*Hokkaido Mathematical Journal Vol. 6 (1977) p. 183-187*

# On a theorem of Manning-Cameron*

By Shiro IWASAKI

(Received December 25, 1975)

In 1929, Manning ([6]) proved that if $G$ is a uniprimitive permutation group on $\Omega$ (i. e., $(G, \Omega)$ is primitive, but not doubly transitive), and if the stabilizer $G_a$ of a point $a\in\Omega$ acts doubly transitively on an orbit of length $k>2$, then $G_a$ has an orbit whose length is greater than $k$ and a divisor of $k(k-1)$. Recently this was reproved more explicitly and strongly by Cameron ([1], [2]). In this short note, we remark that a similar result holds even when $G_a$ does not act doubly transitively on an orbit of length $k$.

DEFINITIONS and NOTATION. All permutation groups and sets considered in this note are finite. For definitions and notation, we follow those of Wielandt [7] and Higman [5]. Let $G$ be a transitive permutation group on a finite set $\Omega$. For $a\in\Omega$, $g\in G$ and a subgroup $H$ of $G$, we denote by $a^g$ the image of $a$ under $g$ and set $a^H=\{a^g|g\in H\}$. For a subset $S$ of $\Omega$, we set $S^g=\{a^g|a\in S\}$, $G_S=\{g\in G|a^g=a$ for all $a\in S\}$, and $G_{(S)}=\{g\in G|S^g=S\}$. If $S=\{a, b,\cdots\}$, $G_S$ is written $G_{ab\cdots}$.

The number of $G_a$-orbits on $\Omega$ counting the trivial orbit $\{a\}$ is independent of the choice of $a\in\Omega$ and is called the rank of $G$. If $(G, \Omega)$ is primitive and has rank greater than 2, it is said uniprimitive. The lengths of the $G_a$-orbits are called the subdegrees of $G$. Any $G_a$-orbit $\Delta(a)$ is chosen so that $\Delta(a)^g=\Delta(a^g)$ for all $a\in\Omega$ and all $g\in G$, and $\Delta$ is called an orbital of $G$. Each $\Delta(a)$ has a paired orbit defined by $\{a^{g^{-1}}|g\in G, a^g\in\Delta(a)\}$, which is also $G_a$-orbit and denoted by $\Delta'(a)$. $|\Delta(a)|=|\Delta'(a)|, \Delta''(a)=\Delta(a)$ by [7, §16], and

$$b\in\Delta(a) \text{ if and only if } a\in\Delta'(b).$$

If $\Delta'(a)=\Delta(a)$, $\Delta$ or $\Delta(a)$ is said self-paired. Following Cameron [1], for orbitals $\Delta$ and $\Gamma$, we define

$$(\Delta\circ\Gamma)(a)=\left\{b\in\Omega|\Delta(a)\cap\Gamma'(b)\neq\phi, b\neq a\right\},$$

which is a union of some $G_a$-orbits.

THEOREM. *Let $G$ be a uniprimitive permutation group on a finite set*

---

* This partly overlaps with "Jikken-Haichi no Kumiawase-Sugaku to Gunron", Res. Inst. Math. Sci., 1974, 75-82 (in Japanese).

*$\Omega$, and for $a\in\Omega$ let $\Delta(a)$ a $G_a$-orbit of length $k\geqq 2$ on which $G_a$ acts as rank $r$ group with subdegrees $1, k_1, \cdots, k_{r-1}$ $(k=1+k_1+\cdots+k_{r-1})$. Suppose, either*

( * ) *$\Delta(a)$ is self-paired, or*

(**) *$|G_a : G_{a\cup\Delta(a)}|$ is even.*

*Then, there exists a $G_a$-orbit $\Gamma(a)$ of length $l$ such that*

(i) *$\Gamma\neq\Delta, \Delta'$ and $\Gamma(a)\subseteq(\Delta'\circ\Delta)(a)$,*

