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Numerical ranges of the tensor products of elements

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Abstract. We will discuss the convexoid, normaloid and spectraloid elements in a unital Banach algebra A, and in the tensor product $A_1 \otimes_\alpha A_2$ of two unital Banach algebras A_1, A_2 under some compatible reasonable norm α. If A is a Hilbert space, the convexoid, normaloid and spectraloid operators on A are investigated by Halmos, Furuta, Nakamoto, Takeda and Saito etc. Moreover, we give necessary and sufficient conditions for the joint convexoidity of n-tuple of operators on Hilbert spaces.

1. Introduction.

Recently, Bonsall and Duncan in [1] developed the numerical range $V(T)$ for general normed linear space A which is defined by

$$V(T) = \{f(Tx) : (x, f) \in \pi\}$$

where $\pi = \{(x, f) \in S(A) \times S(A^*) : f(x) = 1\}$, and $S(A)$ denotes the unit sphere of A and A^* is the dual space of A.

Let A be a normed algebra with unit 1. For $a \in A$, the numerical range $V(a)$ of the element a is defined by

$$V(a) = V(T_a),$$

where T_a is the left regular representation (operator) on A. It is remarkable that $V(a)$ can be expressed by $V(a) = \{f(a) : f \in D(A, 1)\}$, where $D(A, 1) = \{f \in A^* : ||f|| = 1 = f(1)\}$ (cf. Bonsall and Duncan [1]). The numerical range $W(T)$ of the operator T on Hilbert space (cf. Halmos [6]) is convex, but in general $V(T)$ is not convex. While $V(a)$ is known to be a compact convex set (cf. Bonsall and Duncan [1]). In this note we discuss the numerical range of a Banach algebra with unit, and consider $a \in A$ such that the numerical range of the element a coincides with the convex hull of its spectrum; for such element we shall say that a is convexoid. It seems not to be known whether the tensor product of convexoid elements x, y

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in two unital Banach algebras A_1, A_2 respectively is convexoid or not. If A is a Hilbert space, the convexoidity of operators on A was investigated by Halmos [2]. Furuta and Nakamoto [4], Furuta [5] and Saito [9] etc. Now we will investigate the convexoid, normaloid and spectraloid elements in a unital Banach algebra. For convenience, we begin in section 2 to describe the definitions and some notations. Sections 3 and 4 are the main parts, we discuss the convexoid, normaloid and spectraloid elements in the tensor products of two unital Banach algebras for a compatible reasonable norm. This will yield a sharper version of a similar theorems of Furuta and Nakamoto [4] and Saito [9]. Recently Dash [2], Dash and Schechter [3] have discussed the joint numerical range of operators T_i $(1 \leq i \leq n)$ acting on the tensor products of Hilbert spaces. In section 4, we will apply the methods in section 3 to study the joint convexoidity of an n-tuple of operators $T_1, \cdots, T_n$ on the tensor products $H_1 \otimes H_2 \otimes \cdots \otimes H_n$ of Hilbert spaces, and establish necessary and sufficient conditions for joint convexoidity (For the definition see section 4).

2. Preliminaries and notations.

Through out this note, all normed algebras are over complex field $\boldsymbol{C}$.

Let A be a normed algebra with unit 1, such that $\|1\|=1$, *i.e.* a unital normed algebra. Denote by A^* the dual space of A. We define the *state space* of A to be the set:

$$D(A, 1) = \left\{ f \in A^* : f(1) = 1 = \|f\| \right\}.$$

For each $a \in A$, the *numerical range* of a is defined by:

$$V(A, a) = \left\{ f(a) : f \in D(A, 1) \right\},$$

and the radius $v(a)$ of numerical range, called *numerical radius*, is given by $v(a) = \sup \{|\lambda| : \lambda \in V(A, a)\}$. The *spectrum* of a is denoted by $\operatorname{Sp}(A, a)$ and the *spectral radius* by $\rho(a)$. In a unital Banach algebra it is known that the spectrum $\operatorname{Sp}(A, a)$ is contained in the numerical range $V(A, a)$ for any $a \in A$.

