Hokkaido Mathematical Journal Vol. 7 (1978) p. 259-270

A uniqueness theorem for holomorphic functions of exponential type

By Mitsuo MORIMOTO and Kunio YOSHINO

(Received October 28, 1977; Revised November 21, 1977)

§ 1. Introduction.

In this paper we treat a uniqueness theorem for holomorphic functions of exponential type on a half plane from the point of view of the theory of analytic functionals with non-compact carrier.

Avanissian and Gay [1] proved among others the following Theorem 1 using the theorey of analytic functionals of Martineau [4].

THEOREM 1. *Let* $F(\zeta)$ *be an entire function of type* $<\pi$. *If we have* $F(-n)=0$ *for every* $n=1,2,3,\cdots$, *then the entire function* $F(\zeta)$ *vanishes identically.*

Theorem 1 is a corollary to Carlson's theorem (see Boas [2] p. 153).

THEOREM 2. (Carlson) *Let* $F(\zeta)$ *be a holomorphic function on the half plane* $\{\zeta=\xi+i\eta;\ \xi=\mathrm{Re}\,\zeta<0\}$. *Suppose that there exist real numbers* a, k *with* $0\leqslant k<\pi$ *and* $C\geqslant 0$ *such that*

$$\left|F(\zeta)\right|\leqslant C\exp(a\xi+k|\eta|) \qquad \textit{for}\ \mathrm{Re}\,\zeta<0\,.$$

If we have $F(-n)=0$ *for* $n=1,2,3,\cdots$, *then the function* $F(\zeta)$ *vanishes identically.*

We will prove Carlson's theorem by means of the theory of analytic functionals with non-compact carrier, which was introduced by the first named author [5] in connection with the theory of ultra-distributions of exponential growth of Sebastiaõ-è-Silva [6], namely Fourier ultra-hyperfunctions.

Following the sections we outline the results. In § 2 we define the fundamental space $\mathscr{Q}(L;k')$, the element of which is a holomorphic function in a tubular neighborhood L_ϵ of the closed half strip $L=[a,\infty)+i[k_1,k_2]$. A continuous linear functional on the space $\mathscr{Q}(L;k')$ is, by definition, an analytic functional with carrier in L and of (exponential) type $\leqslant k'$. The image of the Laplace transformation of $\mathscr{Q}'(L;k')$ is characterized in Theorem 3. As in Avanissian-Gay [1] we define in § 3 the transformation G_μ

of an analytic functional $\mu\in\mathscr{Q}'(L;k')$ by the formula: $G_\mu(\zeta)=\langle\mu_z, (1-\zeta e^z)^{-1}\rangle$ and call it the Avanissian-Gay transformation. The Avanissian-Gay transformation is defined for $0\leqslant k'<1$ and G_μ is a holomorphic function on the complement of the set $\exp(-L)$, vanishes at the infinity and satisfies a certain growth condition at the origin. We show in § 4 the Avanissian-Gay transformation is injective if the width of the half strip L is less than 2π, proving the inversion formula (Theorem 4). As a corollary, we have the above mentioned Carlson theorem. In the last section, we determine the image of the Avanissian-Gay transformation (Theorem 6).

§ 2. Analytic functionals with half strip carrier and their Laplace transformation.

In this section we recall the definition of analytic functionals with non-compact carrier and characterize their Laplace transformation.

We begin with some notations. In the sequel, L denotes the closed half strip in the complex number plane $\boldsymbol{C}$:
$L=A+iK$, $A=[a,\infty)$, $K=[k_1,k_2]$ and $i=\sqrt{-1}$, namely, $L=\{z=x+iy\in\boldsymbol{C};\ x\geqslant a,\ k_1\leqslant y\leqslant k_2\}$. By L_ε we denote the ε-neighborhood of L:

$$L_\varepsilon=L+[-\varepsilon,\varepsilon]+i[-\varepsilon,\varepsilon].$$

For $\varepsilon>0$, $\varepsilon'>0$ and $0\leqslant k'<\infty$, we define the function space $\mathscr{Q}_b(L_\varepsilon;k'+\varepsilon')$ as follows:

$$\mathscr{Q}_b(L_\varepsilon;k'+\varepsilon')=\left\{f\in\mathscr{O}(\operatorname{int}L_\varepsilon)\cap\mathscr{C}(L_\varepsilon);\ \sup_{z\in L_\varepsilon}\left|f(z)\right|\exp\left((k'+\varepsilon')x\right)<\infty\right\},$$

