Hokkaido Mathematical Journal Vol. 8 (1979) p. 95-102

Q-projective transformations of an almost quaternion manifold

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(Received June 2, 1978; Revised August 25, 1978)

Introduction.

The transformations of an almost quaternion manifold preserving the quaternion structure have been investigated by M. Obata, S. Ishihara, Y. Takemura and others. M. Obata ([6]) obtained the conditions for such transformations to be affine transformations with respect to a certain affine connection, S. Ishihara ([4]) proved some results concerning infinitesimal transformations preserving the quaternion structure of a quaternion Kählerian manifold, and automorphism groups of quaternion Kählerian manifolds were studied by Y. Takemura ([7]).

In this paper, we shall study the transformations which preserve a certain kind of curves on an almost quaternion manifold or a quaternion Kählerian manifold. They are analogous to projective transformations of a Riemannian manifold or holomorphically projective transformations of a Kählerian manifold.

§ 1. Preliminaries.

Let (M, V) be an almost quaternion manifold[1] of dimension $4m$, that is, a manifold M which admits a 3-dimensional vector bundle V consisting of tensors of type (1, 1) over M satisying the following condition: In any coordinate neighborhood U of M, there is a local base $\{J_1, J_2, J_3\}$ of V such that

$$J_p J_q = -\delta_{pq} I + \delta_{pqr} J_r \quad (1.1)$$
[2]

1) Throughout this paper, we assume that manifolds are connected and every geometric object is differentiable and of class C^∞.

2) We use the summation convention. For example, we denote $\sum_{p=1}^{3} J_p \otimes J_p$ by $J_p \otimes J_p$ or $\sum_{i=1}^{4m} g(e_i, e_i)$ by $g(e_i, e_i)$. And sum indices run over the following ranges:

$$p, q, r, s = 1, 2, 3;$$
$$a, b, c = 0, 1, 2, 3;$$
$$h, i, j, k, l = 1, \cdots, 4m.$$

where I, δ_{pq} and δ_{pqr} denote the identity tensor field of type (1, 1) on M, the Kronecker's delta and the generalized Kronecker's delta defined by

$$\delta_{pqr} = \det \begin{pmatrix} \delta_{1p} & \delta_{1q} & \delta_{1r} \\ \delta_{2p} & \delta_{2q} & \delta_{2r} \\ \delta_{3p} & \delta_{3q} & \delta_{3r} \end{pmatrix},$$

respectively. Such a local base $\{J_1, J_2, J_3\}$ of V is called a canonical local base of V in U. And it is well known that $\Lambda = J_p \otimes J_p$ is a tensor field of type (2, 2) defined globally on M ([3]).

We now consider an affine connection Γ and a curve $x(t)$ on (M, V) satisfying

$$\nabla_{\dot{x}(t)} \dot{x}(t) = \phi_a(t) J_a \dot{x}(t) \tag{1.2}$$

where $\dot{x}(t)$ is the vector tangent to $x(t)$, $\phi_a(t)$ $(a=0, 1, 2, 3)$ are certain functions of the parameter t, $J_0 = I$ and ∇ is an operator of covariant differentiation with respect to Γ. Such a curve is called a Q-planar curve with respect to Γ. And two affine connections Γ and Γ' on (M, V) are called to be Q-projectively related if they have all Q-planar curves in common. In [1] and [2], the present author proved

THEOREM A ([1], [2]). *In an almost quaternion manifold (M, V) of dimension $4m$ $(\geqq 8)$, the following conditions are equivalent to each other*:

(1) *Affine connections Γ and Γ' on (M, V) are Q-projectively related.*

(2) *There exist local 1-forms ψ_a $(a=0, 1, 2, 3)$ on M satisfying*

$$S(X, Y) + S(Y, X) = \psi_a(X) J_a Y + \psi_a(Y) J_a X$$

for any vector fields X and Y on M.

(3) *There exist local functions η_a $(a=0, 1, 2, 3)$ on the tangent bundle of M such that*

$$Q(X) = \eta_a(X) J_a X$$

for any vector field X on M, where ∇ and ∇' are operators of covariant differentiation with respect to Γ and Γ' respectively, $S(X, Y) = \nabla'_X Y - \nabla_X Y$ and $Q(X) = S(X, X)$.