For a normed space A, we denote by $S(A)$ the unit sphere of A, and

$$\pi = \left\{ (s, f) \in A \times A^* : x \in S(A) \text{ and } f \in S(A^*), f(x) = 1 \right\}.$$

For each $T \in \mathfrak{B}(A)$, the set of all bounded linear operators on a normed linear space A, we define the *spatial numerical range* $V(T)$ of T as

$$V(T) = \left\{ f(Tx) : (x, f) \in \pi \right\}.$$

If A is a Hilbert space, the classical numerical range $W(T)=\{\langle Tx, x\rangle : x\in S(A)\}$ coincides with $V(T)$. Here $\langle , \rangle$ denotes the scalar product. If A is a unital normed algebra, consider the left regular representation $a\to T_a$ of A in $\mathfrak{B}(A)$, we have (see Bonsall and Duncan [1]) $V(A, a)=V(T_a)$.

Given a bounded linear operator T on a Banach space A, we may regard T as an element of the unital Banach algebra $\mathfrak{B}(A)$, and so the numerical range is given by $V(\mathfrak{B}(A), T)$. In Bonsall and Duncan [1; Theorem 9, 4] and Stampfli and Williams [10; Theorem 6], they give a further result that $\overline{\text{Co}}\, V(T)=V(\mathfrak{B}(A), T)$, where $\overline{\text{Co}}$ means the closure of convex hull. If A is a Hilbert space, then $V(\mathfrak{B}(A), T)=\overline{W(T)}$.

An element $a\in A$ is said to be *convexoid* if $V(a)=\text{Co}\,\text{Sp}\,(a)$, where $V(a)=V(A, a)$, $\text{Sp}\,(a)=\text{Sp}\,(A, a)$. The element $a\in A$ is said to be *normaloid* if $\rho(a)=||a||$ and $a\in A$ is said to be *spectraloid* if $v(a)=\rho(a)$. It is easy to see that our notions of convexoid, normaloid and spectraloid elements extend the definitions given for the cases $\mathfrak{B}(H)$ by which are defined in Furuta [5] and Halmos [6]. Here and henceforth H denotes Hilbert space.

3. Numerical range of the tensor products of elements.

Denote by $A_1\otimes A_2$ the algebraic tensor product of normed algebras A_1 and A_2. Every element u in $A_1\otimes A_2$ can be expressed in the form $u=\sum_{i=1}^{k} x_i\otimes y_i$. There is a natural multiplication in $A_1\otimes A_2$ defined by

$$u_1\cdot u_2=\sum_{i=1}^{n}\sum_{j=1}^{m} x_i s_j\otimes y_i t_j$$

where $u_1=\sum_{i=1}^{n} x_i\otimes y_i$ and $u_2=\sum_{j=1}^{m} s_j\otimes t_j$ are elements in $A_1\otimes A_2$, and then $A_1\otimes A_2$ becomes an algebra under the natural multiplication. If A_1 and A_2 are *-algebras, then we can supply an involution on $A_1\otimes A_2$ by

$$\left(\sum_{i=1}^{n} x_i\otimes y_i\right)^* = \sum_{i=1}^{n} x_i^*\otimes y_i^* .$$

This * defined here is well defined (cf. Laursen [8]), and so $A_1\otimes A_2$ forms a *-algebra. There are several norms on the algebraic tensor product $A_1\otimes A_2$ of normed algebras A_1 and A_2. Among these norms we mention the *least cross norm* ε, defined as follows: for $u=\sum_{i=1}^{n} x_i\otimes y_i$ in $A_1\otimes A_2$,

$$||u||_\varepsilon=\sup\left|\sum_{i=1}^{n} x'(x_i)\, y'(y_i)\right|$$

where the sup is taken over all choices of x', y' in the unit balls of the dual spaces of A_1, A_2, and is independent of the choice of representation for u.

Another natural norm on $A_1 \otimes A_2$ is the *greatest cross norm* π, which is defined by the following manner: for $u \in A_1 \otimes A_2$ define

$$\|u\|_\pi = \inf \sum \|x_i\| \|y_i\|$$

where the inf is taken over all representations of $u = \sum_{i=1}^{n} x_i \otimes y_i$. A norm α on $A_1 \otimes A_2$ is called *reasonable*, if α is a cross norm on $A_1 \otimes A_2$ and the dual norm α' induced by the dual of $A_1 \otimes_\alpha A_2$ is a cross norm on $A_1^* \otimes A_2^*$. It is well known that π and ε are reasonable, and every norm α with $\varepsilon \leq \alpha \leq \pi$ is reasonable.

