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Fixed points for mappings majorized by real functionals

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Abstract.

In this paper the author extends the Caristi-Kirk fixed point theorem to the multi-valued case, as well as give a generalization to the so-called "strictly p-nonexpansive mappings." Specialization yields a stronger form of Edelstein's fixed point theorem. A characterization of asymptotic regularity in terms of fixed point features of cluster points is made. Also given is a nonlinear generalization of Stein's theorem for spectral radius less than one. Furthermore, it is shown that the Browder-Göhde-Kirk fixed point theorem cannot be extended in a "natural way."

1. Introduction.

In recent papers [11], [12], [22] (see also [4], [5], [9], [15], [16], [29], [32]), Caristi and Kirk have shown the following interesting technical sharpenings of the Banach contraction mapping principle: Let (M, d) be a complete metric space and $f: M \to M$ an arbitrary mapping. Suppose that there exists a lower semi-continuous real functional $p: M \to [0, \infty)$ such that for each x in M,

$$d\big(x, f(x)\big) \leqslant p(x) - p\big(f(x)\big).$$

Then f has a fixed point. As pointed out by Kirk and Caristi [22], the strength of this result lies in fact that it typically applies to mappings f which need not be continuous. Caristi-Kirk's theorem includes the Banach contraction theorem as a very special case (by takin $p(x) = (1-k)^{-1} d(x, f(x))$ if $k < 1$ is a Lipschitz constant for f). Because this theorem can be applied to investigate the theory of normal solvability ([7], [8], [22], [23]) and of inward mappings ([11], [12]), Browder [9] pointed out this theorem may well become an important tool in the future development of nonlinear functional analysis. It is the purpose of the present paper to extend this result to

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the multi-valued case and to the so-called "strictly p-nonexpansive mappings." Specialization yields a stronger form of Edelstein's fixed point theorem [14]. A characterization of asymptotic regularity in terms of fixed point features of cluster points is made. Also given is a nonlinear generalization of Stein's theorem [30] for spectral radius less than one. Furthermore, it is shown that the Browder-Göhde-Kirk fixed point theorem [6], [18], [21] cannot be extended in a "natural way". A number of examples are constructed to compare our results with existing related results or to show that certain weakenings of our hypotheses cannot be made.

2. Basic results.

In the sequel of this paper unless otherwise is stated, M will denote a metric space with metric d. Let 2^M be the family of all nonempty subsets of M,

$$C(M)=\left\{A\in 2^M:\ A \text{ is closed}\right\};\ K(M)=\left\{A\in 2^M:\ A \text{ is compact}\right\}.$$

To fix our terminology and to state results concisely, we consider the following definitions ((v) and (vi) below are well-known):

(i) A mapping f: $M\to 2^M$ is said to be a *weak p-contraction* if there exists a real functional p: $M\to[0,\infty)$ such that for each x in M and *some* $y\in f(x)$, $d(x,y)\leqslant p(x)-p(y)$.

(ii) A mapping f: $M\to 2^M$ is said to be a *p-contraction* if there exists a real functional p: $M\to[0,\infty)$ such that for each x in M and *all* $y\in f(x)$, $d(x,y)\leqslant p(x)-p(y)$.

(iii) A mapping f: $M\to 2^M$ is said to be *strictly p-nonexpansive* if there exists a real functional p: $M\to[0,\infty)$ such that for each x in M with $x\notin f(x)$ and all $y\in f(x)$, $p(y)<p(x)$.

(iv) A mapping f: $M\to 2^M$ is said to be *p-nonexpansive* if there exists a real functional p: $M\to[0,\infty)$ such that for each x in M and all $y\in f(x)$, $0\leqslant p(x)-p(y)$.

(v) A mapping f: $M\to 2^M$ is said to be *upper semicontinuous* on M if for each x in M and for any neighborhood G of $f(x)$, there is a neighborhood V of x such that $f(V)=\bigcup_{x\in V} f(x)\subset G$.

(vi) A mapping f: $M\to 2^M$ is said to be *closed* on M if for each x in M and for every sequence $\{x_k\}$ in M, $\lim_{k\to\infty} x_k=x$, $y_k\in f(x_k)$ and $\lim_{k\to 8} y_k=y\in M$ imply $y\in f(x)$.