where $\mathscr{O}(\operatorname{int}L_\varepsilon)$ denotes the space of holomorphic functions on the interior $\operatorname{int}L_\varepsilon$ of L_ε and $\mathscr{C}(L_\varepsilon)$ denotes the space of continuous functions on L_ε. Endowed with the norm

$$\sup_{z\in L_\varepsilon}\left|f(z)\right|\exp\left((k'+\varepsilon')x\right),$$

the space $\mathscr{Q}_b(L_\varepsilon;k'+\varepsilon')$ becomes a Banach space. If $\varepsilon_1<\varepsilon$ and $\varepsilon_1'<\varepsilon'$, the restriction mapping

$$\mathscr{Q}_b(L_\varepsilon;k'+\varepsilon')\longrightarrow\mathscr{Q}_b(L_{\varepsilon_1};k'+\varepsilon_1')\qquad(2.1)$$

is defined and a continuous linear injection. Following the mappings (2.1), we from the locally convex inductive limit:

$$\mathscr{Q}(L;k')=\lim_{\substack{\longrightarrow\\ \varepsilon>0,\ \varepsilon'>0}}\operatorname{ind}\mathscr{Q}_b(L_\varepsilon;k'+\varepsilon').$$

If we put $X_n=\mathcal{O}_b(L_{1/n};\ k'+1/n)$, then with mappings (2. 1) we have a sequence of Banach spaces with compact injective mappings $X_j \longrightarrow X_{j+1}$:

$$X_1 \longrightarrow X_2 \longrightarrow X_3 \longrightarrow \cdots$$

As we have clearly $\mathcal{O}(L;\ k')=\lim \operatorname{ind} X_j$, the locally convex space $\mathcal{O}(L;\ k')$ is a DFS space (namely the dual space of a Fréchet-Schwartz space). We denote the dual space of $\mathcal{O}(L;\ k')$ by $\mathcal{O}'(L;\ k')$, an element of which is, by definition, an analytic functional with carrier in L and of type $\leqslant k'$.

We denote by $h_L(\zeta)$ the supporting function of the half strip L:

$$h_L(\zeta)=\sup_{z\in L} \operatorname{Re} \zeta z=\begin{cases} a\xi-k_1\eta & \text{if } \xi\leqslant 0 \text{ and } \eta\geqslant 0 \\ a\xi-k_2\eta & \text{if } \xi\leqslant 0 \text{ and } \eta\leqslant 0\,. \end{cases}$$

Remark $h_L(\zeta)=\infty$ if $\operatorname{Re}\zeta>0$. For $k'\geqslant 0$ we denote by $\operatorname{Exp}((-\infty,\ -k')+i\boldsymbol{R};\ L)$ the space of all holomorphic functions φ on the open half plane $(-\infty,\ -k')+i\boldsymbol{R}$ for which

$$\sup_{\operatorname{Re}\zeta\leqslant -k'-\varepsilon'} |\varphi(\zeta)| \exp\left(-h_L(\zeta)-\varepsilon|\zeta|\right)<\infty \tag{2. 2}$$

for every $\varepsilon>0$ and $\varepsilon'>0$. An element of the space $\operatorname{Exp}((-\infty,\ -k')+i\boldsymbol{R};\ L)$ is said to be a holomorphic function of exponential type in L. Endowed with the norms (2. 2), the space $\operatorname{Exp}((-\infty,\ -k')+i\boldsymbol{R};\ L)$ is an FS space (namely a Fréchet-Schwartz space). (As for the DFS spaces and FS spaces, we refer the reader to Komatsu [3].)

We define the Laplace transformation of an analytic functional μ with carrier in L and of type $\leqslant k'$ as follows:

$$\tilde{\mu}(\zeta)=\left\langle \mu_z,\ \exp(z\zeta)\right\rangle. \tag{2. 3}$$

Remark that $\tilde{\mu}(\zeta)$ is defined for ζ of the half plane $\{\zeta;\ \operatorname{Re}\zeta<-k'\}$. The next Paley-Wiener type theorem characterizes the Laplace transformation of the analytic functionals with carrier in L and of type $\leqslant k'$.

THEOREM 3. (Morimoto [5]) *The Laplace transformation* (2. 3) *is a linear topological isomorphism of the space* $\mathcal{O}'(L;\ k')$ *onto the space* $\operatorname{Exp}((-\infty,\ -k')+i\boldsymbol{R};\ L)$.

We have the following density theorem.

PROPOSITION 1. *For* $h\in\mathcal{O}(L;\ k')$, *we have*

$$\lim_{\delta\downarrow 0} h(z)\exp(-\delta z^2)=h(z)$$

in the topology of $\mathcal{O}(L;\ k')$.