A cross norm or reasonable norm α on $A_1 \otimes A_2$ is called *uniform*, if for any pair $(T_1, T_2) \in \mathfrak{B}(A_1) \times \mathfrak{B}(A_2)$, we have

$$\sup\left\{\left\|(T_1 \otimes T_2)\, u\right\|_\alpha ;\ \|u\|_\alpha \leq 1,\ u \in A_1 \otimes_\alpha A_2\right\} \leq \|T_1\| \|T_2\| .$$

The greatest and smallest reasonable norm π and ε are uniform, and if α is a reasonable norm, so is α' (cf. Ichinose [7; p. 129]).

In this section we assume that α is a *compatible reasonable norm* on $A_1 \otimes A_2$. This means that

$$\alpha(u_1 \cdot u_2) \leq \alpha(u_1)\, \alpha(u_2), \qquad \text{for all } u_1, u_2 \in A_1 \otimes A_2 ,$$

so that $A_1 \otimes_\alpha A_2$ forms a normed algebra. The greatest cross norm π is always compatible with multiplication, there are some examples of algebras in which the least cross norm ε is not compatible with multiplication. We denote $A_1 \hat{\otimes}_\alpha A_2$ to be the completion of $A_1 \otimes_\alpha A_2$ with the compatible reasonable norm α.

The following two propositions are easy to see and may be known in the product states and product functionals, for convenient which we state as following.

PROPOSITION 3.1. *Let A_1, A_2 be unital normed algebras, then*

$$D(A_1, 1) \otimes D(A_2, 1) \subseteq D(A_1 \otimes_\alpha A_2,\ 1 \otimes 1) .$$

Furthermore

$$\overline{Co}\Big(D(A_1, 1) \otimes D(A_2, 1) \subseteq D(A_1 \otimes_\alpha A_2, 1 \otimes 1\Big) ,$$

where $D(A_1 \otimes_\alpha A_2,\ 1 \otimes 1)$ is the state space of $A_1 \otimes_\alpha A_2$, the closure is taken in weak-topology in $(A_1 \otimes_\alpha A_2)^*$.*

PROPOSITION 3.2. *Let A_1 and A_2 be two unital Banach algebras,*

$x \in A_1$, $y \in A_2$, $x \otimes y \in A_1 \otimes_\alpha A_2$. Then, for a compact set E of complex numbers, $Co(E)$ is compact and so $\overline{Co}(E)=Co(E)$. Also $V(x)\cdot V(y)$ is compact. So $\overline{Co}(V(x)\cdot V(y))=Co(V(x)\cdot V(y))\subseteq V(x\otimes y)$.

If α is a reasonable compatible norm on $A_1 \otimes A_2$, then we have:

THEOREM 3.3. *Let A_1 and A_2 be unital Banach algebras and α a uniform compatible norm on $A_1 \otimes A_2$. Then*

$$Sp(x\otimes y) = Sp(x)\, Sp(y)\,.$$

PROOF. Let $x \in A_1$, $y \in A_2$ and T_x, T_y be left regular representations of A_1, A_2 in $\mathfrak{B}(A_1)$, $\mathfrak{B}(A_2)$ respectively. Evidently, $T_{x\otimes y}=T_x \otimes T_y \in \mathfrak{B}(A_1 \otimes_\alpha A_2)$. Since α is a reasonable norm, $T_x \otimes T_y$ is a bounded operator on $A_1 \otimes_\alpha A_2$ and so $T_{x\otimes y}$ coincides algebraically with $T_x \otimes T_y$. Therefore $T_x \otimes T_y$ can be extended continuously to the completion $A_1 \hat{\otimes}_\alpha A_2$ of $A_1 \otimes_\alpha A_2$. We denote by $\widetilde{T_x \otimes_\alpha T_y}$, the extension of $T_x \otimes T_y$. By virtue of Theorem 1.9 and Theorem 4.3 in Ichinose [7], we have

$$\mathrm{Sp}(T_{x\otimes y}) = \mathrm{Sp}(\widetilde{T_x \otimes_\alpha T_y}) = \mathrm{Sp}(T_x)\,\mathrm{Sp}(T_y)\,.$$