We begin with the following two theorems.

THEOREM 1. *Let (M,d) be a complete metric space and f: $M\to 2^M$ be* *closed. Suppose that f is a weak p-contraction (with p not necessarily lower semi-continuous). Then the sequence* $\{x_k\}$ *generated by* $x_{k+1}\in f(x_k)$, $k=0, 1, \cdots, x_0\in M$, *converges to a fixed point* x^* *of f with the estimate* $d(x_k, x^*)\leqslant p(x_k)$, $k=0, 1, \cdots$.

THEOREM 2. *Let* (M, d) *be a complete metric space and* $f\colon M\to 2^M$ *an arbitrary mapping. Suppose that f is a weak p-contraction with p being lower semi-continuous. Then f has a fixed point.*

Special cases of Theorem 1 and 2 are the weak p-contraction replaced by the p-contraction.

PROOF of THEOREM 1. Since f is a weak p-contraction, $d(x_k, x_{k+1})\leqslant p(x_k)-p(x_{k+1})$, $k=0, 1, \cdots$. Note that the sequence $\{p(x_k)\}$ is monotonically decreasing in $[0, \infty)$. Hence $\{p(x_k)\}$ converges. The estimate

$$(1)\qquad d(x_k, x_{k+m})\leqslant \sum_{j=k}^{k+m-1} d(x_j, x_{j+1}) \leqslant \sum_{j=k}^{k+m-1}\big(p(x_j)-p(x_{j+1})\big) = p(x_k)-p(x_{k+m})$$

shows that $\{x_k\}$ is a Cauchy sequence. By completeness of M, $\{x_k\}$ converges to a point x^* in M. Since $x_{k+1}\in f(x_k)$ and f is closed, $x^*\in f(x^*)$. The estimate follows from (1) as $m\to\infty$.

Geometrically, the estimate

$$d(x_0, x_k)\leqslant \sum_{j=0}^{k-1} d(x_j, x_{j+1})\leqslant p(x_0)-p(x_k)$$

can be regarded as an upper bound for the "path-length" $\sum_{j=0}^{k-1} d(x_j, x_{j+1})$ traversed by the iterates. Suppose that we have a problem of solving a nonlinear operator equation $F(x)=0$ in a Banach space with norm $||\cdot||$ and that an initial point for a certain iterative process (e. g., the Newton-Kantorovich method [20] of the form $x_{k+1}=x_k-(F'(x_k))^{-1}F(x_k)$, $k=0, 1, \cdots$. Here $F'(x)$ denotes the Fréchet derivative of F at x). From the proof of Theorem 1, we see that the problem of finding roots for $F(x)=0$ is reduced to the construction of the "majorant function" p such that

$$||x_k-x_{k+1}||\leqslant p(x_k)-p(x_{k+1}), \qquad k=0, 1, \cdots.$$

PROOF of THEOREM 2. By a result of Ekeland [16, p. 324], there exists some $v\in M$ such that $p(w)>p(v)-d(v, w)$ for all $w\in M$ with $w\neq v$. We assert that $v\in f(v)$. Indeed, suppose not, then $p(w)>p(v)-d(v, w)$ for each $w\in f(v)$, and hence, $d(v, w)>p(v)-p(w)$ for some v in M and all

$w \in f(v)$. This is a contradiction to the weak p-contraction of f.

An interesting extension of the Caristi-Kirk theorem is the following theorem which weakens the p-contraction to be strictly p-nonexpansive.

THEOREM 3. *Let f: $M \to K(M)$ be upper semi-continuous. Suppose that f is strictly p-nonexpansive with p being continuous. Let the sequence $\{x_k\}$ be generated by the recurrence relation*

$$x_{k+1} \begin{cases} = x_k & \text{if} \quad x_k \in f(x_k)\,, \\ \in f(x_k) & \text{if} \quad x_k \notin f(x_k)\,, \end{cases} \qquad k = 0, 1, \cdots, x_0 \in M\,.$$

Then each cluster point $\zeta \in M$ of the successive iterates $\{x_k\}$ is a fixed point of f.

