On infinitesimal projective transformations satisfying the certain conditions

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§ 1. Introduction.

We consider the following problem

**PROBLEM.** Let $M$ be a compact Riemannian manifold with positive constant scalar curvature. If $M$ admits a nonisometric infinitesimal projective transformation, then is $M$ a space of positive constant curvature?

For this problem, the following results are known.

**THEOREM A.** Let $M$ be a complete Riemannian manifold with parallel Ricci tensor. If $M$ admits nonaffine infinitesimal projective transformations, then $M$ is a space of positive constant curvature. [1].

**THEOREM B.** Let $M$ be a compact Riemannian manifold with constant scalar curvature $K$. If the scalar curvature is nonpositive, then an infinitesimal projective transformation is a motion. [2].

**THEOREM C.** Let $M$ be a compact Riemannian manifold satisfying a condition $\nabla_{k}K_{ji}-\nabla_{j}K_{ki}=0$, $(K \neq 0)$, where $\nabla_{k}, K_{ji}$ denote a covariant derivative and Ricci tensor, respectively. The projective Killing vector $v^{h}$ can be decomposed uniquely as follows,

$$v^{h} = w^{h} + q^{h},$$

where $w^{h}$ and $q^{h}$ are Killing vector and gradient projective Killing vector, respectively. [2].

**THEOREM D.** Let $M$ be a compact Riemannian manifold satisfying a condition $\nabla_{k}K_{ji}-\nabla_{j}K_{ki}=0$, $(K \neq 0)$. If $M$ admits nonisometric infinitesimal projective transformations, then $M$ is a space of positive constant curvature. [2].

The purpose of this paper is to prove the following theorems

**THEOREM 1.** Let $M$ be a complete, connected and simply connected Riemannian manifold with positive constant scalar curvature. If a projective Killing vector $v^{h}$ is decomposable as follows,
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\[ v^h = w^h - \frac{n(n-1)}{2K} f^h, \]

where \( w^h \) and \( \frac{n(n-1)}{2K} f^h \) are a Killing vector and a non-zero gradient projective Killing vector, respectively, then \( M \) is isometric to a sphere of radius \( \sqrt{\frac{n(n-1)}{K}} \).

**Theorem 2.** Let \( M \) be a compact Riemannian manifold with constant scalar curvature and let \( v^h \) be a projective Killing vector. Put \( f = \nabla_i v^i / (n + 1) \). Then the following conditions are equivalent.

1. \( w^h = v^h + \frac{n(n-1)}{2K} f^h \) is a Killing vector,
2. \( Z_{kji}^h f^k = 0 \), where \( Z_{kji}^h = K_{kji}^h + \frac{K}{n(n-1)} (\delta^h_j g_{ki} - \delta^h_k g_{ji}) \), and \( K_{kji}^h \) denotes the Riemannian curvature tensor,
3. \( G_{ji} f^j = 0 \), where \( G_{ji} = K_{ji} - \frac{K}{n} g_{ji} \).

A vector field \( v^h \) is called an infinitesimal projective transformation or a projective Killing vector if it satisfies

\[ \mathfrak{L}_{\{h\}_j} = \nabla_j \nabla_i v^h + K_{kji}^h v^k = \delta^h_\iota \varphi^i + \delta^h_i \varphi_\jota, \]

where \( \mathfrak{L}, \{h\}_j, \varphi^i \) denote Lie derivation with respect to \( v^h \), Christoffel's symbol and associated vector, respectively. From these equations, we get following results

\[ \mathfrak{L} K_{kji}^h = -\delta^h_k \nabla_j \varphi^i + \delta^h_j \nabla_k \varphi^i, \]
\[ \mathfrak{L} K_{ji} = -(n-1) \nabla_j \varphi^i, \]
\[ \nabla^i \nabla_i v^j + K_{ji} v^i = 2 \varphi^j, \]
\[ \nabla_j (\nabla_i v^i) = (n + 1) \varphi^j. \]

We have \( f_j = \varphi_j \), where \( f_j \) means \( \nabla_j f \), therefore \( \varphi_j \) is a gradient vector and in the following discussions, we use \( f_j \) instead of \( \varphi_j \).