*and for some $k_i(1\leqq i\leqq r-1)$,*

(ii) *$k_i<l$ and $l$ is a divisor of $kk_i$,*

(iii) *if $b\in\Delta(a)$, $|\Gamma(b)\cap\Delta(a)|=a$ sum of some $k_j$'s containing $k_i$ (so $|\Delta(b)\cap\Delta(a)|$ is 0 or a sum of some $k_j$'s, $j\neq i$).*

*Furthermore, if all the $r$ $G_{ab}$-orbits on $\Delta(a)$ $(b\in\Delta(a))$ are self-paired, $\Gamma(a)$ is self-paired.*

PROOF. Proof is almost trivial. Take a point $b\in\Delta(a)$. By assumption, $G_{ab}$ has $r$ orbits on $\Delta(a)$, say $\{b\}, \Delta_1, \cdots, \Delta_{r-1}$ with $|\Delta_i|=k_i$ $\left(\text{and so } \Delta(a)-\{b\}=\bigcup_{i=1}^{r-1}\Delta_i\right)$. First, we show that $\Delta(b)\not\supseteq\Delta(a)-\{b\}$ in the case (*) and that $\Delta(b)\cup\Delta'(b)\not\supseteq\Delta(a)-\{b\}$ in case $\Delta$ is not self-paired and (**) holds. In the former case, if $\Delta(b)\supseteq\Delta(a)-\{b\}$, then $\{a\}\cup\Delta(a)=\{b\}\cup\Delta(b)$. This implies that, if we take $g\in G$ with $a^g=b$, then $g\in G_{(a\cup\Delta(a))}$ and so $G_a\lneqq G_{(a\cup\Delta(a))}\lneqq G$, which contradicts the primitivity of $(G, \Omega)$. In the latter case, suppose $(\Delta(b)\cup\Delta'(b))\cap\Delta(a)=\Delta(a)-\{b\}$. By Higman [5, (4.2)], $|\Delta(b)\cap\Delta(a)|=|\Delta'(b)\cap\Delta(a)|=(k-1)/2$. Since $\Delta(b)\cap\Delta(a)$ is a union of some $G_{ab}$-orbits, we may set $\Delta(b)\cap\Delta(a)=\bigcup_{i=1}^{t}\Delta_i$. Then we have $\Delta'(b)\cap\Delta(a)=\bigcup_{i=1}^{t}\Delta'_i$, where $\Delta'_i=\{b^{h^{-1}}|h\in G_a, b^h\in\Delta_i\}$, the paired orbit of $\Delta_i$, because for all $i$, $1\leqq i\leqq t$, $\Delta'_i$ is contained in $\Delta'(b)$ and $\Delta(a)$ by definition, $\Delta_i\neq\Delta_j$ implies $\Delta'_i\neq\Delta'_j$, and $|\Delta(b)\cap\Delta(a)|=|\Delta'(b)\cap\Delta(a)|$. Thus $\Delta(a)=\{b\}\cup(\Delta_1\cup\Delta'_1)\cup\cdots\cup(\Delta_t\cup\Delta'_t)$ and the transitive permutation group $(G_a/G_{a\cup\Delta(a)}, \Delta(a))$ has no nontrivial self-paired orbit and so $|G_a/G_{a\cup\Delta(a)}|$ is odd by Wielandt [7, Theorem 16.5]. This contradicts the assumption (**). Therefore, in both cases there exists an element $c$ of some $\Delta_i$ such that $c\notin\Delta(b)\cup\Delta'(b)$. Let $\Gamma(b)$ be a $G_b$-orbit containing $c$ and set $l=|\Gamma(a)|$. Then $\Gamma\neq\Delta, \Delta'$. By definition $(\Delta'\circ\Delta)(b)\ni c$ and so $(\Delta'\circ\Delta)(b)\supseteq c^{G_b}=\Gamma(b)$, proving (i). $\Gamma(b)\cap\Delta(a)$ contains $c$ and so is a union of some $G_{ab}$-orbits containing $c^{G_{ab}}=\Delta_i$, proving (iii). Since $|G_a : G_{abc}|=|G_a : G_{ab}|\cdot|G_{ab} : G_{abc}|=kk_i$ and $|G_b : G_{abc}|=|G_b : G_{bc}|\cdot|G_{bc} : G_{abc}|=l|G_{bc} : G_{abc}|$, it follows that $l$ is a devisor of $kk_i$.