Before coming to the proof of Theorem 3, we remark that a strictly p-nonexpansive mapping f may not have a fixed point, or, alternatively, that there may have arbitrarily many fixed points even if f is single-valued. To illustrate this, let us consider the functions

$$f(x) = \exp\left(\frac{-x}{2}\right) + x, \; p(x) = \exp\left(\frac{-x}{2}\right) \qquad \text{for} \quad x \geqslant 0\,.$$

Then f is strictly p-nonexpansive and f has no fixed point. Let $[x]$ be the greatest integer which is less than or equal to x, the real parameter α varies between 0 and $+\infty$. Consider the function f: $[0, \infty) \to [0, \infty)$ defined by

$$f(x) = \begin{cases} [x] & \text{if} \quad x \leqslant \alpha\,, \\ 0 & \text{if} \quad x > \alpha\,. \end{cases}$$

Let $p(x) = x$ for $x \geqslant 0$. Then f is strictly p-nonexpansive on $[0, \infty)$. If $\alpha = 0$, f has exactly one fixed point and in general, if $\alpha = k$ (positive integer), f has $k+1$ fixed points. Moreover, as $\alpha \to +\infty$, f has denumerably many fixed points. On the other hand, let

$$f(x) = \begin{cases} \dfrac{1}{2} & \text{if} \quad x \in \left[0, \dfrac{1}{2}\right], \\ x & \text{if} \quad x \in \left(\dfrac{1}{2}, 1\right]; \end{cases} \qquad p(x) = \exp(-x) \quad \text{for} \quad x \in [0, 1]\,.$$

Then f is strictly p-nonexpansive on $[0, 1]$ and there is even a continuum of fixed points. We note also that a strictly p-nonexpansive mapping is indeed weaker than p-contraction. To see this, consider the function f: $[0, \infty) \to [0, \infty)$ defined by $f(x) = [x] + 1$. We assert that f is not a p-contraction for any p: $[0, \infty) \to [0, \infty)$. Indeed, suppose that there exists some p such that for each x in M,

$$d(x,f(x)) \leqslant p(x)-p(f(x)) . \tag{2}$$

Then (2) implies that for any sequence $\{x_k\}$ induced by

$$x_{k+1}=f(x_k), \qquad k=0, 1, \cdots, \ x_0\in[0, \infty),$$

$$\sum_{k=0}^{\infty} d(x_k, x_{k+1}) \leqslant p(x_0) < +\infty . \tag{3}$$

But, if we choose the initial value $x_0=1$, then $\sum_{k=0}^{\infty} d(x_k, x_{k+1})=+\infty$. There comes a contradiction to (3). On the other hand if we define $p(x)=\exp(-x)$ for $x\geqslant 0$, then f is strictly p-nonexpansive.

In the proof of the theorem, we shall make use of the following two lemmas. Their proofs can be found in Nikaido [27, pp. 66–67].

LEMMA 1. *Let f: $M\to K(M)$ be upper semi-continuous. Then, if S is a compact subset of M, the image $f(S)=\bigcup_{x\in S} f(x)$ is also compact.*

LEMMA 2. *If f: $M\to C(M)$ is upper semi-continuous, then f is closed.*

PROOF of THEOREM 3. Let $F(f)=\{x\in M: x\in f(x)\}$. If $\{x_k\}\cap F(f)\neq\phi$, then the result follows. So we may assume that $x_k\notin F(f)$ for each k. Let ζ be a cluster point of $\{x_k\}$. Then there exists a subsequence $\{x_{n(k)}\}$ or $\{x_k\}$ such that $x_{n(k)}\to\zeta$ as $k\to\infty$. Because of the continuity of p, $p(x_{n(k)})\to p(\zeta)$ as $k\to\infty$. For every $\varepsilon>0$, there is a positive integer j such that $k\geqslant j$ implies $p(x_{n(k)})-p(\zeta)<\varepsilon$. From antitone property of p,

$$0\leqslant p(x_k)-p(\zeta)=p(x_k)-p(x_{n(k)})+p(x_{n(k)})-p(\zeta)<\varepsilon$$

for every $k\geqslant j$. It follows that

$$p(x_k)\longrightarrow p(\zeta) \quad \text{as} \quad k\longrightarrow\infty . \tag{4}$$

