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On well posedness of mixed problems for Maxwell's equations II

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§ 0. Introduction and main result

The purpose of this paper is to give an extension of the result in a preceding paper [5] to the case where boundary conditions are not necessarily real.

Let us consider the mixed problem for the system P of Maxwell's equations:

$$(P, B)\qquad \begin{cases} P\begin{bmatrix} E \\ H \end{bmatrix} = f & \text{in } (0, \infty)\times G, \\ B\begin{bmatrix} E \\ H \end{bmatrix} = 0 & \text{on } (0, \infty)\times \partial G, \\ E(0, x) = H(0, x) = 0 & \text{for } x\in G, \end{cases}$$

where

$$(0.1)\qquad P\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) = \frac{\partial}{\partial t} + \begin{bmatrix} 0 & -\mathrm{curl} \\ \mathrm{curl} & 0 \end{bmatrix}$$

which will be often denoted by $\frac{\partial}{\partial t} + \sum_{j=1}^{3} A_j \frac{\partial}{\partial x_j}$, G is an open subset of $\boldsymbol{R}^3$ with C^∞ boundary ∂G and $B(t, x)$ is a C^∞ complex 2×6 matrix function defined on $\boldsymbol{R}^1\times\partial G$ which is of rank two everywhere and is constant for $|t|+|x|$ sufficiently large. It is assumed, as in [5], that the problem (P, B) is *reflexive*, i.e., the kernel of $B(t, x)$ contains that of the boundary matrix $A_\nu(x) = \sum_{j=1}^{3} \nu_j(x) A_j$ at each $(t, x)\in\boldsymbol{R}^1\times\partial G$, where $\nu = {}^t(\nu_1, \nu_2, \nu_3)$ is the inner unit normal to ∂G.

When B is real we proved in [5] the following: If the frozen problem $(P, B)_{(t^0, x^0)}$ at an arbitrary boundary point $(t^0, x^0)\in\boldsymbol{R}^1\times\partial G$ (by this we mean the constant coefficients problem (P, B) with B replaced by the constant matrix $B(t^0, x^0)$ and G by the half space $\{x\in\boldsymbol{R}^3;\ \nu(x^0)\cdot x>0\}$) satisfies Kreiss' condition (or the uniform Lopatinskii condition), then the kernel of $B(t^0, x^0)$

is *maximally negative* for $A_\nu(x^0)$, i. e., it is maximally nonpositive for $A_\nu(x^0)$ which is negative definite over the kernel of $B(t^0, x^0)$.

In the present article we shall show that the above fact is also true for the case where B is not necessarily real in the following sense:

THEOREM 1. *Suppose that (P, B) is reflexive and that the frozen problem $(P, B)_{(t^0,x^0)}$ satisfies Kreiss' condition for a point $(t^0, x^0)\in \boldsymbol{R}^1\times\partial G$. Then there exist neighborhoods $U(t^0)$, $U(x^0)$ in $\boldsymbol{R}^1$, $\bar{G}$ respectively and a C^∞ nonsingular 6×6 matrix function $T(t, x)$ defined on $U(t^0)\times U(x^0)$ such that $T^{-1}(t, x)\,A_j T(t, x)$ is hermitian for each $j=1, 2, 3$ and $(t, x)\in U(t^0)\times U(x^0)$ and that the kernel of $B(t, x)\,T(t, x)$ is maximally negative for $T^{-1}(t, x)$ $A_\nu(x)\,T(t, x)$ at each $(t, x)\in U(t^0)\times(\partial G\cap U(x^0))$.*

Since the curl operator is invariant under rotations, the proof of Theorem 1 may be reduced to that of the following special case.

THEOREM 2. *Suppose that B is constant and G is the half space*

$$G_1=\left\{x=(x_1, x_2, x_3)\in\boldsymbol{R}^3\,;\; x_1>0\right\}.$$

Then the statement of Theorem 1 is valid, where $T(t, x)$ is taken to be constant.

In mixed problems for hyperbolic systems, the existence of local symmetrizers such as the matrix $T(t, x)$ in Theorem 1 is very useful even if the system is symmetric, since it yields readily the existence and uniqueness of the solution and the finiteness of the speed of propagation. (See for instance Lax and Phillips [6], Courant and Hilbert [1], and Friedrichs and Lax [3]).

It should be pointed out that, for Maxwell's equations, Theorem 1 is stronger than the result of Majda and Osher [7] who showed for a class of hyperbolic systems the following: If a mixed problem satisfies the hypotheses of Theorem 1 at each boundary point then it is L^2-well posed and an analogue to the main estimate of Kreiss [4] holds (see also (ii) and (iii) in the Appendix below).

In proving Theorem 2 we first derive in section 2 two inequalities from Kreiss' condition for (P, B) (see (2. 1) and (2. 2) below) and then using a method developed in [5] we show in section 3 that these inequalities assure the existence of such a constant matrix T as described in Theorem 1. We assume in sections 1, 2 and 3 that $G=G_1$ and B is constant. In section 4 we prove Theorem 1 and finally we describe in the Appendix the connection between our result and the one in [7].

§ 1. Notations

In order to prove Theorem 2 it is convenient to diagonalize A_1 (which is now the boundary matrix) as usual. Note that the matrices defined by (0.1) may be written as

$$A_j=\begin{bmatrix} 0 & M_j \\ {}^tM_j & 0 \end{bmatrix}, \qquad j=1,2,3, \tag{1.1}$$

where

$$M_1=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},\ M_2=\begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ M_3=\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence we shall make the same change of the dependent variables as in [5] by

$$\begin{bmatrix} E \\ H \end{bmatrix}=T_1u$$

with the orthogonal 6×6 matrix

$$T_1=\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & \sqrt{2} \\ I & -K & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 \\ -K & I & 0 & 0 \end{pmatrix}, \tag{1.2}$$

where

$$I=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad K=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Then (P, B) is transformed into the following problem:

$$\begin{cases} \dfrac{\partial}{\partial t}u+\sum_{j=1}^{3}({}^tT_1A_jT_1)\dfrac{\partial}{\partial x_j}u={}^tT_1f & \text{in } (0,\infty)\times G_1, \\ BT_1u=0 & \text{on } (0,\infty)\times\partial G_1, \\ u(0,x)=0 & \text{for } x\in G_1, \end{cases}$$

and it follows from [5], (3.1) that

$$ {}^tT_1A_1T_1=\begin{bmatrix} I & & \\ & -I & \\ & & 0 \end{bmatrix},\quad {}^tT_1A_2T_1=\frac{1}{\sqrt{2}}\begin{bmatrix} \multicolumn{2}{c}{0} & -K \\ & & KJ \\ -{}^tK & {}^t(KJ) & 0 \end{bmatrix}, \tag{1.3}$$

$$ {}^tT_1A_3T_1=\frac{1}{\sqrt{2}}\begin{bmatrix} \multicolumn{2}{c}{0} & -I \\ & & J \\ -I & J & 0 \end{bmatrix},$$

where

$$J=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Now the reflexiveness of (P, B) means that the right 2×2 block of BT_1 is the zero matrix, because of the form of ${}^tT_1A_1T_1$. Furthermore if (P, B) satisfies Hersh's condition which is implied by Kreiss' condition then the left 2×2 block of BT_1 is nonsingular (see [5], Lemma 2. 10). Thus we may assume in what follows that B is of the form:

(1. 4) $$B=[I, S, 0]\,{}^tT_1,$$

where S is a 2×2 matrix, so that the boundary condition $B\begin{bmatrix} E \\ H \end{bmatrix}=0$ becomes

(1. 5) $$\begin{bmatrix} E_2+H_3 \\ E_3-H_2 \end{bmatrix}+S\begin{bmatrix} H_2+E_3 \\ H_3-E_2 \end{bmatrix}=0,$$

where we have set $E={}^t(E_1, E_2, E_3)$, $H={}^t(H_1, H_2, H_3)$.