$\Gamma(b)\cap\Delta(a)\supseteq\Delta_i$ implies $l\geqq k_i$. If $l=k_i$, then $k=|G_{bc}:G_{abc}|=|a^{G_{bc}}|$. Since $G_{bc}$ acts on $\Delta'(b)\cap\Delta'(c)$ containing $a$, we have $\Delta'(b)\cap\Delta'(c)\supseteq a^{G_{bc}}$ and so $\Delta'(b)=\Delta'(c)$, which implies $G_b\lneqq G_{\{\Delta'(b)\}}\lneqq G$ and contradicts the primitivity of $(G,\Omega)$. Thus we have $l>k_i$, proving (ii).

Next we assume that, for $b\in\Delta(a)$, all the $G_{ab}$-orbits on $\Delta(a)$ are self-paired. Since $\Gamma(b)\cap\Delta(a)$ is nonempty and a union of some $G_{ab}$-orbits on $\Delta(a)$, we may set $\Gamma(b)\cap\Delta(a)=\bigcup_{j=1}^{s}\Delta_j$. Then we have $\Gamma'(b)\cap\Delta(a)=\bigcup_{j=1}^{s}\Delta'_j$ as before. On the other hand, by assumption $\Delta'_j=\Delta_j$, $1\leqq j\leqq s$ and so $\Gamma'(b)\cap\Delta(a)=\Gamma(b)\cap\Delta(a)$. Thus $\Gamma'(b)\cap\Gamma(b)$ is nonempty and $\Gamma'(b)=\Gamma(b)$. This completes the proof.

REMARK 1. If $r=2$ and $k>2$, Cameron [2] asserts $2k\leqq l$ (and so $2(k-1)\leqq l$). However, in general $2k_i\not\leqq l$. For example, let $G$ be the Higman-Sims simple group of degree 100 with subdegrees 1, 22, 77. $G_a$ ($\cong$ the Mathieu group $M_{22}$) acts on the orbit of length 77 (the blocks of the associated Steiner system) as rank 3 group with subdgrees 1, 16, 60. Although $16<22|77\cdot16$, $2\cdot16\not\leqq22$.

From (ii) and the last assertion of Theorem, we have immediately

COROLLARY 1. *Let $G$ be a uniprimitive permutation group on $\Omega$, and for $a\in\Omega$ $\Delta(a)$ a $G_a$-orbit with $|\Delta(a)|\geqq2$. Suppose $|\Delta(a)|<|\Gamma(a)|$ for any $G_a$-orbit $\Gamma(a)$ different from $\{a\}$ and $\Delta(a)$. Then $G_a$ does not act regularly on $\Delta(a)$.*

COROLLARY 2. *There exists no uniprimitive permutation group $(G,\Omega)$ such that $G_a$ $(a\in\Omega)$ has only one nontrivial self-paired orbit $\Delta(a)$ and for $b\in\Delta(a)$, all the $G_{ab}$-orbits on $\Delta(a)$ are self-paired.*

REMARK 2. The simple unitary group $PSU(3,3^2)$ has a representation as a primitive group $G$ of rank 4 such that $G_a\cong PSL(3,2)$ and the subdegrees are 1, 21, 7, 7, and the $G_a$-orbits of lenth 7 are paired. However, $G_a$ on the $G_a$-orbit of length 21 has subdegrees 1, 2, 2, 4, 4, 8 and the orbits of length 4 are paired.