Let $S=\{\zeta, x_{n(1)}, x_{n(2)}, \cdots\}$. Thus S is a compact subset of M. Since $f(x)$ is compact for each x in M and f is upper semicontinuous, from Lemma 1, we see that $f(S)=\bigcup_{x\in S} f(x)$ is a compact subset of M. Since the sequence $\{x_{n(k)+1}\}$ is contained in $f(S)$, there exists a subsequence $\{x_{m(n(k)+1)}\}$ of $\{x_{n(k)+1}\}$ such that $x_{m(n(k)+1)}\to\eta$ as $k\to\infty$. By the same argument as above,

$$p(x_k)\longrightarrow p(\eta) \quad \text{as} \quad k\longrightarrow\infty . \tag{5}$$

From (4) and (5)

$$p(\zeta)=p(\eta) . \tag{6}$$

Because f is upper semi-continuous and $f(x)$ is compact for each x in M (that $f(x)$ is closed is needed only), by Lemma 2, we come to the conclusion that f is closed. Since $x_{n(k)}\to\zeta$, every subsequence of $\{x_{n(k)}\}$ also converges to ζ. Therefore, we can construct a subsequence $\{x_{i(n(k))}\}$ of $\{x_{n(k)}\}$ such that $x_{i(n(k))}\to\zeta$ and $x_{i(n(k))+1}=x_{m(n(k)+1)}$. Hence we have chosen a sequence $\{x_{i(n(k))}\}$ to fulfill

(α) $x_{i(n(k))+1}\in f(x_{i(n(k))})$,

(β) $x_{i(n(k))+1}\longrightarrow\eta$,

(γ) $x_{i(n(k))}\longrightarrow\zeta$.

From closedness of f, $\eta\in f(\zeta)$. Thus, if ζ is not a fixed point of f, then $p(\eta)<p(\zeta)$. This yields a contradiction to (6).

EXAMPLE 1. This example shows that the upper semi-continuity of f cannot be substituted by the closedness in Theorem 3.

Let $M=\{-1\}\cup\left[0, \frac{1}{2}\right]\cup[2, \infty)$. Define f: $M\to K(M)$ by

$$f(x)=\begin{cases}\{-1\} & \text{if} \quad x=0 \text{ or } x=-1\,,\\ \{1/x\} & \text{if} \quad 0<x\leqslant\dfrac{1}{2}\,,\\ \{1/3x\} & \text{if} \quad x\geqslant 2\,.\end{cases}$$

Then f is closed but not upper semi-continuous. Define P: $M\to[0, \infty)$ by

$$p(x)=\begin{cases}\dfrac{1}{2} & \text{if} \quad x=-1\,,\\ x+1 & \text{if} \quad 0\leqslant x\leqslant\dfrac{1}{2}\,,\\ \dfrac{4}{5x}+1 & \text{if} \quad x\geqslant 2\,.\end{cases}$$

Then f is strictly p-nonexpansive with p being continuous on M. Let the sequence $\{x_k\}$ induced by $x_{k+1}=f(x_k)$, $k=0, 1, \cdots, x_0=\frac{1}{4}$; then 0 is a cluster point of $\{x_k\}$, but $0\notin f(0)=\{-1\}$.

We remark that our main idea of the proof of Theorem 3 is due to Zangwill's convergence theorem for nonlinear programming [33, p. 91].

Now we turn to a stronger form of Edelstein's fixed point theorem.

THEOREM 4. *Let f: $M\to M$ be continuous. Suppose that f is strictly p-nonexpansive mapping with p being continuous. Let $x\in M$ and let ζ be a cluster point of the successive iterates $\{f^k(x)\}$ of x. Then*:

(a) *ζ is a fixed point of f.*

(b) *If $p(f(x))<p(x)$ for each x in M with $x\neq\zeta$ and $\{f^k(x)\}$ is precompact, then $f^k(x)\to\zeta$ as $k\to\infty$.*

PROOF. Since f is single-valued and f is continuous, f is upper semicontinuous. Then conclusion (a) follows immediately from Theorem 3. To show (b), suppose that η is any cluster point of $\{f^k(x)\}$. From (a), η is a fixed point of f. Suppose that $\eta \neq \zeta$, then $p(f(\eta)) < p(\eta)$. A contradiction comes up. Hence $\eta = \zeta$. Since all the convergent subsequences of $\{f^k(x)\}$ have ζ as a cluster point, $f^k(x) \to \zeta$ as $k \to \infty$.

