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Some generating functions of modified Laguerre polynomials

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1. Introduction:

We consider in this paper some generating functions of modified Laguerre polynomials of the partial differential equation

$$L\left(x, \frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}\right)u = x\frac{\partial^2 u}{\partial x^2}+(1-x)\frac{\partial u}{\partial x}-y\frac{\partial^2 u}{\partial x\partial y}-z\frac{\partial^2 u}{\partial x\partial z}+z\frac{\partial u}{\partial z}=0$$

which is derived from the linear differential equation

$$L\left(x, \frac{d}{dx}, n, \beta\right)f_n^{(\beta)}(x)=\left[x\frac{d^2}{dx^2}+(1-x-n-\beta)\frac{d}{dx}+n\right]\cdot f_n^{(\beta)}(x)=0$$

by replacing β by $y\frac{\partial}{\partial y}$, n by $z\frac{\partial}{\partial z}$, $f_n^{(\beta)}(x)$ by $u(x, y, z)$. In this case, we find the infinitesimal operators A_{ij}, $i=1, 2$; $j=1, 2, 3$ which generate a Lie algebra. Therefore, we get certain two parameters generating functions of modified Laguerre polynomials. We state such generating functions as follows:

$$(1.1)\quad \exp\left[xt_2+\frac{1}{wt_1}(t_1+t_2-1)\right](1-t_1-t_2)^{-n-\beta}\cdot f_n^{(\beta)}\left\{\left[x+\frac{1}{w}\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\right](1-t_1-t_2)\right\}$$
$$=\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{l=0}^{\infty}\frac{(w)^{-l}}{l!}\sum_{k=0}^{\infty}\frac{(-w)^{-k}}{k!}(\beta-k)_m(n-l+1)_p\cdot f_{n-l+p}^{(\beta-k+m)}(x)\,t_1^{m-k}t_2^{p-l}.$$

$$(1.2)\quad \exp\frac{t_1}{w}f_n^{(\beta)}\left[x-\frac{1}{w}(t_1+t_2)\right]=\sum_{l=0}^{\infty}\frac{(-w)^{-l}}{l!}\sum_{k=0}^{\infty}\frac{(w)^{-k}}{k!}f_{n-l}^{(\beta-k)}(x)\,t_1^k t_2^l.$$

$$(1.3)\qquad \exp\left[xt_2+\frac{t_1}{w}(1-t_2)\right](1-t_2)^{-n-\beta}f_n^{(\beta)}\left[\left(x-\frac{t_1}{w}\right)(1-t_2)\right]$$
$$=\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{k=0}^{\infty}\frac{(w)^{-k}}{k!}(n+1)_p f_{n+p}^{(\beta-k)}(x)\,t_1^k t_2^p\,.$$

$$(1.4)\qquad (1-t_1)^{-n-\beta}f_n^{(\beta)}\left[\left(x-\frac{t_2}{w}\right)(1-t_1)\right]$$
$$=\sum_{m=0}^{\infty}\frac{1}{m1}\sum_{l=0}^{\infty}\frac{(-w)^{-1}}{l!}(\beta)_m f_{n-l}^{(\beta+m)}(x)\,t_1^m t_1^l\,.$$

These generating functions do not seem to appear before, and contain six cases of one parameter generating functions in (I, II). Some other special cases are shown in 4 of this paper. If we change the order of A_{i1}, A_{i2}, A_{i3}, $i=1.2$, we can get other types of two parameters generating functions of Modified Laguerre polynomials. The main result of this paper is in 4 and there we determine all class of generating functions.