§ 2. Necessary conditions for (P, B) to satisfy Kreiss' condition

In this and the next sections we assume that B is a constant matrix of the form (1. 4). The purpose of the present section is then to prove

PROPOSITION 2. 1. *If (P, B) satisfies Kreiss' condition, the following two inequalities hold*:

(2. 1) $$|\det S|<1.$$

(2. 2) $$D\equiv 2\det(I-S^*S)+|a|^2+|b|^2-|a^2+b^2|>0,$$

where

(2. 3) $$a=\operatorname{tr} S, \quad b=\operatorname{tr}(SK),$$

and I, K are the matrices in (1. 2).

Let us recall that Kreiss' condition for (P, B) means

(2. 4) $$R(\tau, \sigma)\neq 0 \qquad \text{for all } (\tau, \sigma)\in(\bar{C}_-\times \boldsymbol{R}^2)\backslash 0,$$

where $R(\tau, \sigma)$ is a Lopatinskii determinant of (P, B), τ or σ is the covariable of t or (x_2, x_3) respectively, $\boldsymbol{C}_-$ denotes the set of complex numbers with negative imaginary parts and $\bar{C}_-$ its closure. For convenience we shall denote σ by (σ_1, σ_2) which is denoted in [5] by (σ_2, σ_3). Then from Lemma 3. 1 in that paper we have

(2. 5) $$R(\tau, \sigma)=(1-\det S)\tau^2-(1+\det S)\tau\lambda^+(\tau, \sigma)-\Phi(\sigma)/2 \quad \text{for } (\tau, \sigma)\in\boldsymbol{C}_-\times(\boldsymbol{R}^2\backslash 0).$$

Here

$$\Phi(\sigma)=(1-\det S)|\sigma|^2-c(\sigma_1^2-\sigma_2^2)+2d\sigma_1\sigma_2\,,$$

$$(2.6)\qquad c=\mathrm{tr}\,(SJ)\,,\quad d=\mathrm{tr}\,(SKJ)\,,$$

J is the matrix in (1.3) and $\lambda^+(\tau,\sigma)$ is the root of the equation $\tau^2-\lambda^2-|\sigma|^2=0$ in λ with positive imaginary part for $\tau\in\boldsymbol{C}_-$, i.e.,

$$(2.7)\qquad \lambda^+(\tau,\sigma)=i\sqrt{|\sigma|^2-\tau^2}\qquad \text{for } \tau\in\boldsymbol{C}_-,$$

where $\sqrt{\zeta}$ denotes the branch of the square roots of ζ such that $\sqrt{1}=1$. Since $|\sigma|^2-\tau^2\notin(-\infty,0]$ for $\tau\in\boldsymbol{C}_-$, we see that for fixed $\sigma\in\boldsymbol{R}^2\backslash 0$, $\lambda^+(\tau,\sigma)$ is an analytic function of $\tau\in\boldsymbol{C}_-$ and is extended up to $\bar{\boldsymbol{C}}_-$ by continuity, hence so is $R(\tau,\sigma)$.

In what follows we shall denote σ with $\sigma\in\boldsymbol{R}^2$ and $|\sigma|=1$ by ξ, and the unit circle in $\boldsymbol{R}^2$ by Σ.

LEMMA 2.2. *Suppose that* (2.4) *holds. Then for every* $\xi\in\Sigma$ *the function defined by*

$$(2.8)\qquad \psi(\xi)=R(|\xi|,\xi)$$

does not vanish and the polynomial in z *defined by*

$$(2.9)\qquad R_1(z,\xi)=\psi(\xi)\,z^2-i(1+\det S)\,z-\big(1-\det S-\psi(\xi)\big)$$

has all its zeros in the open half plane $\mathrm{Im}\, z>0$, *i.e.*,

$$(2.10)\qquad R_1(z,\xi)\neq 0\qquad \textit{for all } z\in\bar{\boldsymbol{C}}_-\,.$$

PROOF. It suffices to prove (2.10). Let $\xi\in\Sigma$ be fixed arbitrarily. For $\tau\in\bar{\boldsymbol{C}}_-\backslash\{1,-1\}$ we set

$$(2.11)\qquad z=\tau/\sqrt{1-\tau^2}\,,$$

where $\sqrt{\ }$ is the same branch as in (2.7). Then we have from (2.5), (2.7) and (2.9)

$$(1-\tau^2)^{-1}R(\tau,\xi)=R_1(z,\xi)\,,$$

since (2.8) and (2.5) imply

$$(2.12)\qquad \psi(\xi)=1-\det S-\Phi(\xi)/2\,.$$

Therefore to derive (2.10) from (2.4) with $\sigma=\xi$ we must only show that the mapping $\tau\to z$ defined by (2.11) is a surjection from $\bar{\boldsymbol{C}}_-\backslash\{1,-1\}$ onto $\bar{\boldsymbol{C}}_-\backslash\{-i\}$, since $R_1(-i,\xi)=-2\neq 0$ by (2.9).

We claim that the relation

$$(2.13)\qquad \tau/\sqrt{1-\tau^2}=-i\sqrt{\tau^2/(\tau^2-1)}$$

holds for $\tau\in C_-$ and hence for $\tau\in\bar{C}\backslash\{1,-1\}$ by continuity. In fact, both the sides are analytic in C_- and coincide on a half line in it, say, on the negative imaginary line, so that they coincide on C_-. We now see from (2.13) that (2.11) is a mapping from $\bar{C}_-\backslash\{1,-1\}$ into $\bar{C}_-\backslash\{-i\}$. To establish that it is also surjective we note that (2.11) implies

$$\tau^2=(z^2+|z|^4)/|1+z^2|^2.$$

We then observe that C_- is mapped onto $C_-\backslash\{z;\ \mathrm{Re}\ z=0 \text{ and } \mathrm{Im}\ z\leqq-1\}$. It is not hard to see that $(-1,1)$ is done onto the real line. Finally we find from (2.13) that $(1,\infty)$ is mapped onto the interval on the imaginary line such that $\mathrm{Im}\, z<-1$, which completes the proof.