Next, if a transformation f of M onto itself leaves the bundle V invariant, then f is called a Q-transformation of (M, V) ([4]). And a vector field X on M is called an infinitesimal Q-transformation of (M, V) if $\exp(tX)$ $(|t| < \varepsilon$, ε being a certain positive number) is a Q-transformation of (M, V). S. Ishihara proved

THEOREM B ([4]). *Let f be a transformation of an almost quaternion manifold (M, V) onto itself. Then the following conditions are equivalent to each other*:

(1) *f is a Q-transformation of (M, V).*

(2) *f preserves the tensor field Λ.*

(3) *$f^*\bar{J}_p = s_{pq} J_q$ in $U \cap f^{-1}U'$,*

where U and U' are any coordinate neighborhoods of M such that $U \cap f^{-1}U'$ is not empty, $\{J_1, J_2, J_3\}$ and $\{\bar{J}_1, \bar{J}_2, \bar{J}_3\}$ are local canonical bases of V in U and U' respectively, $f^*\bar{J}_p$ denotes the tensor field induced by f from $\bar{J}_p$ and $(s_{pq}) \in SO(3)$ at each point in $U \cap f^{-1}U'$.

THEOREM C ([4]). *Let X be a vector field on an almost quaternion manifold (M, V). Then the following conditions are equivalent to each other*:

(1) *X is an infinitesimal Q-transformation of (M, V).*

(2) $\mathcal{L}_X \Lambda = 0$.

(3) *$\mathcal{L}_X J_p = \alpha_{pq} J_q$ and $\alpha_{pq} + \alpha_{qp} = 0$ in each coordinate neighborhood U, where $\mathcal{L}_X$ is the Lie derivative with respect to X, $\{J_1, J_2, J_3\}$ is a local canonical base of V in U and α_{pq} $(p, q = 1, 2, 3)$ are certain functions on U.*

§ 2. *Q*-projective transformations.

Let f and Γ be a transformation of an almost quaternion manifold (M, V) onto itself and an affine connection on M, respectively. If f maps any Q-planar curve with respect to Γ into another one with respect to Γ, f is called a Q-projective transformation with respect to Γ of (M, V). Now let $x(t)$ be a Q-planar curve such that

$$\nabla_{\dot{x}(t)} \dot{x}(t) = \phi_a(t) J_a \dot{x}(t), \quad x(t_0) = x_0 \quad \text{and} \quad \dot{x}(t_0) = u \tag{2.1}$$

for a point $x_0 \in M$, a tangent vector u at x_0 and functions $\phi_a(t)$ $(a = 0, 1, 2, 3)$ of the parameter t, where ∇ and $\{J_1, J_2, J_3\}$ denote the operator of covariant differentiation with respect to Γ and a canonical local base of V in the coordinate neighborhood U of M containing x_0, respectively.

Assume that f is a Q-projective transformation with respect to Γ of (M, V) and put $\bar{x}(t) = f(x(t))$. Then, since $\bar{x}(t)$ is a Q-planar curve with respect to Γ, we have

$$\nabla_{\dot{\bar{x}}(t)} \dot{\bar{x}}(t) = \bar{\phi}_a \bar{J}_a \dot{\bar{x}}(t) \tag{2.2}$$

for certain functions $\bar{\phi}_a$ $(a = 0, 1, 2, 3)$ depending upon $x(t)$, where $\bar{J}_0 = I$ and $\{\bar{J}_1, \bar{J}_2, \bar{J}_3\}$ is the canonical local base of V in a coordinate neighborhood U′ such that $f(x_0) \in U'$. Denoting by $\overset{*}{\nabla}$ the operator of covariant differentiation with respect to an affine connection induced from Γ by f, we have

$$f_*\left(\overset{*}{\nabla}_{\dot{x}(t)}\dot{x}(t)\right)=\nabla_{\dot{\bar{x}}(t)}\dot{\bar{x}}(t)\,. \tag{2. 3}$$

From (2. 2) and (2. 3), we have

$$\overset{*}{\nabla}_{\dot{x}(t)}\dot{x}(t)=\bar{\phi}_a(f^*\bar{J}_a)\,\dot{x}(t) \tag{2. 4}$$