Incidentally we add

PROPOSITION. *Let $(G,\Omega)$ be a transitive permutation group and for $a\in\Omega$ let $\Delta(a)$ and $\Gamma(a)$ be $G_a$-orbits different from $\{a\}$. Let $m$ be the number of $G_a$-orbits contained in $(\Delta'\circ\Gamma)(a)$ and for some $b\in\Delta(a)$ let $t$ be the number of $G_{ab}$-orbits on $\Gamma(a)$. Then we have*

(i) $1\leqq m\leqq t-1$ *if* $\Gamma=\Delta$ (i.e., *$G_a$ acts on $\Delta(a)$ as a group of rank $t$*) *and $|\Gamma(a)|\geqq2$. In particular, if $G_a$ acts doubly transitively on $\Delta(a)$, then $(\Delta'\circ\Delta)(a)$ is a self-paired $G_a$-orbit.*

(ii) $1 \leqq m \leqq t$ *if* $\Gamma \neq \Delta$. *In particular, if* $|\Delta(a)|$ *and* $|\Gamma(a)|$ *are relatively prime and* $|\Delta(a)|$ *or* $|\Gamma(a)| \geqq 2$, *then* $(\Delta \circ \Gamma)(a)$ *is a single* $G_a$*-orbit.*

PROOF. Let $\Gamma_0(a)=\{a\}$, $\Gamma_1(a)=\Delta(a)$, $\Gamma_2(a)=\Gamma(a)$, $\Gamma_3(a)$, $\cdots$ be the set of $G_a$-orbits on $\Omega$. Following Higman [5] we set $\mu_{ij}^{(\alpha)}=|\Gamma_\alpha(c) \cap \Gamma_i(a)|$ for $c \in \Gamma_j(a)$. Then, by definition $(\Delta' \circ \Gamma)(a)= \bigcup_{i \neq 0, \mu_{1'i}^{(2')} \neq 0} \Gamma_i(a)$, where $\Gamma_{j'}(a)=\Gamma'_j(a)$. By [5, (4.2)], $\mu_{1'i}^{(2')} \neq 0$ if and only if $\mu_{i1'}^{(2)} \neq 0$, and so $(\Delta' \circ \Gamma)(a)= \bigcup_{i \neq 0, \mu_{i1'}^{(2)} \neq 0} \Gamma_i(a)$. Set $(\Delta' \circ \Gamma)(a)= \bigcup_{j=1}^{m} \Gamma_{i_j}(a)$. Then, for all $j$, $1 \leqq j \leqq m$, $\Gamma(a) \cap \Gamma_{i_j}(b)$ is nonempty and a union of some $G_{ab}$-orbits on $\Gamma(a)$. Therefor we have $m \leqq$(the number of $G_{ab}$-orbits on $\Gamma(a))=t$. In particular, if $\Gamma=\Delta$, then $b \in \Gamma(a)$ and $m \leqq$(the number of $G_{ab}$-orbits on $\Gamma(a)$ different from $\{b\})=t-1$. Since $\sum_i \mu_{i1'}^{(2)}=|\Gamma(a)|$ and $\Gamma_{01'}^{(2)}=1$ or 0 according as $\Gamma=\Delta$ or $\Gamma \neq \Delta$, $\mu_{i1'}^{(0)} \neq 0$ for some $i \neq 0$ (in case $\Gamma=\Delta$, $|\Gamma(a)| \geqq 2$ by assumption) and so $m \geqq 1$. In general, since $\mu_{1'i}^{(1')}=\mu_{1'i'}^{(1')}$, if $(\Delta' \circ \Delta)(a)$ contains a $G_a$-orbit $\Gamma_i(a)$, then it does also the paired orbit $\Gamma'_i(a)$, so the 'in particular' part of (i) is obvious. If $|\Delta(a)|$ and $|\Gamma(a)|$ are relatively prime and $|\Delta(a)|$ or $|\Gamma(a)| \geqq 2$, then $\Delta \neq \Gamma$ and by [7, theorem 17.3] $G_{ab}$ is transitive on $\Gamma(a)$, i.e., $t=1$, hence $m=1$, i.e., $(\Delta' \circ \Gamma)(a)$ is a $G_a$-orbit. Therefore, by replacing $\Delta'$ by $\Delta$, $(\Delta \circ \Gamma)(a)$ is a $G_a$-orbit.