In order to eliminate z from (2.10) we use a method due to Hermite, as in [8].

LEMMA 2.3. *Let $f(z)$ be a quadratic polynomial. Then the equation $f(z)=0$ has all its roots in the open half plane $\mathrm{Im}\, z>0$ if and only if the hermitian 2×2 matrix $K(f)$ defined by*

$$\frac{1}{i}\frac{f(x)\overline{f(y)}-\overline{f(x)}f(y)}{x-y}=(y,1)\,K(f)\begin{pmatrix}x\\1\end{pmatrix},\qquad x,y\in\boldsymbol{R}^1, \tag{2.14}$$

is positive definite.

For the proof see for instance [2], p. 62. Higher degree cases are also treated there. But in our case letting the leading coefficient of $f(z)$ be 1 and factorizing it as $f(z)=(z-\alpha)(z-\beta)$, we find from (2.14) that

$$K(f)=2\,\mathrm{Im}\begin{bmatrix}\alpha+\beta & -\alpha\beta\\ -\alpha\beta & (\bar{\alpha}+\bar{\beta})\,\alpha\beta\end{bmatrix},$$

$$\det K(f)=4(\mathrm{Im}\,\alpha)(\mathrm{Im}\,\beta)\left(|\alpha|^2+|\beta|^2-2\mathrm{Re}\,(\alpha\bar{\beta})\right).$$

Therefore it is not hard to prove Lemma 2.3.

LEMMA 2.4. *Let $\xi\in\Sigma$. If $\phi(\xi)\neq0$ and (2.10) holds, the hermitian 2×2 matrix defined by*

$$H(\xi)=\begin{bmatrix}\mathrm{Re}\left(\phi(\xi)+\overline{\phi(\xi)}\det S\right) & -\mathrm{Im}\left(\phi(\xi)+\overline{\phi(\xi)}\det S\right)\\ -\mathrm{Im}\left(\phi(\xi)+\overline{\phi(\xi)}\det S\right) & 1-|\det S|^2-\mathrm{Re}\left(\phi(\xi)+\overline{\phi(\xi)}\det S\right)\end{bmatrix} \tag{2.15}$$

is positive definite.

PROOF. For convenience set

$$p=-i(1+\det S),\quad q(\xi)=-\left(1-\det S-\phi(\xi)\right); \tag{2.16}$$

We then find that the matrix $K(R_1)(\xi)$ defined by (2. 14) with $f(z)=R_1(z, \xi)$ is as follows:

$$K(R_1)(\xi)=2\,\mathrm{Im}\begin{bmatrix}\psi\bar{p} & \psi\bar{q}\\ \psi\bar{q} & p\bar{q}\end{bmatrix}(\xi).$$

Hence by (2. 15) and (2. 16) we obtain

$$K(R_1)(\xi)=2H(\xi).$$

The assertion now follows from Lemma 2. 3.

In order to derive (2. 1) and (2. 2) by eliminating ξ from the conclusion of Lemma 2. 4 we use the following elementary facts for complex numbers.

LEMMA 2. 5. *Let α, β, γ and δ be complex numbers. Then*:

$$(|\alpha|^2+|\beta|^2)^2-|\alpha^2+\beta^2|^2=4\left(\mathrm{Im}\,(\alpha\bar{\beta})\right)^2. \tag{2. 17}$$

$$\begin{aligned}|\alpha^2+\beta^2|^2-|\gamma^2+\delta^2|^2&=2\,\mathrm{Re}\left((\gamma^2+\delta^2)\,\bar{z}\right)+|z|^2\\&=2\,\mathrm{Re}\left((\alpha^2+\beta^2)\,\bar{z}\right)-|z|^2,\end{aligned} \tag{2. 18}$$

if $z=\alpha^2+\beta^2-\gamma^2-\delta^2$.

$$\max_{(X,Y)\in\Sigma}|\alpha X-\beta Y|^2=\frac{1}{2}(|\alpha|^2+|\beta|^2+|\alpha^2+\beta^2|). \tag{2. 19}$$

PROOF. Since the left side of (2. 17) is equal to

$$2\left(|\alpha|^2|\beta|^2-\mathrm{Re}\,(\alpha^2\bar{\beta}^2)\right)=2\left(|\alpha\bar{\beta}|^2-\mathrm{Re}\,(\alpha\bar{\beta})^2\right),$$

using the identity $|\gamma|^2-\mathrm{Re}\,\gamma^2=2\,(\mathrm{Im}\,\gamma)^2$ we obtain (2. 17).

Next we shall prove (2. 18). Since it follows from the definition of z that

$$\mathrm{Re}\left\{\left(2(\gamma^2+\delta^2)+z\right)\bar{z}\right\}=|\alpha^2+\beta^2|^2-|\gamma^2+\delta^2|^2,$$

we obtain the first equality. The second follows from the relation

$$2(\gamma^2+\delta^2)+z=2(\alpha^2+\beta^2)-z.$$

Finally we shall prove (2. 19). Let $(X, Y)\in\Sigma$ and set $X=\cos\theta$, $Y=\sin\theta$. Then we have

$$\begin{aligned}&|\alpha X-\beta Y|^2\\&\quad=\frac{1}{2}\left\{|\alpha|^2+|\beta|^2+(|\alpha|^2-|\beta|^2)\cos 2\theta-2\left(\mathrm{Re}\,(\alpha\bar{\beta})\right)\sin 2\theta\right\}.\end{aligned}$$

Therefore using the addition formulas for trigonometric functions we find that

$$\max_{(X,Y)\in\Sigma}|\alpha X-\beta Y|^2=\frac{1}{2}\left\{|\alpha|^2+|\beta|^2+\sqrt{(|\alpha|^2-|\beta|^2)^2+4\left(\mathrm{Re}\,(\alpha\bar{\beta})\right)^2}\right\}.$$

Moreover the last term in the brackets is equal to $|\alpha^2+\beta^2|$. In fact, since $|\alpha\bar{\beta}|=|\alpha\beta|$ we have

$$(|\alpha|^2-|\beta|^2)^2+4\left(\mathrm{Re}\,(\alpha\bar{\beta})\right)^2=(|\alpha|^2+|\beta|^2)^2-4\left(\mathrm{Im}\,(\alpha\bar{\beta})\right)^2,$$

the right side of which is equal to $|\alpha^2+\beta^2|^2$ by (2.17). Thus we complete the proof.