PROOF. By the definition of the space $\mathscr{Q}(L;\ k')$, there exist $\varepsilon>0$ and $\varepsilon'>0$ such that $h\in\mathscr{Q}_b(L_{2\varepsilon};\ k'+2\varepsilon')$.
In particular, we have

$$\sup_{z\in L_{2\varepsilon}}\left|h(z)\right|\exp\left((k'+2\varepsilon')\,x\right)=M>\infty\,.$$

Then we have

$$\sup_{z\in L_\varepsilon}\left|h(z)\right|\left|1-\exp(-\delta z^2)\right|\exp\left((k'+\varepsilon')\,x\right)$$
$$\leqslant M\sup_{z\in L_\varepsilon}\left|1-\exp(-\delta z^2)\right|\exp(-\varepsilon' x)\,.$$

As the righthand side tends to 0 as $\delta\downarrow 0$, $h(z)\exp(-\delta z^2)$ tends to $h(z)$ in the topology of $\mathscr{Q}_b(L_\varepsilon;\ k'+\varepsilon')$ as $\delta\downarrow 0$.

q. e. d.

COROLLARY. *If $k_1'>k'$, then the space $\mathscr{Q}(L;\ k_1')$ is a dense subspace of the space $\mathscr{Q}(L;\ k')$. The dual space $\mathscr{Q}'(L;\ k')$ can be considered as a subspace of $\mathscr{Q}'(L;\ k_1')$.*

PROOF. If $\delta>0$ and $h\in\mathscr{Q}(L;\ k')$, then $h(z)\exp(-\delta z^2)$ belongs to $\mathscr{Q}(L;\ k_1')$. The second assertion results from the Hahn-Banach theorem.

q. e. d.

§ 3. The Avanissian-Gay transformation.

If $0\leqslant k'<1$ and $\zeta\notin\exp(-L)$, then the function of z, $(1-\zeta e^z)^{-1}$ belongs to the space $\mathscr{Q}(L;\ k')$. Following Avanissian-Gay [1] we define the transformation G_μ of an analytic functional $\mu\in\mathscr{Q}'(L;\ k')$ as follows:

$$G_\mu(\zeta)=\left\langle\mu_z,(1-\zeta e^z)^{-1}\right\rangle.$$

$G_\mu(\zeta)$ is a function of $\zeta\notin\exp(-L)$ and has the following properties.

PROPOSITION 2. *Suppose $\mu\in\mathscr{Q}'(L;\ k')$, $0\leqslant k'<1$.*

(i) *$G_\mu(\zeta)$ is a holomorphic function on the complement of* $\exp(-L)$.

(ii) *The following Laurent expansion is valid:*

$$G_\mu(\zeta)=-\sum_{n=1}^{\infty}\zeta^{-n}\tilde{\mu}(-n)$$

for $|\zeta|>e^{-a}$.

(iii) $\lim\limits_{|\zeta|\to\infty}|G_\mu(\zeta)|=0.$

PROOF. (i) can be derived from Morera's theorem.

(ii) We have the following expansion:

$$(1-\zeta e^{z})^{-1} = -\sum_{n=1}^{\infty} \zeta^{-n} \exp(-nz).$$

By elementary calculations, we can show that this series converges uniformly with respect to ζ with $|\zeta| \geqslant e^{-a+\varepsilon}$, $\varepsilon>0$, in the topology of $\mathscr{Q}(L; k')$. Hence we have

$$G_{\mu}(\zeta) = -\sum_{n=1}^{\infty} \zeta^{-n} \langle \mu_{z}, \exp(-nz) \rangle$$
$$= -\sum_{n=1}^{\infty} \zeta^{-n} \tilde{\mu}(-n).$$

(iii) is a trivial consequence of (ii). q. e. d.

If the half strip L has the width $k_2-k_1<2\pi$, the complement of the set $\exp(-L)$ contains the open angular domain

$$A(-k_1, -k_2+2\pi) = \left\{\zeta \in \boldsymbol{C}\backslash(0) ; -k_1 < \arg \zeta < -k_2+2\pi\right\}.$$

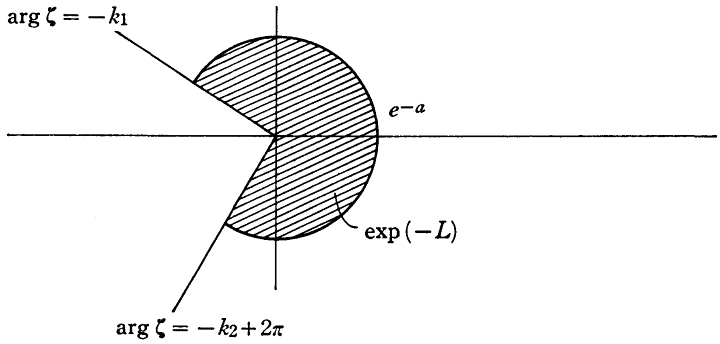

Fig. 1.

