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On the Hall-Higman and Shult theorems

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The purpose of this paper is to improve the proof of the wellknown Hall-Higman and Shult theorems. See also [3, Satz 5. 17. 13] and [5] for alternate proofs of these theorems.

THEOREM 1. ([4, Theorem 3. 1] and [2, Theorem B]). *Assume that the following hold*:

(a) *$G=PQ$ is a finite group, $Q=O_{p'}(G)$, $P=\langle x\rangle$ is a cyclic Sylow p-subgroup of G of order $p^{n}\neq 1$, and $O_{p}(G)=1$;*

(b) *k is a field of characteristic r and V is a faithful kG-module;*

(c) *V_{P} contains no kP-submodule isomorphic to kP, where V_{P} denotes the module V viewed as a kP-module.*

Let d be the degree of the minimal polynomial of x on V. Then there is a prime $q\neq p$ such that a Sylow q-subgroup of G is nonabelian and either

(1) *$q=2$, p is a Fermat prime, and $p^{n}-p^{n-1}\leq d<p^{n}$, or*

(2) *$p=2$, q is a Mersenne prime $<2^{n}$, and $2^{n}q/(q+1)\leq d<2^{n}$.*

We shall prove this theorem. By the induction argument ([1, Theorem 11. 1. 4] and [4, pp. 703-705]), the proof is reducted to the case where the following hypothesis is satisfied.

HYPOTHESIS 1. (1) The assumption of Theorem 1 holds;

(2) Q is an extraspecial q-group of order q^{2t+1}, q a prime, and P acts irreducibly on Q/Q' and trivially on Q';

(3) k is algebraic closed and Q acts irreducibly on V.

We shall prove the following theorem which is the main object in this paper. What it implies Theorem 1 is shown by an easy calculation.

THEOREM 2. *Assume that Hypothesis 1 holds. The V_{P} is isomorphic to the quotient module of a free kP-module by an irreducible submodule.*

To prove this theorem, we need two lemmas which are both wellknown.

LEMMA 3. *Let $G=PQ$ be a Frobenius group whose kernel Q is an elementary abelian minimal normal subgroup of G and complement P is*

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a cyclic p-group. Let k be an algebraic closed field and let V be a kG-module such that $C_V(Q)=0$. *Then* V_P *is a free kP-module.*

This lemma is proved by the same way as the proofs of the elementary abelian cases of Hall-Higman and Shult theorems. See also [1, Theorem 3.4.3].

LEMMA 4. *Let Q be an extraspecial q-group of order* q^{2t+1}, *q a prime, and let f be an automorphism of Q which acts irreducibly on* Q/Q' *and trivially on* Q'. *Then* $|f|\,\big|\,q^t+1$.

This lemma is proved in [1, Lemma 11.2.5]. See also [1, Lemma 5.6.3 and Theorem 5.6.5].

We can now prove Theorem 2.

PROOF OF THEOREM 2. Set $E=End_k(V)$. For $\theta\in E$ and $g\in G$, define $\theta^g\in E$ be $\theta^g(v)=g^{-1}\theta(gv)$ for $v\in V$, so that E is a kG-module. The irreducibility of Q on V yields that elements of $Z(G)=Z(Q)$ acts on V as a scalar transformation. Thus $Q'=Z(Q)$ acts trivially on E, and so E is regarded as a $k\bar{G}$-module, where $\bar{G}=G/Q'$. Since V_Q is an irreducible kQ-module and k is a splitting field for Q, we have that $C_E(\bar{Q})=End_{kQ}(V)=k$. Since $\bar{Q}=Q/Q'$ is an elementary abelian minimal normal subgroup of $\bar{G}$, it follows from Lemma 3 that $E_P=C_E(Q)\oplus[E,\bar{Q}]\cong k\oplus e\cdot kP$ for some integer e, where $e\cdot kP$ is the direct sum of e copies of kP. It is wellknown that $\dim V=q^t$. See [1, Theorem 5.5.5]. Thus $q^{2t}=\dim E=1+eq^n$.

Suppose first that $p=r=char(k)$. Let $V_1,\cdots,V_m$ be indecomposable kP-submodules of V_P such that $V_P=\Sigma_i V_i$. Set $E_{ij}=\mathrm{Hom}_k(V_i,V_j)$ for each i,j. Then $E_P=\sum_{ij}E_{ij}$. Since kP is indecomposable and $E_P\simeq k\oplus e\cdot kP$, there is uniquely (i_0,j_0) such that $E_{ij}\cong e_{ij}\cdot kP$ if $(i,j)\neq(i_0,j_0)$ and $E_{ij}\cong k\oplus e_{ij}\cdot kP$ if $(i,j)=(i_0,j_0)$. Let $d_i=\dim_k V_i$ for each i. Then $1\leq d_i\leq p^n$ and

$$d_i d_j=\dim E_{ij}\equiv\begin{cases}0 & \text{if } (i,j)\neq(i_0,j_0)\\ 1 & \text{if } (i,j)=(i_0,j_0)\end{cases}$$

modulo p^n. Thus $i_0=j_0$ and $d_i\equiv 0 \pmod{p^n}$ for $i\neq i_0$. We may assume that $i_0=j_0=1$. Hence $d_1^2\equiv 1 \pmod{p^n}$ and $d_i=p^n$ for $i\geq 2$. By Lemma 4, $p^n|q^t+1$. Thus $d_1\equiv\dim V=q^t\equiv -1 \pmod{p^n}$, and so $d_1=p^n-1$, whence $V_1\cong kP/Z(kP)$. Hence V_P is isomorphic to the quotient module of $m\cdot kP$ by an irreducible submodule.

Suppose next that $r\neq p$. Let ω be a primitive p^n-th root of unity in k. Let V_i be an irreducible kP-module such that $(x-\omega^i)V_i=0$. Such V_i is uniquely determined up to isomorphism for each i, $0\leq i<p^n$, and $V_{j+p^n}\cong V_j$ for each j. There are d_i, $0\leq i<p^n$, such that $V_P\cong\Sigma_i d_i V_i$. Set $E_{ij}=$

$\mathrm{Hom}_k(V_i, V_j)$. Then $E_{ij} \cong V_{i-j}$, and so $E_P \cong \oplus \Sigma d_i d_j V_{i-j} \cong \oplus \Sigma d_i d_{i-j} V_j$, where i, j run over 0 to p^n-1. Since $E_P \cong k \oplus e \cdot kP$ and $kP \cong \oplus \Sigma V_i$,

$$\Sigma_i d_i d_{i-j} = \begin{cases} 1+e & \text{if } j=0 \\ e & \text{if } 0<j<p^n, \end{cases}$$

where i runs 0 to p^n-1. This implies that there is i_0 such that $d_i = m \pm \delta_{i,i_0}$ for some m. See the proof of [3, Satz 5. 17. 13]. As $p^n | q^t+1$ by Lemma 4, $q^t = (p^n-1)m + d_{i_0} \equiv -1 \pmod{p^n}$. Thus $d_{i_0} = m-1$. Hence $V_P \oplus V_{i_0} \cong \oplus \Sigma m V_i = m \cdot kP$, proving the theorem.

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