2. Linear differential operators

DEFINITION: Modified Laquerre polynomial is defined for all x, β and n by

$$(2.1)\qquad f_n^{(\beta)}(x)=(-1)^n L_n^{-n-\beta}(x)=\frac{(\beta)_n}{n!}\,{}_1F_1\left[\begin{matrix}-n\\1-n-\beta\end{matrix};\,x\right]$$

which is a solution of the ordinary differential equation

$$(2.2)\qquad x\frac{d^2}{dx^2}f_n^{(\beta)}(x)+(1-x-n-\beta)\frac{d}{dx}f_n^{(\beta)}(x)+nf_n^{(\beta)}(x)=0\,.$$

By replacing $\frac{d}{dx}$ by $\frac{\partial}{\partial x}$, β by $y\frac{\partial}{\partial y}$, n by $z\frac{\partial}{\partial z}$ and $f_n^{(\beta)}(x)$ by $u(x,y,z)$ $=f_n^{(\beta)}(x)\,y^\beta z^n$, the differential equation (2. 2) reduces to

$$(2.3)\qquad x\frac{\partial^2 u}{\partial x^2}+(1-x)\frac{\partial u}{\partial x}-y\frac{\partial^2 u}{\partial x\partial y}-z\frac{\partial^2 u}{\partial x\partial z}+z\frac{\partial u}{\partial z}=0\,.$$

From (2. 3), we define the infinitesimal operators A_{ij} ($i=1,2,\ j=1,2,3$)

$$A_{ij}=A_{ij}^{(1)}\frac{\partial}{\partial x}+A_{ij}^{(2)}\frac{\partial}{\partial y}+A_{ij}^{(3)}\frac{\partial}{\partial z}+A_{ij}^{(0)}\qquad i=1,2,\ j=1,2,3$$

of first order linear differential operators as

$$(2.4)\qquad A_{11}=y\frac{\partial}{\partial y}$$

$$A_{12}=y^{-1}\frac{\partial}{\partial x}-y^{-1}$$

$$A_{13} = xy\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} - yz\frac{\partial}{\partial z}$$

$$A_{21} = z\frac{\partial}{\partial z}$$

$$A_{22} = z^{-1}\frac{\partial}{\partial x}$$

$$A_{23} = xz\frac{\partial}{\partial x} - yz\frac{\partial}{\partial y} - z^2\frac{\partial}{\partial z} - xz$$

which satisfy the following rules:

$$A_{11}\left[f_n^{(\beta)}(x)\,y^\beta z^n\right] = \beta f_n^{(\beta)}(x)\,y^\beta z^n .$$

$$A_{12}\left[f_n^{(\beta)}(x)\,y^\beta z^n\right] = -f_n^{(\beta-1)}(x)\,y^{\beta-1} z^n .$$

$$A_{13}\left[f_n^{(\beta)}(x)\,y^\beta z^n\right] = -\beta f_n^{(\beta+1)}(x)\,y^{\beta+1} z^n .$$

$$A_{21}\left[f_n^{(\beta)}(x)\,y^\beta z^n\right] = n f_n^{(\beta)}(x)\,y^\beta z^n .$$

$$A_{22}\left[f_n^{(\beta)}(x)\,y^\beta z^n\right] = f_{n-1}^{(\beta)}(x)\,y^\beta z^{n-1} .$$

$$A_{23}\left[f_n^{(\beta)}(x)\,y^\beta z^n\right] = -(n+1)\,f_{n+1}^{(\beta)}\,y^\beta z^{n+1} .$$

3. Lie algebra

We get easily the following theorem.

THEOREM 1. The set $\{1, A_{ij}\,(i=1, 2, j=1, 2, 3)\}$ generates a Lie algebra $\mathscr{L}$ with commutator rules

$$[A_{ij}, A_{kl}] = 0 \qquad i \neq k \quad \text{for } j,\ l = 1, 2, 3 ,$$

$$[A_{i1}, A_{i2}] = -A_{i2} \qquad i = 1, 2$$

$$[A_{i1}, A_{i3}] = A_{i3} \qquad i = 1, 2 ,$$

$$[A_{12}, A_{13}] = [A_{23}, A_{22}] = 1$$

and

$\{1, A_{ij}\,(j=1, 2, 3)\}$ $i=1, 2$ generate a sub-algebra of $\mathscr{L}$.