Since $\mathrm{Sp}(T_x)=\mathrm{Sp}(x)$ and $\mathrm{Sp}(T_y)=\mathrm{Sp}(y)$ (cf. [7] Theorem 1.6.9), we have

$$\begin{aligned}\mathrm{Sp}(x\otimes y) &= \mathrm{Sp}(T_{x\otimes y}) = \mathrm{Sp}(\widetilde{T_x \otimes_\alpha T_y}) = \mathrm{Sp}(T_x)\,\mathrm{Sp}(T_y) \\ &= \mathrm{Sp}(x)\,\mathrm{Sp}(y).\end{aligned}$$

Q. E. D.

It is known that in the tensor products of operators T and S on a complex Hilbert space, the relation

$$(*) \qquad \overline{W}(T\otimes S) = \mathrm{Co}\Big(W(T)\cdot W(B)\Big)$$

need not be always true (cf. Saito [9]).

It is natural to ask when the relation (*) holds for the tensor products of elements of Banach algebras, that is; when does the relation

$$(**) \qquad V(x\otimes y) = \mathrm{Co}\Big(V(x)\cdot V(y)\Big)$$

hold for the elements x, y of Banach algebras?

We give necessary and sufficient conditions for (**) in the following

THEOREM 3.4. *Let A_1, A_2 be unital Banach algebras and α a uniform compatible norm on $A_1 \otimes A_2$. Suppose that $x \in A_1$ and $y \in A_2$ are convexoid. Then the element $x\otimes y \in A_1 \hat{\otimes}_\alpha A_2$ is convexoid if and only if the following identity*

$$(**) \qquad V(x\otimes y) = Co\Big(V(x)\cdot V(y)\Big)$$

holds.

PROOF. For the necessity, it sufficies to prove that

$$V(x\otimes y)\subseteq \mathrm{Co}\Big(V(x)\cdot V(y)\Big).$$

Since by Theorem 3. 3,

$$V(x\otimes y)=\mathrm{Co}\ \mathrm{Sp}(x\otimes y)=\mathrm{Co}\Big(\mathrm{Sp}(x)\ \mathrm{Sp}(y)\Big)\subseteq \mathrm{Co}\Big(V(x)\cdot V(y)\Big),$$

it follows from Proposition 3. 2 that we have

$$\mathrm{Co}\Big(V(x)\cdot V(y)\Big)=V(x\otimes y).$$

Conversely if $V(x\otimes y)=\mathrm{Co}(V(x)\ V(y))$, we will prove that $x\otimes y$ is convexoid. That is

$$V(x\otimes y)=\mathrm{Co}\ \mathrm{Sp}(x\otimes y).$$

This follows at once from the elementary observation that for sets E, F of complex numbers

$$\mathrm{Co}\Big(\mathrm{Co}(E)\ \mathrm{Co}(F)\Big)=\mathrm{Co}(F,E).$$

Q. E. D.

It is known that $\overline{W(T)}=V(\mathfrak{B}(H),T)$, for $T\in\mathfrak{B}(H)$. It follows that the convexoid, normaloid and spectraloid elements of the Banach algebra $\mathfrak{B}(H)$ are convexoid, normaloid and spectraloid operators on Hilbert space H (cf. Halmos [6], Furuta [5]).

COROLLARY 3. 5. (Saito [9]). *Let T, S be operators on a Hilbert space and convexoid. Then the following conditions are equivalent*:

(i) $\overline{W}(T\otimes S)=\overline{Co}(W(T)\ W(s))$

(ii) *$T\otimes S$ is convexoid.*

The results for the normaloid and spectraloid elements are immediately by virtue of [4]. For convenience, we state and prove the following theorem

THEOREM 3. 6. *Let A_1, A_2 be unital Banach algebras if $x\in A_1$ and $y\in A_2$ are normaloid, then $x\otimes y\in A_1\hat{\otimes}_\alpha A_2$ is also normaloid and vice versa.*

PROOF. It x and y are normaloid, then $\rho(x)=||x||$, $\rho(y)=||y||$. Since α is a cross norm,

$$\begin{aligned}\rho(x\otimes y)&=\lim_{n\to\infty}||(x\otimes y)^n||_\alpha^{\frac{1}{n}}=\lim_{n\to\infty}||x^n\otimes y^n||_\alpha^{\frac{1}{n}}\\&=\lim_{n\to\infty}(||x^n||^{\frac{1}{n}}||y^n||^{\frac{1}{n}})=\lim_{n\to\infty}||x^n||^{\frac{1}{n}}\lim_{n\to\infty}||y^n||^{\frac{1}{n}}\\&=\rho(x)\ \rho(y)=||x||\ ||y||=||x\otimes y||_\alpha.\end{aligned}$$