Hence, from (2. 1) and (2, 4), we have

$$Q\left(\dot{x}(t)\right)=\left(\phi_a(t)\,J_a-\bar{\phi}_a(f^*\bar{J}_a)\right)\dot{x}(t)$$

where $Q(\dot{x}(t))=\nabla_{\dot{x}(t)}\dot{x}(t)-\overset{*}{\nabla}_{\dot{x}(t)}\dot{x}(t)$. Since $\phi_a(t)$ $(a=0, 1, 2, 3)$ are arbitrary functions of t, $u=\dot{x}(t_0)$ is an arbitrary vector in $T_{x_0}(M)$ and $(Q(\dot{x}(t)))_{t=t_0}$ depends upon u but not $\phi_a(t)$, we have

$$(f^*\bar{J}_a)\,u=\psi_{ab}(u)\,J_b u\,, \tag{2. 5}$$
$$Q(u)=\psi_a(u)\,J_a u$$

for any vector $u\in T_{x_0}(M)$ and certain functions ψ_{ab} and ψ_a $(a, b=0, 1, 2, 3)$ on $T_{x_0}(M)$. From (2, 5), we have

$$(\psi_{ab})\in\begin{pmatrix}1 & 0\\ 0 & SO(3)\end{pmatrix}$$

Therefore, from Theorems A and B, we can obtain

THEOREM 1. *Let (M, V) be an almost quaternion manifold of dimension $4m$ $(\geqq 8)$ with an affine connection Γ. Then, a transformation f of M onto itself is a Q-projective transformation with respect to Γ of (M, V) if and only if*

(1) *f is a Q-transformation of (M, V)*
and

(2) *Γ and the affine connection induced by f from Γ are Q-projectively related.*

Let X be a vector field on (M, V) with an affine connection Γ. If $\exp(tX)$ $(|t|<\varepsilon$, ε being a certain positive number) is a Q-projective transformation with respect to Γ of (M, V), X is called an infinitesimal Q-projective transformation with respect to Γ of (M, V). From Theorem 1, we can obtain

THEOREM 2. *Let (M, V) be an almost quaternion manifold of dimension $4m$ $(\geqq 8)$ with an affine connection Γ. Then, a vector field X on M is an infinitesimal Q-projective transformation with respect to Γ if and only if*

(1) *X is an infinitesimal Q-transformation of (M, V)*

and

(2) *X satisfies*

$$\mathcal{L}_X(\Gamma^h_{ji}+\Gamma^h_{ij})/2=\psi_{a,j}J_{a,i}{}^h+\psi_{a,i}J_{a,j}{}^h$$

for certain local 1-forms ψ_a $(a=0,1,2,3)$ on each coordinate neighborhood U, where Γ^h_{ji}, $\psi_{a,i}$ and $J_{a,i}{}^h$ denote coefficients of Γ, components of ψ_a and ones of a canonical local base $\{J_1, J_2, J_3\}$ of V in U with respect to local coordinates, respectively.

§ 3. Infinitesimal Q-projective transformations on a compact quaternion Kählerian manifold.

Let (M, g, V) be a quaternion Kählerian manifold of dimension $4m$ $(\geqq 8)$, that is, an almost quaternion manifold (M, V) which admits a Riemannian metric g satisfying

$$g(X, \phi Y)+g(\phi X, Y)=0\,, \tag{3.1}$$

$$\nabla_X \Lambda=0 \tag{3.2}$$

for any cross-section ϕ of V and any vector fields X and Y on M, where Λ is the tensor field mentioned in § 1. And (3.2) is equivalent that there exist local 1-forms β_{pq} $(p, q=1, 2, 3)$ such that

$$\nabla_X J_p=\beta_{pq}(X)\,J_q \quad \text{and} \quad \beta_{pq}+\beta_{qp}=0 \tag{3.3}$$

for any vector field X and a canonical local base $\{J_1, J_2, J_3\}$ of V.