**§ 2. Proof of Theorem 1.**

**Lemma 1.** Let \( M \) be a complete, connected and simply connected Riemannian manifold of dimension \( n \). In order that \( M \) admits a nontrivial solution \( \phi \) for the system of differential equations

\[ \nabla^i \nabla_i \phi^i + K (2 \phi_k g_{ji} + \phi_j g_{ik} + \phi_i g_{kj}) = 0, \quad K > 0, \quad \phi^i = \nabla_i \phi, \]
it is necessary and sufficient that $M$ be isometric with a sphere $S^n$ of radius $\frac{1}{\sqrt{K}}$ in Euclidean $(n+1)$-space.

For this Lemma, see [3].

**Lemma 2.** If $v_h = w_h - \frac{n(n-1)}{2K} f_h$, then we have
\[
\nabla_h \nabla_i f_i + \frac{K}{n(n-1)} (2f_h g_{ji} + f_j g_{hi} + f_i g_{hf}) = 0.
\]

**Proof.** Substituting $v_h$ into (1.1), since $w_h$ is the Killing vector, we obtain
\[
(2.1) \quad \nabla_i \nabla_j f_i + K g_{ij} f^k = -\frac{2K}{n(n-1)} (g_{ij} f_i + g_{ji} f_j).
\]

Since $\nabla_i f_h = \nabla_h f_i$, we have
\[
0 = \nabla_i \nabla_j f_i - \nabla_j \nabla_i f_i
\]
\[
= -K g_{ij} f^k - \frac{2K}{n(n-1)} (g_{ij} f_i + g_{ji} f_j) + K g_{ij} f^k
\]
\[
+ \frac{2K}{n(n-1)} (g_{ji} f_j + g_{ij} f_i)
\]
\[
= -2K g_{ij} f^k - \frac{2K}{n(n-1)} (g_{ij} f_i - g_{ji} f_j).
\]

Substituting this result into (2.1), we get
\[
\nabla_i \nabla_j f_i + \frac{K}{n(n-1)} (2f_i g_{ji} + f_j g_{hi} + f_i g_{hf}) = 0.
\]

From **Lemma 1**, and Lemma 2, we have **Theorem 1**.

**§ 3. Proof of Theorem 2.**

In this section we assume $M$ is compact and the scalar curvature is constant.

**Lemma 3.** If $w^k = v^k + \frac{n(n-1)}{2K} f^k$ is a Killing vector, then we have $Z_{kji} f^k = 0$.

This is obvious from the proof of **Theorem 1**.

**Lemma 4.** If $Z_{kji} f^k = 0$, then we obtain $G_{ji} f^j = 0$.

This proof is trivial.

**Lemma 5.** There is the following equation,
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\[(n-1)\Delta f + 2K \Delta f + 2K_{ji} \nabla^{f} f^{i} = 0,\]

where \(\Delta\) means \(g^{ji} \nabla_{j} \nabla_{i}\).

For this Lemma, see [2].

Lemma 6. If \(G_{ji} f^{j} = 0\), then we have \(\Delta f = -\frac{2(n+1)}{n(n-1)} K f\).

Proof is the same as that in page 266, [2].

Lemma 7. A necessary and sufficient condition for a vector field \(w^{i}\) in \(M\) to be a Killing vector is \(\nabla_{i} w^{i} = 0\) and \(\nabla^{j} \nabla_{j} w^{i} + K_{i}^{i} w^{i} = 0\).

For this Lemma, see [4].

Lemma 8. If \(G_{ji} f^{j} = 0\), then we get \(v^{i} = w^{i} - \frac{n(n-1)}{2K} f^{i}\).

Proof. If we put \(w^{i} = v^{i} + \frac{n(n-1)}{2K} f^{i}\), then we have

\[\nabla^{i} w_{i} = \nabla^{i} v_{i} + \frac{n(n-1)}{2K} \Delta f = (n+1)f - (n+1)f = 0,\]

\[\nabla^{j} \nabla_{j} w^{i} + K_{i}^{j} w_{j} = \nabla^{j} \nabla_{j} v^{i} + K_{i}^{j} v_{j} + \frac{n(n-1)}{2K} \{\nabla^{j} \nabla_{j} f^{i} + K_{i}^{j} f_{j}\} = 2f_{i} + \frac{n(n-1)}{2K} \left\{-\frac{2(n+1)}{n(n-1)} K f_{i} + \frac{2K}{n} f_{i}\right\} = 0.\]

Therefore \(w_{i}\) is a Killing vector from the Lemma 7. Consequently we arrive at the complete proof of Theorem 2 by means of Lemma 3, Lemma 4 and Lemma 8.

Bibliography