LEMMA 2.6. *Let $H(\xi)$ be positive definite for every $\xi\in\Sigma$. Then* (2.1) *and* (2.2) *hold.*

PROOF. Since each of the diagonal components of $H(\xi)$ is positive by hypothesis, we obtain (2.1) by (2.15). Moreover for every $\xi\in\Sigma$ we have $\det H(\xi)>0$, which may be written as

$$2(1-|\det S|^2)\,\mathrm{Re}\left(2\left(\psi(\xi)+\overline{\psi(\xi)}\det S\right)\right)-\left|2\left(\psi(\xi)+\overline{\psi(\xi)}\det S\right)\right|^2>0\,. \tag{2.20}$$

In order to apply (2.19) to (2.20) let us now put

$$X=\mathrm{Re}\,(\xi_1+i\xi_2)^2=\xi_1{}^2-\xi_2{}^2\,,\quad Y=\mathrm{Im}\,(\xi_1+i\xi_2)^2=2\xi_1\xi_2\,.$$

Then from (2.12) and the definition of Φ in (2.5) we have for $\xi\in\Sigma$

$$2\psi(\xi)=1-\det S+cX-dY\,,$$

so that

$$2(\psi+\bar{\psi}\det S)=1-|\det S|^2+(c+\bar{c}\det S)\,X-(d+\bar{d}\det S)\,Y\,.$$

Hence setting

$$\alpha=c+\bar{c}\det S\,,\quad \beta=d+\bar{d}\det S \tag{2.21}$$

we find that (2.20) becomes

$$(1-|\det S|^2)^2-|\alpha X-\beta Y|^2>0\,.$$

Therefore using (2.19) we obtain

$$2(1-|\det S|^2)^2-(|\alpha|^2+|\beta|^2+|\alpha^2+\beta^2|)>0\,, \tag{2.22}$$

since (X, Y) revolves twice on Σ as ξ does once. Thus the proof will be complete if we prove the following

LEMMA 2.7. (2.1) *and* (2.22) *imply* (2.2). *Here α and β are the numbers defined by* (2.21).

PROOF. We first claim that

$$4\det S=a^2+b^2-c^2-d^2\,, \tag{2.23}$$

$$(2.24) \qquad D=2(1+|\det S|^2)-|c|^2-|d|^2-|a^2+b^2| .$$

In fact, the former follows from (2. 3) and (2. 6) (see also (3. 10) below) and the latter from the definition of D in (2. 2) and (3. 6) below.

Now transpose $|\alpha^2+\beta^2|$ in (2. 22), sequare both the resulting sides and apply (2. 17 . Then we obtain

$$(2.25) \qquad 4(1-|\det S|^2)^4-4(1-|\det S|^2)^2(|\alpha|^2+|\beta|^2)+4\left(\operatorname{Im}(\alpha\bar{\beta})\right)^2>0 .$$

Moreover from (2. 21) we have

$$(2.26) \qquad \operatorname{Im}(\alpha\bar{\beta})=(1-|\det S|^2)\operatorname{Im}(c\bar{d}),$$

$$(2.27) \qquad |\alpha|^2+|\beta|^2=(1+|\det S|^2)(|c|^2+|d|^2)+2\operatorname{Re}\left((c^2+d^2)\overline{\det S}\right).$$

The former and (2. 1) imply that (2. 25) is equivalent to

$$(2.28) \qquad 4(1-|\det S|^2)^2-4(|\alpha|^2+|\beta|^2)+4\left(\operatorname{Im}(c\bar{d})\right)^2>0 .$$

Furthermore from (2. 23) and the first half of (2. 18) we have

$$-8\operatorname{Re}\left((c^2+d^2)\overline{\det S}\right)=-|a^2+b^2|^2+|c^2+d^2|^2+16|\det S|^2 .$$

Therefore inserting (2. 27) into (2. 28) and using (2. 17) we obtain

$$D(D+2|a^2+b^2|)>0 ,$$

since (2. 24) implies

$$\begin{aligned}&4(1+|\det S|^2)^2-4(1+|\det S|^2)(|c|^2+|d|^2)+(|c|^2+|d|^2)^2\\ &\quad -|a^2+b^2|^2=D(D+2|a^2+b^2|) .\end{aligned}$$

Thus it suffices to prove

$$(2.29) \qquad D+2|a^2+b^2|>0 .$$

From (2. 22) we have

$$(2.30) \qquad 2(1-|\det S|^2)^2>|\alpha|^2+|\beta|^2 .$$

On the other hand (2. 23) and the second half of (2. 18) imply

$$2\operatorname{Re}\left((c^2+d^2)\overline{\det S}\right)=2\operatorname{Re}\left((a^2+b^2)\overline{\det S}\right)-8|\det S|^2 .$$

Substituting this into (2. 27) and using (2. 24) we find that (2. 30) is equivalent to

$$(1+|\det S|^2)(D+2|a^2+b^2|)>(1+|\det S|^2)|a^2+b^2|+2\operatorname{Re}\left((a^2+b^2)\overline{\det S}\right).$$

Thus we obtain (2.29), since the right side is not less than $|a^2+b^2|(1-|\det S|)^2$. This completes the proof.

Now Proposition 2.1 follows immediately from Lemmas 2.2, 2.4 and 2.6.

§ 3. Construction of symmetrizers for constant coefficients problems

In this section we shall complete the proof of Theorem 2. In doing so an essential role will be played by the following

LEMMA 3.1. *There exists a nonsingular* 6×6 *matrix* T, *such that* $T^{-1}A_jT$ *is hermitian for* $j=1, 2, 3$ *and the kernel of* BT *is maximally negative for* $T^{-1}A_1T$, *if and only if the hermitian matrix defined by*

$$(3.1)\qquad Q(y)=I-S^*S+iy(K+S^*KS)$$

is positive definite for some real number $y\in(-1, 1)$. *In this case, such a matrix* T *may be given by*

$$(3.2)\qquad T=T_1W,\quad W=\begin{bmatrix} W_1 & & \\ & W_2 & \\ & & W_1\end{bmatrix},$$

where

$$W_1=(T_0)^*\begin{bmatrix}\sqrt{1+y} & 0\\ 0 & \sqrt{1-y}\end{bmatrix}T_0,\quad W_2=(T_0)^*\begin{bmatrix}\sqrt{1-y} & 0\\ 0 & \sqrt{1+y}\end{bmatrix}T_0$$

and $T_0=(I+iJ)/\sqrt{2}$. *Here* T_1, I, J *and* K *are the matrices in* (1.2) *or* (1.3). *In particular, the kernel of* B *is maximally negative for* A_1 *if and only if* $Q(0)=I-S^*S$ *is positive definite.*

For the proof see [5], Lemma 3.2, its proof and the end of that of part (*a*) of Theorem.

In order to clarify the relation between the matrix $Q(y)$ and conditions (2.1) and (2.2) we now set

$$(3.3)\qquad \begin{aligned}\det Q(y)&=py^2+qy+r,\\ \operatorname{tr}Q(y)&=my+n,\end{aligned}$$

and represent these coefficients in terms of the matrix S.

LEMMA 3.2.

$$(3.4)\qquad p=-\det(K+S^*KS)=-(1+|\det S|^2)-(|a|^2+|b|^2-|c|^2-|d|^2)/2,$$

where c *and* d *are the quantities defined by* (2.6).