Theorem is a little available in dealing with primitive extensions of rank 3 of some permutation groups. Here we say a permutation group $(G, \Omega)$ is a primitive extension of rank 3 of a transitive permutation group $(H, \Delta)$ if $(G, \Omega)$ is primitive and has rank 3 and there exists an orbit $\Delta(a)$ of a stabilizer $G_a$, $a \in \Omega$, such that $G_a$ is faithful on $\Delta(a)$ and $(G_a, \Delta(a))$ is isomorphic to $(H, \Delta)$. For example, the following simple groups have no primitive extensions of rank 3 (Here, for a group $H$ and its subgroup $K$, "$H>K$" denotes the representation of $H$ on $K$).

1) $PSU(3, 5^2)>A_7$
2) The Janko's simple group of order $175560>PSL(2, 11)$
3) Mclaughlin's simple group$>PSU(4, 3^2)$
4) Higman-Sims simple group$>M_{22}$

Indeed, assume that each one of the above groups has a primitive rank 3 extension $(G, \Omega)$ and let $\{a\}$, $\Delta(a)$, $\Gamma(a)$ be the $G_a$-orbits with $|\Delta(a)|=k$, $|\Gamma(a)|=l$, and set $\lambda=|\Delta(a) \cap \Delta(b)|$ for $b \in \Delta(a)$. As to 1), by Theorem, $7<l|50 \cdot 7$ with $\lambda=0$ or 42, or $42<l|50 \cdot 42$ with $\lambda=0$ or 7. However, by Higman [4, Lemma 7] we have $l=30$, 45 or 48 if $l<50$, and $l=50 \cdot 49/2$ if $l \geqq 50$ and $\lambda=0,7$ or 42. These are inconsistent. As to 3), take $c \in \Gamma(a)$. $G_{ac}$ is contained in a maximal subgroup of $G_a$ and by Finkelstein [3, Theorem

1] we have $l \geqq k=275$. By [4, Lemma 7] and Theorem, the case $l=330$, $\lambda=112$ remains. However, by [3] $G_a$ has no maximal subgroup whose index is a divisor of 330. Likewise, in another cases we have a contradiction. Here we use a table of parameters of possible rank 3 permutation groups made by Dr. H. Enomoto on the basis of [4, Lemma 7] with the help of a computer, and the author thanks him.

## References

[1] P. J. CAMERON: Proofs of some theorems of W. A. Manning, Bull. London Math. Soc. 1 (1969), 349–352.

[2] P. J. CAMERON: Permutation groups with multiply transitive suborbits, Proc. London Math. Soc. (3), 25 (1972), 427–440.

[3] L. FINKELSTEIN: The maximal subgroups of Conway's group $C_3$ and McLaughlin's group, J. Alg. 25 (1973), 58–89.

[4] D. G. HIGMAN: Finite permutation groups of rank 3, Math. Z. 86 (1964), 145–156.

[5] D. G. HIGMAN: Intersection matrices for finite permutation groups, J. Alg. 6 (1967), 22–42.

[6] W. A. MANNING: A theorem concerning simply transitive primitive groups, Bull. Amer. Math. Soc. 35 (1929), 330–332.

[7] H. WIELANDT: Finite permutation groups, Academic Press, New York and London, 1964.

Mathematics, Faculty of commerce,
Hitotsubashi University

Department of Mathematics,
Hokkaido University