We shall investigate, for further purposes, the behavior of the function $G_{\mu}(\zeta)$ in this angular domain.

PROPOSITION 3. *Suppose the half strip* $L=[a, \infty)+i[k_1, k_2]$ *has the width* $k_2-k_1<2\pi$ *and* $0 \leqslant k' < 1$. *If* $\mu \in \mathscr{Q}'(L; k')$, *then, for any* ε *with* $0<2\varepsilon<2\pi+k_1-k_2$ *and any* ε' *with* $0<\varepsilon'<1-k'$, *there exists a constant* $C \geqslant 0$ *such that*

$$\left|G_{\mu}(\zeta)\right| \leqslant C|\zeta|^{-k'-\varepsilon'}$$

in the closed angular domain

$$\bar{A}(-k_1+\varepsilon, -k_2+2\pi-\varepsilon) = \left\{\zeta \in \boldsymbol{C}\backslash(0) ; -k_1+\varepsilon \leqslant \arg \zeta \leqslant -k_2+2\pi-\varepsilon\right\}.$$

PROOF. By the continuity of $\mu \in \mathscr{Q}'(L; k')$, there exists a constant $C' \geqslant 0$ such that, for $\zeta \notin \exp(-L_{\varepsilon/2})$, we have

$$\begin{aligned}|G_\mu(\zeta)| &= \langle \mu_z, (1-\zeta e^z)^{-1}\rangle \\ &\leqslant C' \sup_{z\in L_{\varepsilon/2}} |1-\zeta e^z|^{-1} \exp\big((k'+\varepsilon')\,x\big) \\ &= C' \sup_{z\in L_{\varepsilon/2}} |e^{-z}-\zeta|^{-1} \exp\big((k'+\varepsilon'-1)\,x\big) \\ &\leqslant C' \sup_{z\in L_{\varepsilon/2}} |e^{-z}-\zeta|^{-k'-\varepsilon'} \sup_{z\in L_{\varepsilon/2}} |e^{-z}-\zeta|^{k'+\varepsilon'-1} \exp\big((k'+\varepsilon'-1)\,x\big).\end{aligned}$$

Therefore with another constant $C''\geqslant 0$, we have

$$\begin{aligned}&|G_\mu(\zeta)| \\ &\qquad \leqslant C'' \operatorname{dist}\big(\zeta, \exp(-L_{\varepsilon/2})\big)^{-k'-\varepsilon'} \sup_{z\in L_{\varepsilon/2}} |1-\zeta e^z|^{k'+\varepsilon'-1}\end{aligned}$$

for $\zeta \notin \exp(-L_{\varepsilon/2})$. On the other hand, as the set $\exp(-L_{\varepsilon/2})$ is contained in the closed angular domain $\bar{A}(-k_2-\varepsilon/2,\ -k_1+\varepsilon/2)$, we have

$$\operatorname{dist}\big(\zeta, \exp(-L_{\varepsilon/2})\big) \geqslant |\zeta| \sin(\varepsilon/2)$$

for $\zeta\in\bar{A}(-k_1+\varepsilon,\ -k_2+2\pi-\varepsilon)$.
If $z\in L_{\varepsilon/2}$ and $\zeta\in\bar{A}(-k_1+\varepsilon,\ -k_2+2\pi-\varepsilon)$, then

$$|\arg \zeta e^z| \geqslant \varepsilon/2 \qquad \bmod 2\pi.$$

Therefore we have

$$\inf_{z\in L_{\varepsilon/2}} |1-\zeta e^z| \geqslant \sin(\varepsilon/2) \qquad \text{for } \zeta\in\bar{A}(-k_1+\varepsilon,\ k_2+2\pi-\varepsilon).$$

As $-k'-\varepsilon'<0$ and $k'+\varepsilon'-1<0$ by the choice of ε', putting $C=C''(\sin(\varepsilon/2))^{-1}$ we obtain the desired estimate of $G_\mu(\zeta)$. q. e. d.

Suppose always $L=[a,\infty)+i[k_1, k_2]$ has the width $k_2-k_1<2\pi$ and $0\leqslant k'<1$. We denote by $\mathcal{O}_0(\boldsymbol{C}\backslash\exp(-L);\ k')$ the space of all holomorphic functions φ on the domain $\boldsymbol{C}\backslash\exp(-L)$ which satisfy following two conditions:

(1) $|\varphi(\zeta)|\to 0$ as $|\zeta|\to\infty$.