When $p(x) = d(x, f(x))$, Theorem 3 specializes to the following theorem of Edelstein [14] :

COROLLARY 1. *Let f: $M \to M$ and suppose that f is strictly nonexpansive, i. e., $d(f(x), f(y)) < d(x, y)$ for each x, y in M with $x \neq y$. Let $x \in M$ and let $\zeta \in M$ be a cluster point of the succesive iterates $\{f(x)\}$ of x. Then ζ is a unique fixed point of f and $f^k(x) \to \zeta$ as $k \to \infty$.*

COROLLARY 2. *Let f: $M \to M$ be continuous and $\zeta = f(\zeta)$. Suppose that there exists a continuous real functional p: $M \to [0, \infty)$ such that*

$$(7) \qquad p\big(f(x)\big) < p(x) \qquad \text{for each } x \text{ in } M \text{ with } x \neq \zeta .$$

If $\{f^k(x)\}$ is pre-compact, then $f^k(x) \to \zeta$ as $k \to \infty$.

Let $p(x) = d(x, \zeta)$. Then (7) implies that $d(f(x), \zeta) < d(x, \zeta)$ for each x in M with $x \neq \zeta$. Therefore, Corollary 2 generalizes a recent result of Măruster [26] (and hence, an earlier result of Tricomi, see [13]). We note that an example [13, p. 471] shows that the conclusion of Corollary 2 is false if a convergent subsequence is not assumed. The following example shows that our result is a proper generalization of Edelstein's fixed point theorem.

EXAMPLE 2. Let R^2 be 2-dimensional real linear space of column vectors $x = (x_1, x_2)^T$ with the metric induced by the norm

$$\|x\|_\infty = \max\big\{|x_1|, |x_2|\big\}, \qquad \text{for} \quad x = (x_1, x_2)^T \in R^2 .$$

Let $M = \left\{x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 \geqslant 0,\ x_2 \geqslant 0\right\}$. Define f: $M \to M$ by

$$f(x) = \begin{pmatrix} \frac{3}{5} x_1 + \frac{1}{2} x_2 \\ \frac{1}{10} x_1 + \frac{3}{10} x_2 \end{pmatrix}.$$

Since

$$\left\| \begin{pmatrix} \frac{3}{5} & \frac{1}{2} \\ \frac{1}{10} & \frac{3}{10} \end{pmatrix} \right\|_\infty = \frac{3}{5} + \frac{1}{2} > 1 ,$$

$\|f(x)-f(y)\|_\infty \geqslant \|x-y\|_\infty$ for some x, y in M with $x \neq y$, *i.e.*, f is not a strictly nonexpansive mapping. Let p: $M \to [0, \infty)$ be a quadratic form defined by

$$p(x)=x^T\begin{pmatrix} \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{3} \end{pmatrix}x, \qquad \text{for} \quad x=(x_1, x_2)^T \in M.$$

Then

$$\begin{aligned} p\big(f(x)\big) &= \frac{37}{300}x_1^2+\frac{57}{500}x_1x_2+\frac{67}{400}x_2^2 \\ &< \frac{1}{4}x_1^2+\frac{1}{2}x_1x_2+\frac{1}{3}x_2^2 \\ &= p(x), \qquad \text{for each} \quad x \neq 0. \end{aligned}$$

It is clear that 0 is a cluster point of $\{f^k(x)\}$ for $x \neq 0$. Therefore, all the conditions of Theorem 4 are fulfilled.

By refining the proofs of Theorem 3 and 4, we can obtain the following result concerning the weak convergence of successive approximations.