We shall consider the partial differential operator $L_i\,(i=1, 2)$ which can be defined by two forms

$$L_1 = A_{12}A_{13} - A_{11}$$

$$L_2 = A_{22}A_{23} + A_{21} - 1$$

which commutes with A_{ij} $i=1, 2,$ $j=1, 2, 3,$ *i. e.*

(3. 1) $\quad [A_{ij}, L_k]=0 \qquad i, k=1, 2, \; j=1, 2, 3\,.$

If $\phi_{ij}(x, y, z)$ is a solution of $A_{ij}\phi(x, y, z)=0$ and if we transform the form of each A_{ij} to B_{ij} such that

$$B_{ij}=A_{ij}^{(1)}\frac{\partial}{\partial x}+A_{ij}^{(2)}\frac{\partial}{\partial y}+A_{ij}^{(3)}\frac{\partial}{\partial z}$$

then $B_{ij}=\phi_{ij}^{-1}(x, y, z)\, A_{ij}\,\phi_{ij}(x, y, z)$. Finally we transform each B_{ij} to $D=\frac{\partial}{\partial x}$ by change of variables from x, y, z to X, Y, Z.
By means of Taylor's Theorem, we have

$$\begin{aligned} e^{a_{ij}A_{ij}}f(x, y, z) &= \phi_{ij}(x, y, z)\, e^{a_{ij}B_{ij}}\left[\phi_{ij}^{-1}(x, y, z) f(x, y, z)\right] \\ &= \phi_{ij}(x, y, z)\, e^{a_{ij}\frac{\partial}{\partial x}}\left[F_{ij}(X, Y, Z)\right] \\ &= \phi_{ij}(x, y, z)\, F_{ij}\left[X+a_{ij}, Y, Z\right] \\ &= \phi_{ij}(x, y, z)\, g_{ij}(a_{ij}, x, y, z) \end{aligned}$$

where a_{ij} $(i=1, 2,\; j=1, 2, 3)$ are constants.

We shall introduce some functions as follows.

1) $\phi_{11}(x, y, z)=xz\,, \quad x=X+Y\,, \quad y=e^{X-Y}\,, \quad z=Y-Z$

$$e^{a_{11}A_{11}}u(x, y, z)=u(x, e^{a_{11}}y, z)\,.$$

2) $\phi_{21}(x, y, z)=xy\,, \quad x=Y+Z\,, \quad y=Y-Z\,, \quad z=e^{X-Y}$

$$e^{a_{21}A_{21}}u(x, y, z)=u(x, y, e^{a_{21}}z)\,.$$

3) $\phi_{12}=yze^{x}\,, \quad x=\dfrac{X-Y}{Y+Z}\,, \quad y=Y+Z\,, \quad z=Y-Z\,,$

$$e^{a_{12}A_{12}}u(x, y, z)=e^{-\frac{a_{12}}{y}}u\left(\frac{a_{12}}{y}+x, y, z\right).$$

4) $\phi_{22}(x, y, z)=yz \quad x=\dfrac{X-Y}{Y-Z}\,, \quad y=Y+Z\,, \quad z=Y-Z$

$$e^{a_{22}A_{22}}u(x, y, z)=u\left(\frac{a_{22}}{y}+x, y, z\right).$$

5) $\phi_{13}(x, y, z)=x^2yz\,, \quad x=(X-Y)\,Z\,, \quad y=\dfrac{1}{X-Y}\,, \quad z=\dfrac{Y}{(X-Y)\,Z^2}$

$$e^{a_{13}A_{13}}u(x, y, z)=u\left(x(1+a_{13}y), \frac{y}{1+a_{13}y}, \frac{z}{1+a_{13}y}\right).$$