(2) $\sup\{|\varphi(\zeta)\,\zeta^{k'+\varepsilon'}|\,;\ \zeta\in\bar{A}(-k_1+\varepsilon,\ -k_2+2\pi-\varepsilon)\}<\infty$

for any ε with $0<2\varepsilon<2\pi+k_1-k_2$ and any ε' with $0<\varepsilon'<1-k'$. The space $\mathcal{O}_0(\boldsymbol{C}\backslash\exp(-L);\ k')$ equipped with the seminorms $\sup\{|\varphi(\zeta)|\,;\ |\zeta|\geqslant e^{-a+\varepsilon}\}$ and $\sup\{|\varphi(\zeta)\,\zeta^{k'+\varepsilon'}|;\ \zeta\in\bar{A}(-k_1+\varepsilon,\ -k_2+2\pi-\varepsilon)\}$, is clearly a Fréchet (-Schwartz) space. As a corollary to Propositions 2 and 3, we have the following proposition.

PROPOSITION 4. *Suppose the width of L is less than 2π and $0\leqslant k'\leqslant 1$. Then the Avanissian-Gay transformation G is a continuous linear mapping of $\mathcal{Q}'(L;\ k')$ into $\mathcal{O}_0(\boldsymbol{C}\backslash\exp(-L);\ k')$.*

PROOF. The continuity results from the boundedness of the set $\{(1-\zeta e^z)^{-1};\ |\zeta|\geqq e^{-a+\varepsilon}\}$ and the set $\{\zeta^{k'+\varepsilon'}(1-\zeta e^z)^{-1};\ \zeta\in\bar{A}(-k_1+\varepsilon,\ -k_2+2\pi-\varepsilon)\}$ in the space $\mathscr{Q}(L;\ k')$. q. e. d.

§ 4. Inversion formula for $G_\mu(\zeta)$.

In the sequel we suppose the half strip L has the form

$$L=[a,\infty)+i[k_1,k_2],\quad k_2-k_1<2\pi$$

and $0\leqq k'<1$.

LEMMA 1. (An integral formula) *Let $h\in\mathscr{Q}(L;\ k')$. Choose positive numbers ε and ε' so small that $0<2\varepsilon<2\pi+k_1-k_2$, $0<\varepsilon'<1-k'$ and that $h\in\mathscr{Q}_b(L_\varepsilon;\ k'+\varepsilon')$.*

(i) *For any $R>0$, the function of z*

$$H_R(z)=\int_{\partial L_{\varepsilon,R}}h(w)\left(1-\exp(z-w)\right)^{-1}dw$$

belongs to the space $\mathscr{Q}(L;\ 1)$, consequently to the space $\mathscr{Q}(L;\ k')$, where we denote $\partial L_{\varepsilon,R}=\partial L_\varepsilon\cap\{w;\ \mathrm{Re}\,w\leqq R\}$.

(ii) *We have*

$$2\pi i\,h(z)=\int_{\partial L_\varepsilon}h(w)\left(1-\exp(z-w)\right)^{-1}dw\qquad \textit{for } z\in\mathrm{int}\,L_\varepsilon\,.$$

(iii) *In the topology of $\mathscr{Q}(L;\ k')$, we have*

$$\lim_{R\to\infty}\int_{\partial L_{\varepsilon,R}}h(w)\left(1-\exp(z-w)\right)^{-1}dw=2\pi i\,h(z)\,.$$

PROOF. (i) It is clear the function $H_R(z)$ is holomorphic in int L_ε. On the other hand we have

$$\begin{aligned}&\sup_{z\in L_{\varepsilon/2}}\left|H_R(z)\,e^z\right|\\&=\sup_{z\in L_{\varepsilon/2}}\left|\int_{\partial L_{\varepsilon,R}}h(w)\left(e^{-z}-e^{-w}\right)^{-1}dw\right|\\&\leqq\int_{\partial L_{\varepsilon,R}}\left|h(w)\right|\mathrm{dist}\left(e^{-w},\ \exp(-L_{\varepsilon/2})\right)^{-1}|dw|<\infty\,,\end{aligned}$$

because the integrand is continuous and $\partial L_{\varepsilon,R}$ is compact.