Let X be an infinitesimal Q-projective transformation with respect to the Riemannian connection of (M, g, V) and $\left\{ {h \atop ji} \right\}$ be local coefficients of its connection. From Theorem 3 in [1] and Theorem 2 in the present paper, we see that there exists a local 1-form η such that

$$\mathcal{L}_X\left\{ {h \atop ji} \right\}=I^h_j\eta_i+I^h_i\eta_j-\Lambda^{kh}_{ji}\eta_k-\Lambda^{kh}_{ij}\eta_k \tag{3.4}$$

where η_i and Λ^{kh}_{ji} are local components of η and Λ, respectively. On the other hand, we have known that

$$\mathcal{L}_X\left\{ {h \atop ji} \right\}=\nabla_j\nabla_i X^h+R_{kji}{}^h X^k \tag{3.5}$$

where $\{\partial/\partial x^1, \cdots, \partial/\partial x^{4m}\}$ is a local natural frame, $\nabla_i=\nabla_{\partial/\partial x^i}$ and $R_{kji}{}^h$ denote local components of the curvature tensor field of (M, g, V). Transvecting (3.4) and (3.5) by g^{ji}, we have

(3. 6) $$\nabla^k \nabla_k X^h + SX^h/4m = -4\eta^h,$$

because (M, g, V) is an Einstein space ([3]), where g_{ji}, (g^{ji}) and S denote local components of g, the inverse matrix of (g_{ji}) and the scalar curvature, respectively, $\nabla^k = g^{kj}\nabla_j$ and $\eta^h = g^{kh}\eta_k$. Contracting (3. 4) and (3. 5) for h and i, we have

(3. 7) $$\nabla_j \nabla_h X^h = 4(m+1)\eta_j.$$

Therefore, from (3. 6) and (3. 7), we have

$$\begin{aligned}\nabla^i \nabla_i ||X||^2/2 &= ||\nabla X||^2 + X^h \nabla^i \nabla_i X_h \\ &= ||\nabla X||^2 - S||X||^2/4m - 4X^h \eta_h \\ &= ||\nabla X||^2 - S||X||^2/4m + (\nabla_h X^h)^2/(m+1) \\ &\qquad - \nabla_h (X^h \nabla_i X^i)/(m+1),\end{aligned}$$

where $||X||^2 = g_{ji} X^j X^i$ and $||\nabla X||^2 = g_{kj} g_{ih} \nabla^k X^i \cdot \nabla^j X^h$. Assume that M is compact. Since a quaternion Kählerian manifold is orientable ([3]), we have

$$\int_M \left[||\nabla X||^2 - S||X||^2/4m + (\nabla_h X^h)^2/(m+1) \right] *1 = 0$$

where $*1$ is the volume element of (M, g, V). Thus, we can obtain

THEOREM 3. *Let (M, g, V) be a compact quaternion Kählerian manifold of dimension $4m$ $(\geqq 8)$. If the scalar curvature S is negative, an infinitesimal Q-projective transformation X of (M, g, V) is a zero vector field, and if S is vanishes, X is a parallel vector field.*

§ 4. Remarks.

REMARK 1. In [5], Y. Maeda obtained the following theorem:

THEOREM D. *Let (M, g, V) be a complete quaternion Kählerian manifold of dimension $4m$ $(\geqq 8)$. In order that (M, g, V) be isometric to the quaternion projective space with constant Q-sectional curvature $4K$ (>0), it is necessary and sufficient that (M, g, V) admits a non-trivial solution f of the following differential equations:*

$$\nabla_j \nabla_i f_h + K(2f_j g_{ih} + f_i g_{jh} + f_h g_{ji} - g_{ki} A_{jh}^{kl} f_l - g_{kh} A_{ji}^{kl} f_l) = 0,$$

and such a grad$f$ *is a non-trivial infinitesimal Q-transformation of (M, g, V), where $f_h = \partial f/\partial x^h$.*

Now let $\left\{ {h \atop ji} \right\}$ be a Riemannian-Christoffel's symbol induced from g. Then we have

$$\mathcal{L}_{\mathrm{grad} f}\left\{ {h \atop ji} \right\} = -2K(I^h_j f_i + I^h_i f_j - \Lambda^{kh}_{ji} f_k - \Lambda^{kh}_{ij} f_k)\,,$$

from which, it follows that such a grad$f$ is a non-trivial infinitesimal Q-projective transformation of (M, g, V).