Conversely, if $\rho(x\otimes y)=\|x\otimes y\|_\alpha$, $\rho(x)\,\rho(y)=\|x\|\,\|y\|$, and $\rho(x)\leq\|x\|$, $\rho(y)\leq\|y\|$, then $\rho(x)=\|x\|$ and $\rho(y)=\|y\|$, i. e. x and y are normaloid.

THEOREM 3.7. *Let A_1 and A_2 be unital Banach algebras. If $x\in A_1$ and $y\in A_2$ are spectraloid satisfying $V(x\otimes y)=Co(V(x)\cdot V(y))$ then $x\otimes y\in A_1\hat{\otimes}_\alpha A_2$ is spectraloid.*

PROOF. Since $V(x)\,V(y)\subseteq\{\lambda : |\lambda|\leq v(x)\,x(y)\}=D$,

$$\mathrm{Co}\Big(V(x)\cdot V(y)\Big)\subseteq D\,,$$

and

$$V(x\otimes y)\subseteq D,\ \text{i. e.}\ v(x\otimes y)\leq v(x)\,v(y)\,.$$

By Proposition 3.2, $v(x)\,v(y)\leq v(x\otimes y)$. Consequently, $v(x\otimes y)=v(x)\,v(y)=\rho(x)\,\rho(y)=\rho(x\otimes y)$. Q. E. D.

4. The joint convexoidity of an n-tuple of operators on Hilbert spaces.

In this section we investigate the joint convexoidity of an n-tuple of operators T_i $(1\leq i\leq n)$ on Hilbert spaces. Let $\{H_i\}_{i=1}^n$ be Hilbert spaces, I_i be the identity operator on H_i, A_i be an arbitrary bounded linear operator on H_i $(1\leq i\leq n)$. We introduce the operators T_i $(i=1, 2, \cdots, n)$ of tensor products acting on the tensor product space $H_1\otimes H_2\otimes\cdots\otimes H_n$ defined by

$$(1)\qquad T_i=I_1\otimes\cdots\otimes I_{i-1}\otimes A_i\otimes I_{i+1}\otimes\cdots\otimes I_n$$

$(n=1, 2, \cdots, n)$. *The joint numerical range* of T_i $(i=1, 2, \cdots, n)$ is defined to be the set of n-tuple $z=(z_1, \cdots, z_n)$ in C^n given by

$$W(T_1, T_2, \cdots, T_n)=\Big\{(\langle T_1 u, u\rangle, \cdots, \langle T_n u, u\rangle)\,;$$

$$u \text{ is a unit vector in } H_1\otimes\cdots\otimes H_n\Big\}\,.$$

The *joint spectrum* of $T_1, T_2, \cdots T_n$ is a subset in C^n, denoted by $\mathrm{Sp}(T_1, \cdots, T_n)$ which is explaining as following:

Let $\mathfrak{A}$ be the set of all double (or second) commutants of $T_1, \cdots, T_n$, that is the set of all operators on $H_1\otimes\cdots\otimes H_n$ that commute with every operator which commutes with every T_i. Since the operators $T_1, \cdots, T_n$ commute with each other, $\mathfrak{A}$ is a commutative Banach algebra. A complex vector $z=(z_1, \cdots, z_n)$ of C^n belongs to the *joint spectrum* $\mathrm{Sp}(T_1, \cdots, T_n)$ of $T_1, \cdots, T_n$ if and only if for any operator $B_1, \cdots, B_n$ in $\mathfrak{A}$, the following relation holds

$$\sum_{i=1}^n B_i(T_i-z_i)\neq I_1\otimes\cdots\otimes I_n$$

In Dash and Schechter [3] they proved that the joint spectrum of $T_1, \cdots, T_n$ is given by

$$\text{(a)} \qquad \mathrm{Sp}(T_1, \cdots, T_n) = \prod_{i=1}^{n} \mathrm{Sp}(T_i)$$

and

$$\mathrm{Sp}(T_i) = \mathrm{Sp}(A_i)\,.$$

Furthermore Dash proved in [2] that

$$\text{(b)} \qquad W(T_1, \cdots, T_n) = \prod_{i=1}^{n} W(T_i) = \prod_{i=1}^{n} W(A_i)$$

is convex and contains $\mathrm{Sp}(T_1, \cdots, T_n)$.