(ii) By the residue theorem, if $z\in\mathrm{int}\,L_\varepsilon$ and $\mathrm{Re}\,z<R$, then we have

$$\int_{\partial L_\varepsilon(R)}h(w)\left(1-\exp(z-w)\right)^{-1}dw=2\pi i\,h(z)\,,$$

where $\partial L_\varepsilon(R)$ denotes the boundary of the rectangle

$$L_\varepsilon(R) = L_\varepsilon \cap \{w\,;\ \mathrm{Re}\, w \leqslant R\}\,.$$

Let us denote by $C_\varepsilon(R)$ the boundary of $L_\varepsilon \cap \{w\,;\ \mathrm{Re}\, w \geqslant R\}$. We have to show, for z fixed in int L_ε,

$$\int_{C_\varepsilon(R)} h(w)\left(1-\exp(z-w)\right)^{-1} dw$$

tends to 0 as $R \to \infty$.

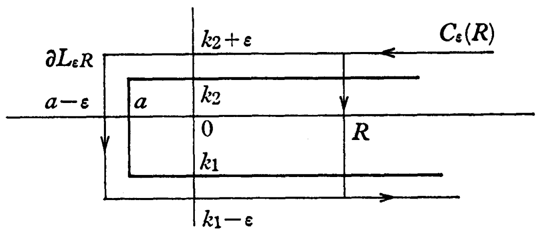

Fig. 2.

As $h \in \mathscr{O}_b(L\,;\ k'+\varepsilon')$, we have with some constant $C \geqslant 0$,

$$\left|\int_{C_\varepsilon(R)} h(w)\left(1-\exp(z-w)\right)^{-1} dw\right|$$
$$\leqslant C\int_{C_\varepsilon(R)} e^{-(k'+\varepsilon')u}(1-e^{x-R})^{-1}|dw| \longrightarrow 0$$

as $R \to \infty$.

(iii) We have to show, putting $C'_\varepsilon(R) = \partial L_\varepsilon - \partial L_{\varepsilon,R}$

$$\int_{C'_\varepsilon(R)} h(w)\left(1-\exp(z-w)\right)^{-1} dw$$

tends to 0 in the topology of $\mathscr{O}(L\,;\ k')$ as $R \to \infty$. Remark that e^{-w} belongs to the closed angular domain $\bar{A}(-k_1+\varepsilon,\ -k_2+2\pi-\varepsilon)$ if $w \in C'_\varepsilon(R)$. Therefore as in the proof of Proposition 3, we can show

$$\inf_{z \in L_{\varepsilon/2}} |e^{-z}-e^{-w}| \geqslant e^{-u}\sin(\varepsilon/2)$$

and

$$\inf_{z \in L_{\varepsilon/2}} |1-e^{-w}e^{z}| \geqslant \sin(\varepsilon/2) \qquad \text{for } w \in C'_\varepsilon(R)\,.$$

Therefore we have with some constants C and $C' \geqslant 0$

$$\sup_{z \in L_{\varepsilon/2}} \left|e^{(k'+\varepsilon'/2)z}\int_{C'_\varepsilon(R)} h(w)\left(1-\exp(z-w)\right)^{-1} dw\right|$$
$$= \sup_{z \in L_{\varepsilon/2}} \left|\int_{C'_\varepsilon(R)} h(w)\,(e^{-z}-e^{-w})^{-k'-\varepsilon'/2}(1-e^{-w}e^{z})^{-1+k'+\varepsilon'/2}\, dw\right|$$

$$\leqslant C\int_{C'_\varepsilon(R)}\left|h(w)\right|e^{((k'+\varepsilon'/2)u)}|dw|$$
$$\leqslant C'\int_{C'_\varepsilon(R)}\left|\exp\left((-\varepsilon'/2)\,w\right)\right||dw|\,.$$

The last term converges to 0 as $R\to\infty$. q. e. d.

THEOREM 4. (Inversion formula) *Let* $\mu\in\mathscr{Q}'(L;\,k')$ *and* $h\in\mathscr{Q}(L;\,k')$ *with* $0\leqslant k'<1$ *and* $L=[a,\infty)+i[k_1,k_2]$, $k_2-k_1<2\pi$. *Choose positive numbers* ε *and* ε' *so small that* $0<2\varepsilon<2\pi+k_1-k_2$, $0<\varepsilon'<1-k'$ *and that* $h\in\mathscr{Q}_b(L;\,k'+\varepsilon')$. *Then we have the inversion formula*:

$$\langle\mu, h\rangle=(2\pi i)^{-1}\int_{\partial L_\varepsilon}G_\mu(e^{-w})\,h(w)\,dw\,.$$

PROOF. We have by Lemma 1 (i)

$$\int_{\partial L_{\varepsilon,R}}\left\langle\mu_z,\left(1-\exp(z-w)\right)^{-1}\right\rangle h(w)\,dw$$
$$=\left\langle\mu_z,\int_{\partial L_{\varepsilon,R}}\left(1-\exp(z-w)\right)^{-1}h(w)\,dw\right\rangle$$

for $R>0$. By Lemma 1 (iii), the righthand side converges to $\langle\mu_z, 2\pi i\,h(z)\rangle$. As the lefthand side converges because of Proposition 3, we obtain the inversion formula. q. e. d.