(3. 5) $$q=i\,\mathrm{tr}\left((K+S^*KS)\,\mathrm{adj}\,(I-S^*S)\right)=2\,\mathrm{Im}\,(a\bar{b})\,,$$

where for a square matrix A *we denote* ${}^t(\mathrm{cof}\,A)$ *by* $\mathrm{adj}\,A$.

(3. 6) $$r=\det(I-S^*S)=1+|\det S|^2-(|a|^2+|b|^2+|c|^2+|d|^2)/2\,.$$

(3. 7) $$m=i\,\mathrm{tr}\,(S^*KS)=\mathrm{Im}\,(a\bar{b}-c\bar{d})\,.$$

(3. 8) $$n=\mathrm{tr}\,(I-S^*S)=2-(|a|^2+|b|^2+|c|^2+|d|^2)/2\,.$$

PROOF. The first equalites of (3. 7) and (3. 8) follow immediately from (3. 1) and (3. 3), and those of (3. 4), (3. 5) and (3. 6) may be obtained by using also the following identity for general 2×2 matrices A and B:

(3. 9) $$\det(A+B)=\det A+\det B+\mathrm{tr}\,(B\,\mathrm{adj}\,A)\,.$$

We shall now derive the second equalities. For convenience set $S=[s_{ij}\,;\ i\downarrow 1, 2,\ j\rightarrow 1, 2]$. Then (2. 3) and (2. 6) imply that

$$a=s_{11}+s_{22}\,,\qquad b=s_{21}-s_{12}\,,$$
$$d=s_{11}-s_{22}\,,\qquad c=s_{21}+s_{12}\,,$$

i. e.,

(3. 10) $$S=\begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix}=\frac{1}{2}\begin{bmatrix} a+d & c-b \\ c+b & a-d \end{bmatrix}.$$

Hence we have

(3. 11) $$S^*S=\begin{bmatrix} |s_{11}|^2+|s_{21}|^2 & \bar{s}_{11}s_{12}+\bar{s}_{21}s_{22} \\ s_{11}\bar{s}_{12}+s_{21}\bar{s}_{22} & |s_{12}|^2+|s_{22}|^2 \end{bmatrix}$$

and

(3. 12) $$S^*KS=\begin{bmatrix} -2i\,\mathrm{Im}\,(s_{11}\bar{s}_{21}) & \bar{s}_{11}s_{22}-\bar{s}_{21}s_{12} \\ -s_{11}\bar{s}_{22}+s_{21}\bar{s}_{12} & -2i\,\mathrm{Im}\,(s_{12}\bar{s}_{22}) \end{bmatrix}.$$

From (3. 10) and (3. 11) we obtain

(3. 13) $$\mathrm{tr}\,(S^*S)=(|a|^2+|b|^2+|c|^2+|d|^2)/2\,,$$

which implies (3. 8). Similarly (3. 10) and (3. 12) give (3. 7). Since (3. 9) yields

$$\det(I-S^*S)=\det I+\det(S^*S)-\mathrm{tr}\,(S^*S)\,,$$

we obtain (3. 6) by (3. 13). To derive (3. 4) we note that $\mathrm{adj}\,K=-K$ and hence (3. 9) yields

$$\det(K+S^*KS)=\det K+\det(S^*KS)-\mathrm{tr}\,(S^*KSK).$$

Moreover it follows from (3. 12) that

$$\operatorname{tr}(S^* KSK) = -2 \operatorname{Re}(s_{11}\bar{s}_{22} - s_{21}\bar{s}_{12}) .$$

Therefore using $\det K=1$ and (3. 10) we obtain (3. 4). Finally we shall derive (3. 5). Since

$$\operatorname{adj}(I-S^* S) = I - \operatorname{adj}(S^* S) ,$$

the first half of (3. 5) implies

$$q = i \operatorname{tr}\left(K + S^* KS - K \operatorname{adj}(S^* S) - S^* KS \operatorname{adj}(S^* S)\right) . \tag{3.14}$$

Now (3. 9) with $A=S^*S$ and $B=S^*KS$ gives

$$\operatorname{tr}\left(S^* KS \operatorname{adj}(S^* S)\right) = 0 ,$$

since $\det(I+K)=2$. Moreover from (3. 11) we have

$$\operatorname{adj}(S^* S) = \begin{bmatrix} |s_{12}|^2 + |s_{22}|^2 & -(\bar{s}_{11}s_{12} + \bar{s}_{21}s_{22}) \\ -(s_{11}\bar{s}_{12} + s_{21}\bar{s}_{22}) & |s_{11}|^2 + |s_{21}|^2 \end{bmatrix},$$

which yields

$$\begin{aligned} \operatorname{tr}\left(K \operatorname{adj}(S^* S)\right) &= -2i \operatorname{Im}(s_{11}\bar{s}_{12} + s_{21}\bar{s}_{22}) \\ &= i \operatorname{Im}(a\bar{b} + c\bar{d}) . \end{aligned}$$

Here the second equality is due to (3. 10). Thus we obtain (3. 5) by (3. 7) and (3. 14). The proof is complete.

A sufficient condition for $Q(y)$ to be positive definite for some $y \in (-1, 1)$ will be given by the following two lemmas.

LEMMA 3. 3. *Suppose that* (2. 2) *holds. Then the function* $\det Q(y)$ *is strictly conave and takes on its maximum at* $y = -q/2p \in (-1, 1)$. *Moreover the maximum value is positive.*

PROOF. In view of (3. 3) it suffices to prove the following three inequalities:

$$p < 0 . \tag{3.15}$$

$$|q| < -2p . \tag{3.16}$$

$$q^2 - 4pr > 0 . \tag{3.17}$$

From (3. 4) and (2. 24) we have

$$-2p = D + |a^2 + b^2| + |a|^2 + |b|^2 , \tag{3.18}$$

which implies (3. 15). Next using (2. 17) we have from (3. 5)

(3. 19) $$q^2=(|a|^2+|b|^2)^2-|a^2+b^2|^2.$$

Hence by (3. 18) we obtain

$$(2p)^2-q^2=(D+|a^2+b^2|)(D+|a^2+b^2|+2|a|^2+2|b|^2)+|a^2+b^2|^2.$$

This together with (3. 15) yields (3. 16). Finally we shall derive (3. 17). Since (3. 6) and (2. 24) imply

$$2r=D+|a^2+b^2|-(|a|^2+|b|^2),$$

it follows from (3. 18) that

$$-4pr=(D+|a^2+b^2|)^2-(|a|^2+|b|^2)^2.$$

Therefore from (3. 19) we have

$$q^2-4pr=D(D+2|a^2+b^2|),$$

which implies (3. 17).

LEMMA 3. 4. *Suppose that* (2. 1) *and* (2. 2) *hold. Then* $\mathrm{tr}\, Q(y)>\det Q(y)$ *at* $y=-q/2p$.

