-block of defect* 0, *or*

2) *$|N_G(P)|$ is even for $P\in Syl_p(G)$, in particular P is abelian.*

PROOF. Suppose 1) does not hold. Theorem 2$'$ implies that there exist involutions x, y such that xy is a p-element. So the p-local subgroup $N_G(\langle xy\rangle)$ has even order. Therefore Theorem 6 is valid if we prove the following lemma.

LEMMA 7. *Let G be a $(p, 2)$-group. Suppose there exists a p-local subgroup $N_G(P_1)$ whose order is even, then $N_G(P)$ has even order.*

PROOF. We may assume $P_1\subsetneqq P$. Put $H_1=N_G(P_1)$. If there exists a subgroup A of H_1 isomorphic to $Z_2\times Z_2$, then $P_1=\prod_{a\in A^\#} C_{P_1}(a)\neq 1$ and $C_{P_1}(a)\neq 1$ for some $a\in A^\#$. This contradicts that G is a $(p, 2)$-group. Therefore the 2-rank of H_1 is 1, and a Sylow 2-subgroup S_1 of H_1 is cyclic or generalized quaternion. Let P_2 be a Sylow p-subgroup of H_1. If S_1 is cyclic, then $P_2\subseteq O(H_1)$ by a theorem of Burnside. If S_1 is generalized quaternion, then $|Z(H_1/O(H_1))|=2$ by a theorem of Brauer-Suzuki ([3]), and since G is a $(p, 2)$-group, $P_2\subseteq O(H_1)$. Therefore by Frattini argument $H_1=N_{H_1}(P_2)\cdot O(H_1)$. Since $|H_1|$ is even, $N_G(P_2)$ has even order. $P_1\subsetneqq P$ implies $P_1\subsetneqq P_2$. By repeating this argument we find $|N_G(P)|$ is even.

In particular some involution acts fixed-point-freely on P, so P is abelian. This completes the proof of Theorem 6.

Theorem 6 yields the following theorem which is a generalization of [6].

THEOREM 8. *Let be a $(p, 2)$-group. Suppose G is a transitive permutation group of degree p^r. Then*

1) *G possesses a p-block of defect* 0, *or*

2) *G is p-solvable,* $O_p(G)\neq 1$ *and G possesses a 2-block of defect* 0.

PROOF. Let H be a one point stabilizer of G. We may assume that G acts on cosets of G by H. Assume 1) does not hold. Let P be a fixed Sylow p-subroup of G, then $G=PH$ and we may chose element of P as coset representatives of G by H. Since G is a $(p, 2)$-group and 1) does not hold, by Theorem 2′ the involutions of G form a single conjugate class. So we may chose involution $x\in H$ such that $z^x=z^{-1}$ for every $z\in P^{\#}$ by Theorem 6. Suppose x fixes a coset zH, then

$$xzH=zH \Leftrightarrow xxz^{-1}xz\in H \Leftrightarrow z^2\in H \Leftrightarrow z\in H.$$

This implies that the number of fixed cosets by every involution is 1, and H is a strongly embedded subgroup of G.

By Bender's results ([1]) we may prove in the following two cases.

Case 1. A Sylow 2-subgroup of G is cyclic or generalized quaternion. In this case $P\subseteq O(G)$ as in the proof of lemma 7. So G is p-solvable by the result of Feit-Thompson. Since G is transitive permutation group of degree p^r, $O_{p'}(G)=1$. Therefore $O_p(G)\neq 1$, and G possesses a 2-block of defect 0 by Corollary 5.

Case 2. G has the normal subgroup L such that $|G:L|$ is odd and $L\cong PSL(2, 2^k)$ with $2^k+1=p^r$, $S_z(2^l)$ with $2^{2l}+1=p^r$ (l is odd) or $U_3(2^m)$ with $2^{3m}+1=p^r$. Since $2^{2l}+1$ and $2^{3m}+1$ cannot be a power of prime number, we have $L\cong PSL(2, 2^k)$ with $2^k+1=p$ or $PSL(2, 8)$ with $p=3$, $r=2$. If $L\cong PSL(2, 2^k)$ with $2^k+1=p$, then G is a permutation group of degree p and $|G|_p=p$. Therefore $|G:L|\not\equiv 0 \pmod p$. Since $PSL(2, 2^k)$ has an irreducible complex character of degree $2^k+1=p$, L possesses a p-block of defect 0. This yields a p-block of defect 0 of G by the Theorem of Clifford. This contradicts our assumption. If $L\cong PSL(2, 8)$ with $p=3$, $r=2$, then $|G|_3\leqq 3^4$. Since a Sylow 3-subgroup of G is abelian by Theorem 6, $|G|_3=3^2$ or 3^3. As above argument $|G|_3\neq 3^2$, so $|G|_3=3^3$ and $3\top|G:L|$, $3^2\top|L|$. Now $G\subseteq \mathrm{Aut}\ PSL(2, 8)\cong PGL(2, 8)\cdot \mathrm{Aut}\ F_8$, where $|PGL(2, 8)|=7\cdot|PSL(2, 8)|$, and F_8 is the finite field of order 8 and $|\mathrm{Aut}\ F_8|=3$. Therefore an element of order 3 of G/L is a field automorphism of $PSL(2, 8)$, and fixes an involution of $PSL(2, 8)$. This contradicts that G is a $(p, 2)$-group. Thus Case 2 does not occur and Theorem 8 has been proved.

REMARK. *A $(p, 2)$-group has the following property.*

1) *G possesses a p-block of defect* 0, *or*

2) *there exists* $\chi\in Irr(G)$ *such that* $\chi(1)$ *is even.*

PROOF. Assume 1) does not hold. By Theorem 6 every p-element of

G is strongly real. Let z be an element of order p, and $N=N_G(\langle z\rangle)$, $C=C_G(z)$. Then $|N|$ is even and $|C|$ is odd. Set

$$\psi=(1_{\langle z\rangle})^N-\lambda^N,$$

where $\lambda\in\mathrm{Irr}(\langle z\rangle)$, $\lambda\neq 1_{\langle z\rangle}$. Since $\langle z\rangle$ is a T.I. set in G, $\psi_{|N-\langle z\rangle}=0$ and $\psi(1)=0$, we have $||\psi^G||=||\psi||$. Let $\psi^G=\sum_{\chi_i\in\mathrm{Irr}(G)} a_i\chi_i$. If $\chi_i(1)$ is odd for every $\chi_i\in\mathrm{Irr}\,(G)$, then

$$||\psi^G||=\sum a_i^2\equiv\sum a_i\equiv\sum a_i\chi_i(1)=\psi^G(1)\qquad(\mathrm{mod}\ 2).$$

Since $\psi^G(1)=0$, and $||\psi^G||=||\psi||=||(1_{\langle z\rangle})^N||+||\lambda^N||=|N:\langle z\rangle|+|C:\langle z\rangle|=\mathrm{odd}$, we have a contradiction. Therefore there exists $\chi\in\mathrm{Irr}\,(G)$ such that $\chi(1)$ is even.

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Otaru University of Commerce