THEOREM 5. *Suppose* $0\leqslant k'<1$ *and* $L=[a,\infty)+i[k_1,k_2]$, $k_2-k_1<2\pi$. *If the function* $F\in\mathrm{Exp}\left((-\infty,-k')+i\boldsymbol{R};\,L\right)$ *satisfies the condition*

$$F(-n)=0\qquad\textit{for every } n=1,2,3,\cdots,$$

then the function $F(\zeta)$ *vanishes identically.*

PROOF. By Theorem 3, there exists an analytic functional $\mu\in\mathscr{Q}'(L;\,k')$ such that $F(\zeta)=\tilde{\mu}(\zeta)$. By Proposition 2 (ii), we have the Laurent expansion:

$$G_\mu(\zeta)=-\sum_{n=1}^{\infty}\zeta^{-n}\tilde{\mu}(-n)=-\sum_{n=1}^{\infty}\zeta^{-n}F(-n)$$

for $|\zeta|>e^{-a}$. By the assumption, $G_\mu(\zeta)=0$. By Theorem 4, we conclude $\mu=0$ and $F(\zeta)=0$. q. e. d.

Putting $-k_1=k_2=k$, $0\leqslant k<\pi$, we obtain Theorem 2 as a corollary.

§ 5. The image of the Avanissian-Gay transformation.

We determine in this section the image of the Avanissian-Gay transformation.

THEOREM 6. *Suppose the width of* L *is less than* 2π *and* $0\leqslant k'<1$.

Then the Avanissian-Gay transformation G is a linear topological isomorphism of $\mathscr{Q}'(L;\, k')$ onto $\mathscr{O}_0(\boldsymbol{C}\backslash\exp(-L);\, k')$.

PROOF. We have proved the Avanissian-Gay transformation G is a continuous linear mapping of $\mathscr{Q}'(L;\, k')$ into $\mathscr{O}_0(\boldsymbol{C}\backslash\exp(-L);\, k')$ in Proposition 4. If we can prove the bijectivity of G, the continuity of the inverse mapping results from the closed graph theorem for Fréchet spaces. The injectivity of G is a consequence of the inversion formula (Theorem 4). Let us prove the surjectivity of G. Let $\varphi\in\mathscr{O}_0(\boldsymbol{C}\backslash\exp(-L);\, k')$ be given. We put, for $h\in\mathscr{Q}(L;\, k')$,

$$\begin{aligned}\langle\mu(\varphi), h\rangle &= \int_{\Gamma_\varepsilon}\varphi(\tau)\, h(-\log\tau)\, d\tau/\tau \\ &= -\int_{\partial L_\varepsilon}\varphi(e^{-z})\, h(z)\, dz\end{aligned} \tag{5.1}$$

where $\varepsilon>0$ is a sufficiently small number and $\Gamma_\varepsilon=\Gamma_\varepsilon^1+\Gamma_\varepsilon^2+\Gamma_\varepsilon^3=\exp(-\partial L_\varepsilon)$ is the path in the τ-plane depicted in the figure 3.

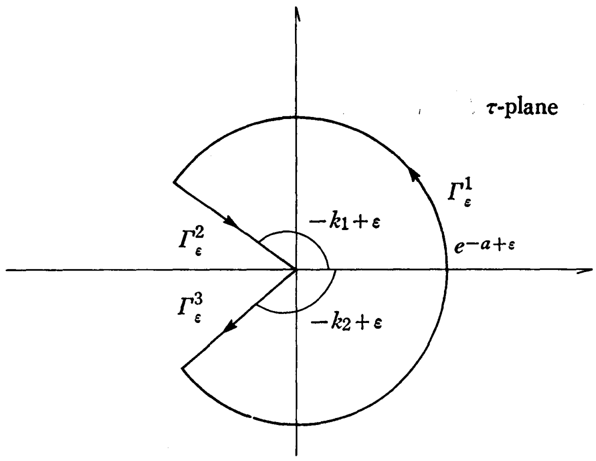

Fig. 3.