PROOF. Note that (2. 2) implies (3. 15). From (3. 3) we have

$$\det Q(-q/2p)=(q^2-4pr)/(-4p),$$
$$\mathrm{tr}\, Q(-q/2p)=(2mq-4np)/(-4p).$$

Moreover, since $2m-q=-2\,\mathrm{Im}\,(c\bar{d})$ and $n-r=1-|\det S|^2$ according to Lemma 3. 2, we have

$$2mq-4np-(q^2-4pr)=2\left\{-2p(1-|\det S|^2)-q\,\mathrm{Im}\,(c\bar{d})\right\}.$$

Therefore in view of (2. 1) and (3. 16) it suffices to prove

(3. 20) $$\left(1-|\det S|^2\right)^2>\left(\mathrm{Im}\,(c\bar{d})\right)^2.$$

Since (2. 17) implies

$$4\left(\mathrm{Im}\,(c\bar{d})\right)^2=(|c|^2+|d|^2)^2-|c^2+d^2|^2$$

and since (2. 2) and (2. 24) yield

$$(|c|^2+|d|^2)^2<4(1+|\det S|^2)^2-4(1+|\det S|^2)|a^2+b^2|+|a^2+b^2|^2,$$

we have

$$\begin{aligned}&4(1-|\det S|^2)^2-4\left(\mathrm{Im}\,(c\bar{d})\right)^2\\&\qquad>4(1+|\det S|^2)|a^2+b^2|+|c^2+d^2|^2-|a^2+b^2|^2-16|\det S|^2.\end{aligned}$$

Moreover the right side except for the first term is equal to $-8\,\mathrm{Re}\,((a^2+b^2)\det\bar{S})$ according to (2.18) and (2.23). Thus we obtain

$$(1-|\det S|^2)^2-\left(\mathrm{Im}\,(c\bar{d})\right)^2>(1+|\det S|^2)|a^2+b^2|-2\,\mathrm{Re}\left((a^2+b^2)\det\bar{S}\right),$$

which yields (3.20). This completes the proof.

Now theorem 2 is a direct consequence of Proposition 2.1, Lemmas 3.1, 3.3 and 3.4.

REMARK. We shall give an interpretation of conditions (2.1) and (2.2). It follows from (3.6) and (3.8) that (2.1) is equivalent to

$$\mathrm{tr}\,(I-S^*S)>\det\,(I-S^*S). \tag{3.21}$$

Now suppose the ratio between a and b is real, i.e., that $\mathrm{Im}\,(a\bar{b})=0$. Then by (2.17) we have $D=2\det\,(I-S^*S)$, so that (2.2) is equivalent to the inequality $\det\,(I-S^*S)>0$. Thus (2.1) and (2.2) imply that the kernel of B is maximally negative for A_1 if $\mathrm{Im}\,(a\bar{b})=0$, according to (3.21) and Lemma 3.1.

§ 4. Construction of local symmetrizers for variable coefficients problems

The purpose of this section is to prove Theorem 1. Let x^0 be an arbitrary point of ∂G. Then there exists a C^∞ orthogonal 3×3 matrix function $\theta(x)$ defined on a neighbourhood $U(x^0)$ in $\bar{G}$ such that

$$\det\theta(x)=1 \qquad \text{for all } x\in U(x^0) \tag{4.1}$$

and

$$^t\theta(x)\nu(x)={}^t(1,0,0) \qquad \text{for all } x\in\partial G\cap U(x^0). \tag{4.2}$$

Using this matrix $\theta(x)$ we shall reduce the frozen problem $(P,B)_{(t,x)}$ to the case where $G=G_1$ and B is essentially of the form (1.4) (see (4.8) and (4.9) below) and then apply the arguments employed in the proof of Theorem 2. To do so we need the following lemma which is a representation of the rotational invariance of the curl operator.

LEMMA 4.1. *Let*

$$\theta(x)=\begin{bmatrix}\theta_{11} & \theta_{12} & \theta_{13}\\ \theta_{21} & \theta_{22} & \theta_{23}\\ \theta_{31} & \theta_{32} & \theta_{33}\end{bmatrix}(x). \tag{4.3}$$

Then the relations

$$^t\begin{bmatrix}\theta(x) & \\ & \theta(x)\end{bmatrix}A_i\begin{bmatrix}\theta(x) & \\ & \theta(x)\end{bmatrix}=\sum_{j=1}^{3}\theta_{ij}(x)\,A_j, \qquad i=1,2,3 \tag{4.4}$$

hold for all $x \in U(x^0)$. *In particular*

$$(4.5)\qquad {}^t\begin{bmatrix}\theta(x) & \\ & \theta(x)\end{bmatrix} A_\nu(x) \begin{bmatrix}\theta(x) & \\ & \theta(x)\end{bmatrix} = A_1 \qquad \text{for all } x \in \partial G \cap U(x^0).$$

PROOF. Since (4.5) is a direct consequence of (4.2), (4.4) and the definition of A_ν, we shall prove only (4.4). In view of (1.1) it suffices to derive

$$(4.6)\qquad {}^t\theta M_i \theta = \sum_{j=1}^{3} \theta_{ij} M_j, \qquad i = 1, 2, 3.$$

Since curl $u = \nabla \times u$ by definition, we have from (0.1) and (1.1)

$$(4.7)\qquad \sum_{j=1}^{3} M_j \sigma_j u = -{}^t(\sigma_1, \sigma_2, \sigma_3) \times u$$

for all $\sigma = {}^t(\sigma_1, \sigma_2, \sigma_3) \in \boldsymbol{R}^3$ and all $u \in \boldsymbol{C}^3$, which implies

$$M_i u = -e_i \times u, \qquad i = 1, 2, 3.$$

where $\{e_1, e_2, e_3\}$ is the canonical basis for $\boldsymbol{R}^3$. Therefore according to (4.1) and the rotational invariance of the vector product we have

$$M_i \theta u = -\theta\big(({}^t\theta e_i) \times u\big), \qquad i = 1, 2, 3.$$

Thus we obtain the desired relations (4.6) by (4.3) and (4.7) since u is arbitrary.

LEMMA 4.2. *Suppose that* (P, B) *is reflexive and that the frozen problem* $(P, B)_{(t^0, x^0)}$ *satisfies Hersh's condition for a point* $(t^0, x^0) \in \boldsymbol{R}^1 \times \partial G$. *Then we may assume that* B *is of the form*:

$$(4.8)\qquad B(t, x) = \big[I, S(t, x), 0\big]\, {}^tT_1\, {}^t\begin{bmatrix}\theta(x) & \\ & \theta(x)\end{bmatrix}$$

for (t, x) *near* (t^0, x^0), *where* $S(t, x)$ *is a* 2×2 *matrix function and* T_1 *is the matrix defined by* (1.2).