REMARK 2. Let Γ be an affine connection on an almost quaternion manifold (M, V) of dimension $4m$ ($\geqq 8$). Γ is called a Q-connection on (M, V) if Γ satisfies (3. 2). By virture of Theorem 1. 3 in [6], we see that the affine connection induced from a Q-connection by a Q-transformation of (M, V) is a Q-connection. And it is easy to see that the set consisting of all infinitesimal Q-projective transformations with respect to a symmetric Q-connection of (M, V) is a Lie subalgebra of the Lie algebra consisting of all infinitesimal Q-transformations of (M, V).

REMARK 3. Let (M, V) be an almost quaternion manifold of dimension $4m$. If two symmetric Q-connections Γ and $\bar{\Gamma}$ are projectively related, that is, if there exists a 1-form ω on M such that

$$\bar{\nabla}_X Y = \nabla_X Y + \omega(X)\, Y + \omega(Y)\, X$$

for any vector fields X and Y, we see easily that Γ and $\bar{\Gamma}$ are affinely related, where $\bar{\nabla}$ and ∇ are operators of covariant differentiation with respect to $\bar{\Gamma}$ and Γ, respectively.

REMARK 4. Let (M, g, V) be a quaternion Kählerian manifold of dimension $4m$ and $\bar{g} = \exp(2\rho)\cdot g$ be a conformal change of g for a certain function ρ on M. Then, we have

$$\bar{\nabla}_X Y = \nabla_X Y + X(\rho)\, Y + Y(\rho)\, X - g(X, Y)\,\mathrm{grad}\,\rho \tag{4. 1}$$

for any vector fields X and Y on M, where $\bar{\nabla}$ and ∇ denote the operators of covariant differentiation with respect to the Riemannian connections induced from $\bar{g}$ and g, respectively, and grad ρ is a gradient vector field of ρ with respect to g, that is, a vector field such that

$$g\,(\mathrm{grad}\,\rho, X) = X(\rho)\,.$$

From (4. 1), we have

$$\begin{aligned}(\bar{\nabla}_X J_p)\, Y = (\nabla_X J_p)\, Y + (J_p Y)\,(\rho)\, X - Y(\rho)\, J_p X \\ - g(X, J_p Y)\,\mathrm{grad}\,\rho + g(X, Y)\, J_p\,\mathrm{grad}\,\rho\,.\end{aligned} \tag{4. 2}$$

Now assume that $m > 1$ and $(M, \bar{g}, V)$ is a quaternion Kählerian manifold. Let vectors $J_a u$ and $J_a v$ $(a = 0, 1, 2, 3)$ be mutually orthogonal with respect to g. Then, from (3. 3) and (4. 2), we have

$$\bar{\beta}_{pq}(u)\, J_q v = \beta_{pq}(u)\, J_q v + (J_q v)\,(\rho)\, u - v(\rho)\, J_p u\,,$$

from which, we see that ρ is constant, that is, $\bar{g}$ is a homothetic change of g.

Next assume that $m=1$ and $\{Y, J_1 Y, J_2 Y, J_3 Y\}$ is a local frame field on an arbitrary coordinate neighborhood of M which is orthonormal with respect to g. Then, from (4.2), we have

$$g\left((\bar{\nabla}_X J_p)\, Y, Y\right) = g\left((\nabla_X J_p)\, Y, Y\right),$$

$$\begin{aligned} g\left((\bar{\nabla}_X J_p)\, Y, J_q Y\right) = g\left((\nabla_X J_p)\, Y, J_q Y\right) &+ (J_p Y)(\rho)\, g(X, J_q Y) \\ &- (J_q Y)(\rho)\, g(X, J_p Y) \\ &+ \delta_{pqr}\left\{Y(\rho)\, g(X, J_r Y) - (J_r Y)(\rho)\, g(X, Y)\right\}, \end{aligned}$$

from which, we have

$$\bar{\nabla}_X J_p = \nabla_X J_p - \delta_{pqr} (J_r d\rho)(X)\, J_q .$$

Therefore, we see that $(M, \bar{g}, V)$ is a quaternion Kählerian manifold. Really, this fact is obvious because all orientable Riemannian manifolds of dimension 4 are quaternion Kählerian manifolds.

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