6) $\phi_{23}(x,y,z)=x^2yze^x, \quad x=(X-Y)Z, \quad y=\dfrac{Y}{(X-Y)Z^2}, \quad z=\dfrac{1}{X-Y}$

$$e^{a_{23}A_{23}}u(x,y,z)=e^{-a_{23}xz}u\left(x(1+a_{23}z),\ \frac{y}{1+a_{23}z},\ \frac{z}{1+a_{23}z}\right).$$

From above stated functions, we easily get

$$\text{(3. 2)} \qquad \exp\left[\sum_{i=1}^{2}(a_{i3}A_{i3}+a_{i2}A_{i2}+a_{i1}A_{i1})\right]f(x,y,z)$$
$$=\exp\left[-a_{23}xz-\frac{a_{12}}{y}(1+a_{13}y+a_{23}yz)\right]f(\xi,\eta,\zeta)$$

where

$$\xi=\left(\frac{a_{12}}{y}+\frac{a_{22}}{z}+x\right)(1+a_{13}y+a_{23}z)$$

$$\eta=\frac{e^{a_{11}}y}{1+a_{13}y+a_{23}z}$$

$$\zeta=\frac{e^{a_{21}}z}{1+a_{13}y+a_{23}z}$$

and the order of A_{i3}, A_{i2}, A_{i1}. can not be changed for $i=1,2$, respectively.

4. Gernerating functions

From (2. 3), $u(x,y,z)=f_n^{(\beta)}(x)y^\beta z^n$ is a solution of the systems

$$\begin{cases}L_ju=0\\(A_{11}-\beta)u=0\end{cases},\quad \begin{cases}L_ju=0\\(A_{21}-n)u=0\end{cases},\quad \begin{cases}L_ju=0\\(A_{11}+A_{21}-\beta-n)u=0\end{cases},\quad j=1,2.$$

From (3. 1), we sasily get

$$SL_j\left(f_n^{(\beta)}(x)y^\beta z^n\right)=L_jS\left(f_n^{(\beta)}(x)y^\beta z^n\right)=0. \qquad j=1,2.$$

where

$$S=\exp\left[\sum_{i=1}^{2}(a_{i3}A_{i3}+a_{i2}A_{i2}+a_{i1}A_{i1})\right]$$

Therefore, the transformation $S[f_n^{(\beta)}(x)y^\beta z^n]$ is also annulled by L_j, $j=1,2$, and the order of A_{i3}, A_{i2}, A_{i1} can not be changed for each $i=1,2$, respectively.

By setting $a_{11}=a_{21}=0$ in (3. 2), we get

$$\text{(4. 1)} \qquad e^{a_{23}A_{23}}e^{a_{13}A_{13}}e^{a_{22}A_{22}}e^{a_{12}A_{12}}\left[f_n^{(\beta)}(x)y^\beta z^n\right]$$

$$= e^{-a_{23}xz} e^{-\frac{a_{12}}{y}(1+a_{13}z+a_{23}z)} (1+a_{13}y+a_{23}z)^{-n-\beta}$$

$$\cdot f_n^{(\beta)}\left[\left(\frac{a_{12}}{y}+\frac{a_{22}}{z}+x\right)(1+a_{13}y+a_{23}z)\right] y^\beta z^n$$

$$= \sum_{p=0}^{\infty}\frac{(a_{23})^p}{p!}\sum_{m=0}^{\infty}\frac{(a_{13})^m}{m!}\sum_{l=0}^{\infty}\frac{(a_{22})^l}{l!}\sum_{k=0}^{\infty}\frac{(a_{12})^k}{k!}(-1)^{k+m+p}(\beta-k)_m$$

$$\cdot (n-l+1)_p f_{n-l+p}^{(\beta-k+m)}(x)\, y^{\beta-k+m} z^{n-l+p}.$$

THEOREM 2: Every generating functions induced by the partial differential equation (2. 3)

$$L\left(x, \frac{\partial}{\partial x}, y\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}\right) u(x,y,z)$$

$$= x\frac{\partial^2 u}{\partial x^2}+(1-x)\frac{\partial u}{\partial x}-y\frac{\partial^2 u}{\partial x \partial y}-z\frac{\partial^2 u}{\partial x\partial z}+z\frac{\partial u}{\partial z}=0$$

are classified into four classes which are also devided by fifteen cases. The details are as follows:

(I) Class 1.
by setting $-y=t_1$, $-z=t_2$ in (4. 1)

Case 1. $a_{12}=a_{22}=-\frac{1}{w}$, $a_{13}=a_{23}=1$

$$\text{(4. 2)}\quad e^{xt_2} e^{(t_1+t_2-1)/wt_1}(1-t_1-t_2)^{-n-\beta} f_n^{(\beta)}\left\{\left[x+\frac{1}{w}\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\right](1-t_1-t_2)\right\}$$

$$= \sum_{p=0}^{\infty}\frac{1}{p!}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{l=0}^{\infty}\frac{(w)^{-l}}{l!}\sum_{k=0}^{\infty}\frac{(-w)^{-k}}{k!}(\beta-k)_m$$

$$\cdot (n-l+1)_p f_{n-l+p}^{(\beta-k+m)}(x)\, t_1^{m-k} t_2^{p-l}.$$

Case 2. $a_{12}=a_{22}=-\frac{1}{w}$, $a_{13}=1$, $a_{23}=0$ implies $p=0$.

$$\text{(4. 3)}\quad e^{\frac{t_1-1}{wt_1}}(1-t_1)^{-n-\beta} f_n^{(\beta)}\left\{\left[x+\frac{1}{w}\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\right](1-t_1)\right\}$$

$$= \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{l=0}^{\infty}\frac{(w)^{-l}}{l!}\sum_{k=0}^{\infty}\frac{(-w)^{-k}}{k!}(\beta-k)_m f_{n-l}^{(\beta-k+m)}(x)\, t_1^{m-k} t_2^{-l}.$$

Case 3. $a_{12}=a_{22}=-\frac{1}{w}$, $a_{13}=0$, $a_{23}=1$ implies $m=0$

$$\text{(4. 4)}\quad e^{xt_2} e^{\frac{t_2-1}{wt_1}}(1-t_2)^{-n-\beta} f_n^{(\beta)}\left\{\left[x+\frac{1}{w}\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\right](1-t_2)\right\}$$

$$= \sum_{p=0}^{\infty}\frac{1}{p!}\sum_{l=0}^{\infty}\frac{(w)^{-l}}{l!}\sum_{k=0}^{\infty}\frac{(-w)^{-k}}{k!}(n-l+1)_p f_{n-l+p}^{(\beta-k)}(x)\, t_1^{-k} t_2^{-l+p}.$$

Case 4. $a_{12}=-\dfrac{1}{w}$ $a_{22}=0$, $a_{13}=a_{23}=1$ implies $l=0$

(4.5) $$e^{xt_2}e^{(t_1+t_2-1)/wt_1}(1-t_1-t_2)^{-n-\beta}f_n^{(\beta)}\left[\left(x+\frac{1}{wt_1}\right)(1-t_1-t_2)\right.$$
$$=\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k=0}^{\infty}\frac{(-w)^{-k}}{k!}(\beta-k)_m(n+1)_p f_{n+p}^{(\beta-k+m)}(x)\,t_1^{m-k}t_2^p\,.$$

Case 5. $a_{12}=0$, $a_{22}=-\dfrac{1}{w}$, $a_{13}=a_{23}=1$ implies $k=0$

(4.6) $$e^{xt_2}(1-t_1-t_2)^{-n-\beta}f_n^{(\beta)}\left[\left(x+\frac{1}{wt_2}\right)(1-t_1-t_2)\right]$$
$$=\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{l=0}^{\infty}\frac{(w)^{-l}}{l!}(\beta)_m(n-l+1)_p f_{n-l+p}^{(\beta+m)}(x)\,t_1^m t_2^{p-l}\,.$$