PROOF. It suffices to show that the right 2×2 block of $B(t, x)\begin{bmatrix}\theta(x) & \\ & \theta(x)\end{bmatrix} T_1$ is equal to the zero matrix for (t, x) near (t^0, x^0) and the left 2×2 block is nonsingular at $(t, x) = (t^0, x^0)$.

Let us consider the frozen problem $(P, B)_{(t^0, x^0)}$ and transform the independent variable x into $\tilde{x}$ by $x = \theta(x^0)\, \tilde{x}$ and the dependent variables E, H into $\tilde{E}$, $\tilde{H}$ by $E(t, x) = \theta(x^0)\, \tilde{E}(t, \tilde{x})$, $H(t, x) = \theta(x^0)\, \tilde{H}(t, \tilde{x})$ respectively. Then the normal $\nu(x^0)$ is transformed into ${}^t\theta(x^0)\, \nu(x^0) = {}^t(1, 0, 0)$, so that according to (4.4) and the orthogonality of $\theta(x^0)$ $(P, B)_{(t^0, x^0)}$ is done into the following problem:

$$(4.9)\qquad \begin{cases} P\left(\dfrac{\partial}{\partial t}, \dfrac{\partial}{\partial \tilde{x}}\right)\begin{bmatrix}\tilde{E}(t,\tilde{x})\\ \tilde{H}(t,\tilde{x})\end{bmatrix} = {}^t\begin{bmatrix}\theta(x^0) & \\ & \theta(x^0)\end{bmatrix} f\left(t, \theta(x^0)\,\tilde{x}\right) \quad \text{in } (0,\infty)\times G_1, \\ \left(B(t^0,x^0)\begin{bmatrix}\theta(x^0) & \\ & \theta(x^0)\end{bmatrix}\right)\begin{bmatrix}\tilde{E}(t,\tilde{x})\\ \tilde{H}(t,\tilde{x})\end{bmatrix} = 0 \quad \text{on } (0,\infty)\times\partial G_1, \\ \tilde{E}(0,\tilde{x}) = \tilde{H}(0,\tilde{x}) = 0 \qquad \text{for } \tilde{x}\in G_1. \end{cases}$$

Since the zeros of the Lopatinskii determinant are invariant under non-singular transformations of the dependent variables, the problem (4. 9) satisfies Hersh's condition by hypothesis. Moreover according to (4. 5) the reflexiveness of $(P, B)_{(t^0,x^0)}$ implies that of (4. 9), and the same is still valid with (t^0, x^0) replaced by $(t, x)\in \boldsymbol{R}^1\times(\partial G\cap U(x^0))$. Therefore the procedure which derived (1. 4) shows that $B(t, x)\begin{bmatrix}\theta(x) & \\ & \theta(x)\end{bmatrix}T_1$ has the desired properties.

PROOF OF THEOREM 1. According to Lemma 4. 2 we may assume B to be of the form (4. 8). Since (4. 9) satisfies Kreiss' condition by hypothesis as we have seen above, applying Proposition 2. 1 to it we obtain (2. 1) and (2. 2) with $S=S(t, x)$ for $(t, x)=(t^0, x^0)$ and hence by continity for all $(t, x)\in U(t^0)\times(\partial G\cap U(x^0))$. Hereafter let $U(t^0)$, $U(x^0)$ denote appropriate neighborhoods of t^0, x^0 in $\boldsymbol{R}^1$, $\bar{G}$ respectively.

We now extend $S(t, x)$ up to $U(t^0)\times U(x^0)$ to be sufficiently smooth. Note that (2. 1) and (2. 2) still hold there. Let $p(t, x)$ and $q(t, x)$ be the functions defined by (3. 18) and (3. 5) with $S=S(t, x)$ respectively. Moreover for every $(t, x)\in U(t^0)\times U(x^0)$ let us define a 6×6 matrix $T(t, x)$ as follows:

$$(4.10)\qquad T(t, x) = \begin{bmatrix}\theta(x) & \\ & \theta(x)\end{bmatrix} T_1 W(t, x),$$

where $W(t, x)$ is the matrix in (3. 2) with $y=-(q/2p)(t, x)$. Then (2. 2) and (3. 16) imply $|(q/2p)(t, x)|<1$ for all $(t, x)\in U(t^0)\times U(x^0)$, so that $T(t, x)$ and $T^{-1}(t, x)$ are sufficiently smooth. Moreover according to (4. 10), (4. 4) and Lemma 3. 1 we see that $T^{-1}(t, x)\,A_j T(t, x)$ is hermitian for each $j=1, 2, 3$ and $(t, x)\in U(t^0)\times U(x^0)$. Finally we find from (4. 8), (4. 10), (4. 5) and Lemma 3. 1 that the kernel of $(BT)(t, x)$ is maximally negative for $(T^{-1}A_\nu T)(t, x)$ at each $(t, x)\in U(t^0)\times(\partial G\cap U(x^0))$. Thus we have proved Theorem 1.

REMARK. The boundary condition $B\begin{bmatrix}E\\ H\end{bmatrix}=0$ with (4. 8) may be written as

$$(4.11)\qquad E-(E\cdot\nu)\,\nu-\nu\times H+\theta\begin{bmatrix}0 & 0\\ 0 & S\end{bmatrix}{}^t\theta\left(H-(H\cdot\nu)\,\nu-\nu\times E\right)=0.$$

In fact, multiplying this from left by ${}^t\theta$ we find from (4. 1), (4. 2) and the rotational invariance of the vector porduct that (4. 11) is equivalent to (1. 5) with E and H replaced by ${}^t\theta E$ and ${}^t\theta H$ respectively. Therefore the desired conclusion follows easily. This fact will be used at (ii) in the Appendix.

Appendix

(i). The converse of the statement of Theorem 1 is also valid, that is, if there exists such a matrix $T(t, x)$ as described in the theorem, the frozen problem $(P, B)_{(t,x)}$ is reflexive and satisfies Kreiss' condition at each $(t, x) \in U(x^0) \times (\partial G \cap U(x^0))$. This can be proved using simple modifications of the arguments in [4], as pointed out in [7], p. 624. (For the reflexiveness see [6]). It may be also established by following conversely the proof of Theorem 2 with (P, B) replaced by (4. 9).

(ii). It is showed in [7] that all local boundary conditions for the system P such that the resulting mixed problems are reflexive may be expressed invariantly in the form:

$$\pi_+ \begin{bmatrix} E \\ H \end{bmatrix} + \tilde{S}\pi_- \begin{bmatrix} E \\ H \end{bmatrix} = 0, \tag{A. 1}$$

where

$$\begin{aligned} \pi_+ \begin{bmatrix} E \\ H \end{bmatrix} &= \begin{bmatrix} E-(E\cdot\nu)\,\nu-\nu\times H \\ H-(H\cdot\nu)\,\nu+\nu\times E \end{bmatrix}, \\ \pi_- \begin{bmatrix} E \\ H \end{bmatrix} &= \begin{bmatrix} E-(E\cdot\nu)\,\nu+\nu\times H \\ H-(H\cdot\nu)\,\nu-\nu\times E \end{bmatrix} \end{aligned} \tag{A. 2}$$

and $\tilde{S}=\tilde{S}(t, x)$ is a 6×6 matrix function satisfying

$$[{}^t\nu, 0]\,\tilde{S}\pi_- = 0, \qquad [0, {}^t\nu]\,\tilde{S}\pi_- = 0 \tag{A. 3}$$

and

$$\pi_-\tilde{S}\pi_- = 0. \tag{A. 4}$$

(See [7], p. 627 and [9], p. 432).