We say that an n-tuple of operators $T_1, \cdots, T_n$ is *joint convexoid* if

$$\mathrm{Co}\,\mathrm{Sp}(T_1, \cdots, T_n) = \overline{W}(T_1, \cdots, T_n)\,.$$

By (a) and (b) we see that the joint convexoidity follows from the following identity:

$$\mathrm{Co}\Big(\prod_{i=1}^{n} \mathrm{Sp}(\mathrm{T}_i)\Big) = \prod_{i=1}^{n} \overline{W}(T_i)\,.$$

We will give a necessary and sufficient condition for the joint convexoidity of an n-tuple of operators $T_1, \cdots, T_n$ which we state as follows

THEOREM 4.1. *An n-tuple of operators $T_1, \cdots, T_n$ on $H_1 \otimes \cdots \otimes H_n$ given by* (1) *at the first paragraph of this section is joint convexoid if and only if each T_i $(1 \leq i \leq n)$ is convexoid.*

PROOF. For necessity, we assume that an n-tuple of operators $T_1, \cdots, T_n$ is joint convexoid, then we have

$$\prod_{i=1}^{n} \overline{W}(T_i) = \mathrm{Co}\Big(\prod_{i=1}^{n} \mathrm{Sp}(T_i)\Big)\,.$$

It follows from $\mathrm{Co}\Big(\prod_{i=1}^{n} \mathrm{Sp}(T_i)\Big) \subseteq \prod_{i=1}^{n} \mathrm{Co}\,\mathrm{Sp}(T_i)$ that we have

$$\overline{W}(T_i) \subseteq \mathrm{Co}\,\mathrm{Sp}(T_i) \qquad \text{for each } i\,.$$

but since $\mathrm{Co}\,\mathrm{Sp}(T_i) \subseteq \overline{W}(T_i)$, we have

$$\overline{W}(T_i) = \mathrm{Co}\,\mathrm{Sp}(T_i)\,,$$

it follows that each T_i is convexoid.

Conversely, since

$$D = \Big\{z = (z_1, \cdots, z_n) \in \boldsymbol{C}^n\,;\ |z - \lambda| \leq r, r \in R, \lambda \in \boldsymbol{C}^n\Big\}$$

is a polydisk containing $\prod_{i=1}^{n} \mathrm{Sp}(T_i) = \mathrm{Sp}(T_1, \cdots, T_n) = \prod_{i=1}^{n} \mathrm{Sp}(A_i)$, thus we have

$$\left(\sum_{i=1}^{n} |\rho(A_i) - \lambda_i|^2\right)^{\frac{1}{2}} \leq r ,$$

where $\rho(A_i)$ is the spectral radius of A_i. Now if each T_i is convexoid, then $\mathrm{Co}\,\mathrm{Sp}(T_i) = \overline{W}(T_i)$, and $\sup_{\|f_i\|=1} \{|\langle A_i f_i, f_i\rangle|\} = \rho(A_i)$, we have

$$\left(\sum_{i=1}^{n} |\langle (A_i - \lambda_i) f_i, f_i\rangle|^2\right)^{\frac{1}{2}} \leq \left(\sum_{i=1}^{n} |\rho(A_i) - \lambda_i|^2\right)^{\frac{1}{2}} \leq r .$$

Since

$$\mathrm{Co}\,\mathrm{Sp}(T_1, \cdots, T_n)$$

is the intersection of all such polydisk containing $\mathrm{Sp}(T_1, \cdots, T_n)$, it follows that

$$\overline{W}(T_1, \cdots, T_n) \subseteq \mathrm{Co}\,\mathrm{Sp}(T_1, \cdots, T_n) .$$

Consequently

$$\mathrm{Co}\,\mathrm{Sp}(T_1, \cdots, T_n) = \overline{W}(T_1, \cdots, T_n) .$$

This shows that the n-tuple of operators $T_1, \cdots, T_n$ is joint convexoid.

Q. E. D.

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