Case 6. $a_{12}=a_{22}=0$, $a_{13}=a_{23}=1$ implies $k=l=0$

(4.7) $$e^{xt_2}(1-t_1-t_2)^{-n-\beta}f_n^{(\beta)}\left[x(1-t_1-t_2)\right]$$
$$=\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{m=0}^{\infty}\frac{1}{m!}(\beta)_m(n+1)_p f_{n+p}^{(\beta+m)}(x)\,t_1^m t_2^p\,.$$

Case 7. $a_{12}=-\dfrac{1}{w}$, $a_{22}=0$, $a_{13}=1$, $a_{23}=0$ implies $l=p=0$
the generating function is shown as [I, (6.4)].

Case 8. $a_{12}=0$, $a_{22}=-\dfrac{1}{w}$, $a_{13}=0$, $a_{23}=1$ implies $k=m=0$
the generating functions is shown as [II. ch. 3. (1.2)].

Case 9. $a_{12}=a_{22}=a_{23}=0$, $a_{23}=1$ implies $k=l=p=0$
the generating functions is shown as [I, (6.3)].

Case 10. $a_{12}=a_{22}=a_{13}=0$, $a_{23}=1$ implies $k=l=m=0$
the generating functions is shown as [II. ch. 3, (8)].

(II) Class 2.
by setting $y^{-1}=t_1$, $z^{-1}=t_2$

Case 11. $a_{12}=a_{13}=-\dfrac{1}{w}$, $a_{13}=a_{23}=0$ implies $m=p=0$

(4.8) $$e^{t_1/w}f_n^{(\beta)}\left[x-\frac{1}{w}(t_1+t_2)\right]$$
$$=\sum_{l=0}^{\infty}\frac{(-w)^{-l}}{l!}\sum_{k=0}^{\infty}\frac{(w)^{-k}}{k!}f_{n-l}^{(\beta-k)}(x)\,t_1^k t_2^l\,.$$

Case 12. $a_{12}=1$, $a_{22}=a_{13}=a_{23}=0$ implies $l=m=p=0$

the generating function is shown as [I, (6. 1)].

Case 13. $a_{22}=1$, $a_{12}=a_{13}=a_{23}=0$ implies $k=m=p=0$ the generating function is shown as [II, ch. 3. (7)].

(III) Class 3.

By setting $y^{-1}=t_1$, $-z=t_2$

Case 14. $a_{12}=-\frac{1}{w}$, $a_{22}=a_{13}=0$, $a_{23}=1$ implies $l=m=0$

$$e^{xt_2}e^{\frac{t_1}{w}(1-t_2)}(1-t_2)^{-n-\beta}f_n^{(\beta)}\left[\left(x-\frac{t_1}{w}\right)(1-t_2)\right]\right]$$
$$=\sum_{p=0}^{\infty}\frac{1}{p!}\sum_{k=0}^{\infty}\frac{(w)^{-k}}{k!}(n+1)^p f_{n+p}^{(\beta-k)}(x)\,t_1^k t_2^p\,. \tag{4.9}$$

(IV) Class 4.

By setting $-y=t_1$, $z^{-1}=t_2$

Case 15. $a_{12}=0$, $a_{22}=-\frac{1}{w}$, $a_{13}=1$, $a_{23}=0$ implies $k=p=0$

$$(1-t_1)^{-n-\beta}f_n^{(\beta)}\left[\left(x-\frac{t_2}{w}\right)(1-t_1)\right]$$
$$=\sum_{m=0}^{\infty}\frac{1}{m!}\sum_{l=0}^{\infty}\frac{(-w)^{-l}}{l!}(\beta)_m f_{n-l}^{(\beta+m)}(x)\,t_1^m t_2^l\,. \tag{4.10}$$

So we have found fifteen generating functions, these two parameters generating functions do not seem to appear before, in which six cases appear in [I, II] as one parameter generating functions.

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