We shall show that the boundary condition (A. 1) with (A. 3) and (A. 4) can be reduced to (4. 11). To do it we may assume G to be G_1 and B to be constant so that (4. 11) coincides with (1. 5), according to the argument which derived (4. 9). Then, since $\nu={}^t(1, 0, 0)$, (A. 2) becomes

$$\text{(A. 5)}\qquad \pi_+\begin{bmatrix} E \\ H \end{bmatrix}=\begin{bmatrix} 0 \\ E_2+H_3 \\ E_3-H_2 \\ 0 \\ H_2-E_3 \\ H_3+E_2 \end{bmatrix}, \qquad \pi_-\begin{bmatrix} E \\ H \end{bmatrix}=\begin{bmatrix} 0 \\ E_2-H_3 \\ E_3+H_2 \\ 0 \\ H_2+E_3 \\ H_3-E_2 \end{bmatrix}.$$

Note that π_+ or π_- may be regarded as the following symmetric 6×6 matrix

$$[0,\ e_2+e_6,\ e_3-e_5,\ 0,\ -e_3+e_5,\ e_2+e_6]$$

or

$$[0,\ e_2-e_6,\ e_3+e_5,\ 0,\ e_3+e_5,\ -e_2+e_6]$$

respectively, where $\{e_1, \cdots, e_6\}$ is the canonical basis for $\boldsymbol{R}^6$. Therefore, setting

$$\tilde{S}=[\tilde{s}_{ij}\,;\ i\downarrow 1, \cdots, 6,\ j\rightarrow 1, \cdots, 6]\,,$$

we find that (A. 3) may be written as

$$\text{(A. 6)}\qquad \begin{aligned} &\tilde{s}_{12}-\tilde{s}_{16}=\tilde{s}_{13}+\tilde{s}_{15}=0\,, \\ &\tilde{s}_{42}-\tilde{s}_{46}=\tilde{s}_{43}+\tilde{s}_{45}=0\,, \end{aligned}$$

and (A. 4) as

$$\text{(A. 7)}\qquad \begin{aligned} &\tilde{s}_{22}-\tilde{s}_{26}-(\tilde{s}_{62}-\tilde{s}_{66})=0\,, \qquad && \tilde{s}_{23}+\tilde{s}_{25}-(\tilde{s}_{63}+\tilde{s}_{65})=0\,, \\ &\tilde{s}_{32}-\tilde{s}_{36}+\tilde{s}_{52}-\tilde{s}_{56}=0\,, && \tilde{s}_{33}+\tilde{s}_{35}+\tilde{s}_{53}+\tilde{s}_{55}=0\,, \end{aligned}$$

since π_- is a symmetric matrix and

$$\tilde{S}\pi_-=[0,\ \tilde{s}_{i2}-\tilde{s}_{i6},\ \tilde{s}_{i3}+\tilde{s}_{i5},\ 0,\ \tilde{s}_{i3}+\tilde{s}_{i5},\ -\tilde{s}_{i2}+\tilde{s}_{i6}\,;\ i\downarrow 1, \cdots, 6]\,.$$

It follows from (A. 5) and (A. 6) that the first and 4-th components on the left in (A. 1) vanish. Moreover we find from (A. 5) and (A. 7) that the 6-th or 5-th component on the left in (A. 1) is equal to the second or (-1) times the third respectively. Thus (A. 1) with (A. 3) and (A. 4) coincides with (1. 5), if we set

$$\text{(A. 8)}\qquad S=\begin{bmatrix} \tilde{s}_{25} & \tilde{s}_{26} \\ \tilde{s}_{35} & \tilde{s}_{36} \end{bmatrix}-\begin{bmatrix} \tilde{s}_{22} & \tilde{s}_{23} \\ \tilde{s}_{32} & \tilde{s}_{33} \end{bmatrix}K,$$

where K is the matrix in (1. 2).

(iii). It is also asserted in [7] the following: (2. 4) with (A. 8) holds if and only if $\rho(\tilde{S})<1$, where $\rho(\tilde{S})$ denotes the spectral radius of the matrix $\tilde{S}$, and hence for Maxwell's equations the class of mixed problems satisfying

(2. 4) is definitely larger than that of mixed problems with maximally negative boundary conditions. But this assertion seems to be not necessarily valid, as seen from the following

EXAMPLE. With the notations in (ii) let $G=G_1$ and let $\tilde{S}$ be a constant 6×6 matrix such that

$$(A. 9)\qquad \tilde{s}_{33}=\tilde{s}_{35}=-1/2\,,\qquad \tilde{s}_{53}=\tilde{s}_{55}=1/2 \quad\text{and}\quad (\text{the other entries})=0.$$

It is clear that (A. 9) implies (A. 6) and (A. 7), i.e., (A. 3) and (A. 4). Moreover we find that (2. 4) with (A. 8) does not hold while $\rho(\tilde{S})=0$. In fact, it is not hard to see that all eigenvalues of $\tilde{S}$ are equal to zero. Furthermore from (A. 8) and (A. 9) we have

$$S=\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}.$$

Hence (2. 5) yields

$$R(\tau, \sigma)=\tau^2-\tau\lambda^+(\tau, \sigma)-\sigma_1^2\,.$$

Thus we find that $R(\tau, \sigma)$ vanishes for some (τ, σ) with $\tau\in\bar{C}_-$ and $\sigma\in\boldsymbol{R}^2\backslash 0$, say, with $\tau=0$, $\sigma_1=0$ and $\sigma_2\neq 0$ or with $\tau=|\sigma|$, $\sigma_2=0$ and $\sigma_1\neq 0$.

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ADDED IN PROOF: Let $\partial\Omega$ be compact and for every $(t^0, x^0)\in \boldsymbol{R}^1\times \partial\Omega$ let the hypotheses of Theorem be satisfied. Then we can find without difficulty a C^∞ global symmetrizer $T(t, x)$ defined on $\boldsymbol{R}^1\times\bar{\Omega}$ which is a uniformly positive definite hermitian 6×6 matrix such that $T(t, x)A_j$ is hermitian for each $j=1, 2, 3$ and $(t, x)\in\boldsymbol{R}^1\times\bar{\Omega}$ and that the kernel of $B(t, x)$ is maximally negative for $T(t, x)\,A_\nu(x)$ at each $(t, x)\in\boldsymbol{R}^1\times\partial\Omega$.

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