THEOREM 5. *Let E be a normed linear space, K a nonempty weakly closed subset of E and f: $K \to K$ a weakly continuous mapping. Suppose that f is strictly p-nonexpansive with p being weakly continuous. Let $x \in M$ and let u be any weak cluster point of the successive iterates $\{f^k(x)\}$ of x. Then*:

(a) *u is a fixed point of f.*

(b) *If $\zeta = f(\zeta)$ and $p(f(x)) < p((x)$ for each x in M with $x \neq \zeta$ and $\{f^k(x)\}$ is weakly pre-compact, then $\{f^k(x)\}$ converges weakly to ζ.*

3. A characterization of asymptotic regularity.

A mapping f: $M \to 2^M$ is said to be asymptotically regular at x if there exists an orbit $O(x)=\{x_k : x_{k+1} \in f(x_k),\ k=0, 1, \cdots,\ x=x_0 \in M\}$ of x such that $\{d(x_{k+1}, x_k)\}$ converges to 0. The concept of asmyptotic regularity for multi-valued mappings generalizes the asymptotic regularity introduced by Browder and Petryshyn [10] for single-valued mappings. The following result is a characterization of asymptotic regularity in terms of fixed point features of cluster points.

THEOREM 6. *Let f: $M \to K(M)$ be upper semi-continuous. Let $x \in M$ be such that its orbits are relatively compact in M. Then the following are equivalent*:

(a) *There is an orbit $O(x)$ of x such that each cluster point of $O(x)$ is a fixed point of f.*

(b) *f is asymptotically regular at x.*

PROOF of (a)$\Rightarrow$(b). Let $O(x)$ be an orbit of x such that for each cluster point of $O(x)$ is a fixed point of f. On the contrary suppose that $\{d(x_{k+1}, x_k)\}$ does not converge to 0. Then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_k\}$ such that

$$d(x_{n(k)+1}, x_{n(k)}) \longrightarrow \delta \in (0, \infty) .$$

By the relative compactness of $O(x)$, we may (by taking a subsequence, if necessary) assume that $\{x_{n(k)}\}$ converges, say $x_{n(k)} \to \zeta$. By the hypothesis, $\zeta \in f(\zeta)$. Since $f(x)$ is compact for each x in M and f is upper semi-continuous, $x_{n(k)+1} \to f(\zeta)$. Hence

$$\begin{aligned} d\big(f(\zeta), \zeta\big) &= d\big(\lim_{k\to\infty} x_{n(k)+1}, \lim_{k\to\infty} x_{n(k)}\big) \\ &= \lim d(x_{n(k)+1}, x_{n(k)}) \\ &= \delta > 0 . \end{aligned}$$

There occurs a contradiction.

PROOF of (b)$\Rightarrow$(a). Since f is asymptotically regular at x, there exists an orbit $O(x)$ of x such that $d(x_{k+1}, x_k) \to 0$. Let ζ be a cluster point of $O(x)$. Then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_k\}$ such that $x_{n(k)} \to \zeta$. Since $f(x)$ is compact for each x in M and f is upper semi-continuous, we see that $x_{n(k)+1} \to f(\zeta)$. Since

$$d\big(\zeta, f(\zeta)\big) \leqslant d(\zeta, x_{n(k)}) + d(x_{n(k)}, x_{n(k)+1}) + d\big(x_{n(k)+1}, f(\zeta)\big) ,$$

letting k tend to infinity, we obtain $\zeta \in f(\zeta)$.

Combining Theorem 4 and 6 we have

PROPOSITION 1. *Let f: $M \to M$ be continuous. Suppose that f is a strictly p-nonexpansive mapping with p being continuous. Let $x \in M$ be such that its successive iterates $\{f^k(x)\}$ is relatively compact. The f is asymptotically regular at x.*

Let $M = [-1, 1]$ and $f(x) = -x$ for $x \in M$. Let $p(x) = |x|$ for $x \in M$. Then f is p-nonexpansive. Clearly, $\frac{1}{2}$ is a cluster point of $\left\{f^k\left(\frac{1}{2}\right)\right\}$, but $\frac{1}{2} \neq f\left(\frac{1}{2}\right)$. By Theorem 6, we conclude that f is not asymptotically regular at $\frac{1}{2}$. From this simple example and Proposition 1, we see that the difference between strictly p-nonexpansive and p-nonexpansive mapping in a compact setting is a concept of asymptotic regularity.