First we show the improper integral of the righthand side of (5.1) exists and is independent of sufficiently small $\varepsilon>0$. If $0<\varepsilon_1<\varepsilon<\pi+(k_1-k_2)/2$ and $0<\varepsilon'<1-k'$, we have

$$\sup_{z\in L_\varepsilon\backslash L_{\varepsilon_1}}\left|\varphi(e^{-z})\, e^{-(k'+\varepsilon')z}\right|<\infty\,.$$

If $h\in\mathscr{Q}_b(L_{\varepsilon_0};\, k'+\varepsilon_0')$, then the righthand side of (5.1) converges clearly for $0<\varepsilon<\min(\pi+(k_1-k_2)/2, \varepsilon_0)$ and is independent of such ε by the Cauchy integral theorem. Therefore $\langle\mu(\varphi), h\rangle$ is well defined by (5.1) and $\mu(\varphi)$ is continuous linear on the space $\mathscr{Q}_b(L_{\varepsilon_0};\, k'+\varepsilon_0')$ for any $\varepsilon_0>0$ and $\varepsilon_0'>0$. By the definition of the inductive limit topology, $\mu(\varphi)$ is a continuous linear functional on $\mathscr{Q}(L;\, k')$.

We shall compute the Avanissian-Gay transformation of the functional $\mu(\varphi)$. By the definition, we have

$$\begin{aligned} G_{\mu(\varphi)}(\zeta) &= \langle \mu(\varphi)_z, (1-\zeta e^z)^{-1} \rangle \\ &= \int_{\Gamma_\varepsilon} \varphi(\tau) \left(1-\zeta \exp(-\log\tau)\right)^{-1} d\tau/\tau \\ &= \int_{\Gamma_\varepsilon} \varphi(\tau)(\tau-\zeta)^{-1} d\tau \\ &= \lim_{\delta\to 0} \int_{\Gamma_{\varepsilon,\delta}} \varphi(\tau)(\tau-\zeta)^{-1} d\tau, \end{aligned}$$

where $\Gamma_{\varepsilon,\delta}=\Gamma_\varepsilon \cap \{\tau;\ |\tau|\geqslant\delta\}$. For a sufficiently large number $R>0$ and sufficiently small number $\delta>0$, we put

$$C_R=\left\{\tau;\ |\tau|=R\right\}$$

and

$$C'_\delta(\varepsilon)=\left\{\tau;\ |\tau|=\delta,\ -k_1+\varepsilon\leqslant \arg\tau\leqslant -k_2+2\pi-\varepsilon\right\}.$$

By Cauchy's integral formula, we have

$$\frac{1}{2\pi i}\int_{C'_\delta(\varepsilon)+C_R+\Gamma_{\varepsilon,\delta}} \varphi(\tau)(\tau-\zeta)^{-1}d\tau=\varphi(\zeta).$$

We will show the integral over the path C_R tends to 0 as $R\to\infty$ and that the integral over the path $C'_\delta(\varepsilon)$ tends to 0 as $\delta\to 0$. If $|\tau|=R$ and $R>|\zeta|$, we have $|\tau-\zeta|\geqslant|\tau|-|\zeta|=R-|\zeta|>0$. Therefore

$$\begin{aligned} \left|\int_{C_R}\varphi(\tau)(\tau-\zeta)^{-1}d\tau\right| &\leqslant \int_{C_R}\left|\varphi(\tau)\right||\tau-\zeta|^{-1}|d\tau| \\ &\leqslant \sup_{|\tau|=R}\left|\varphi(\tau)\right|(R-|\zeta|)^{-1}2\pi R \longrightarrow 0 \text{ as } R\longrightarrow\infty. \end{aligned}$$

If $|\zeta|>\delta$, then

$$\begin{aligned} \left|\int_{C'_\delta(\varepsilon)}\varphi(\tau)(\tau-\zeta)^{-1}d\tau\right| &\leqslant \int_{C'_\delta(\varepsilon)}\left|\varphi(\tau)\right|(|\zeta|-\delta)^{-1}|d\tau| \\ &\leqslant C_1\delta^{-k'-\varepsilon'}(|\zeta|-\delta)^{-1}2\pi\delta \\ &= C_1 2\pi(|\zeta|-\delta)^{-1}\delta^{1-(k'+\varepsilon')} \longrightarrow 0 \text{ as } \delta\longrightarrow 0, \end{aligned}$$

because we may choose ε' so that $1-(k'+\varepsilon')>0$. We have thus proved

$$G_{\mu(\varphi)}=\varphi$$

and the surjectivity of the Avanissian-Gay transformation G. q. e. d.

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Department of Mathematics
Sophia University
Tokyo 102 JAPAN