The next result follows from Proposition 1 and a result of Hillam [19].

PROPOSITION 2. *Let* $f: [a, b] \subset R^1 \to [a, b]$ *be continuous. Suppose that* f *is strictly* p*-nonexpansive with* p *being continuous. Then every* $x \in [a, b]$ *the successive iterates* $\{f^k(x)\}$ *of* x *converges to a fixed point of* f.

4. A nonlinear generalization of Stein's theorem.

Let A be an $n \times n$ complex matrix, A^* the conjugate transpose of A, and $\rho(A)$ the spectral radius of A. A is said to be Hermitian if $A = A^*$. We denote by C^n the complex n-dimensional normed linear space of all column vector $z = (z_1, z_2, \cdots, z_n)^T$. A is said to be positive definite Hermitian if A is Hermitian and satisfies

$$z^* Az > 0 , \qquad \text{for all} \quad z \neq 0 .$$

In [30] (see also Varga [32, p. 16]) Stein claimed the following result:

THEOREM 7. *Let* $A = (a_{ij})$ *be an* $n \times n$ *complex matrix. If there exists an* $n \times n$ *complex matrix* B *such that* B *and* $B - A^*BA$ *are both positive definite Hermitian, then* $\rho(A) < 1$.

In this section we provide a nonlinear (not necessarily linear) generalization of the above theorem. Our approach is an application of Theorem 5. The result is the following:

THEOREM 8. *Let* E *be a reflexive Banach space,* K *a nonempty weakly closed subset of* E*, and* $f: K \to K$ *a weakly continuous mapping. Suppose that* f *is strictly* p*-nonexpansive with* p *being weakly continuous and* $p(x) \to +\infty$ *as* $||x|| \to +\infty$. *Let* $x \in K$. *Then*:

(a) *The successive iterates* $\{f^k(x)\}$ *of* x *has weak cluster points in* K *and each weak cluster point of* $\{f^k(x)\}$ *is a fixed point of* f.

(b) *If* $\zeta = f(\zeta)$ *and* $p(f(x)) < p(x)$ *for each* x *in* K *with* $x \neq \zeta$, *then* $f^k(x)$ *converges weakly to* ζ.

Let E be a normed linear space, K a nonempty subset of E, and p a real functional on K. As usual, any nonempty set of the form

$$\Phi[p, \alpha] = \left\{x \in K : p(x) \leqslant \alpha\right\} , \qquad \alpha \in R^1 ,$$

is a level set of p. To prove Theorem 8, we need the following basic lemma.

LEMMA 3. *Let* E *be a reflexive Banach space and* K *a nonempty weakly closed subset of* E. *Suppose that* p *is weakly lower semi-continuous* *on K and* $p(x)\to+\infty$ *as* $||x||\to+\infty$. *Then each level set* $\Phi[p,\alpha]$ *is weakly compact.*

Proof. By the weak lower semi-continuity of p, $\Phi[p,\alpha]$ is weakly closed in K. Since K is weakly closed in E, $\Phi[p,\alpha]$ is weakly closed in E. To show that $\Phi[p,\alpha]$ is bounded, on the contrary suppose that

$$\sup\left\{||x||:\ x\in\Phi[p,\alpha]\right\}=+\infty\,.$$

Then there exists a sequence $\{y_k\}\subset\Phi[p,\alpha]$ such that $||y_k||\to+\infty$, then, by hypothesis, $p(y_k)\to+\infty$, which is a contradiction to $p(x)\leqslant\alpha$ for all $x\in\Phi[p,\alpha]$. By reflexivity of E, we conclude that $\Phi[p,\alpha]$ is weakly compact.

Proof of Theorem 8. Let $x\in K$. We may assume thet $f^{k+1}(x)\neq f^k(x)$ for each $k=1,2,\cdots$. (The convention $f^0(x)=x$ is understood.) Since f is strictly p-nonexpansive, we see that $\{f^k(x)\}$ is contained in the level set

$$\Phi[p,p(x)]=\left\{y\in K:\ p(y)\leqslant p(x)\right\}.$$

By Lemma 3, $\Phi[p,p(x)]$ is weakly compact. By the Eberlein-Šmulian theorem [24, p. 303], $\Phi[p,p(x)]$ is weakly sequentially compact. Let u be any weak cluster point of $\{f^k(x)\}$. For Theorem 5 (a), u is a fixed point of f. If $\zeta=f(\zeta)$ and $p(f(x))<p(x)$ for each x in K with $x\neq\zeta$, by Theorem 5 (b), $f^k(x)$ converges weakly to ζ.

Proof of Theorem 7 using Theorem 8. Let $E=K=C^n$ be with norm $||\cdot||$. Define $f:\ C^n\to C^n$ by $f(z)=Az$, and define $p:\ C^n\to[0,\infty)$ by $p(z)=z^*Bz$. By the hypothsis,

$$p(Az)<p(z)\,,\qquad\text{for all}\quad z\neq 0\,.$$

Let $||z||_*=(z^*Bz)^{\frac{1}{2}}$. Then $||\cdot||_*$ is a norm on C^n. Since all norms on C^n are equivalent, $p(z)\to+\infty$ as $||z||\to+\infty$. According to Theorem 8, $A^kz\to0$ for any $z\in C^n$. We assert that $\rho(A)<1$. On the contrary suppose that A has an eigenvalue λ with $|\lambda|\geqslant1$ and corresponding eigenvector $z\neq0$. Then $A^kz=\lambda^kz$ for all k, so that A^kz does not tend to zero, which is absured.

We note that Theorem 8 fails in nonreflexive Banach space with the strong continuity of f and p.

Example 3. Let E be the Banach space C_0 consisting of all real sequences $x=(x_k)$ with $\lim\limits_{k\to\infty}x_k=0$ and normed by $||x||=\sup\{|x_k|:\ k=1,2,\cdots\}$. Let

$$K=\left\{x=(x_k):\ 0\leqslant x_k\leqslant1,\ k=1,2,\cdots\right\}.$$

Then K is a weakly closed subset of C_0. Define $f:\ K\to K$ by

$$f(x)=(1, x_1, x_2, \cdots), \qquad \text{for each} \quad x=(x_k)\in K.$$

Then f is continuous on K, in fact, f is nonexpansive. Define $p: K\to[0, \infty)$ by

$$p(x)=\sum_{k=1}^{\infty}(1-x_k)\,2^{1-k}.$$

Then p is continuous on K and $p(f(x))=\sum_{k=1}^{\infty}(1-x_k)\,2^{1-k}$,

$$\begin{aligned} p(x)-p\big(f(x)\big) &= \sum_{k=1}^{\infty}(1-x_k)\,2^{1-\mathrm{k}}-\sum_{k=1}^{\infty}(1-x_k)\,2^{-k} \\ &= \sum_{k=1}^{\infty}(1-x_k)\,2^{-k}. \end{aligned}$$

Since $\lim_{k\to\infty} x_k=0$, there exists some positive integer k_0 such that $x_{k_0}<1$, and hence,

$$\sum_{k=1}^{\infty}(1-x_k)\,2^{-k}\geqslant(1-x_{k_0})\,2^{-k_0}>0.$$

Hence $p(f(x))<p(x)$ for each x in K. Furthermore, the growth condition $p(x)\to+\infty$ as $||x||\to+\infty$ is automatically satisfied. However, the mapping f has no fixed point, for if $f(x)=x$, where $x=(x_k)\in K$, then $x_k=1$ for $k\geqslant 1$, a contradication.

As an example of the use of Theorem 8, we give the following:

EXAMPLE 4. Consider the 2-dimensional system of difference equation

$$u_{k+1}=\frac{\frac{1}{2}v_k}{1+3v_k^2}$$

$$v_{k+1}=\frac{-\frac{1}{3}u_k}{1+3v_k^2}.$$

Define $f: R^2\to R^2$ by $f\begin{pmatrix}u\\ v\end{pmatrix}=\left(\frac{\frac{1}{2}v}{1+2u^2}, \frac{-\frac{1}{3}v}{1+3v^2}\right)^T$. Let $p(x)=||x||=u^2+v^2$ for each $x=(u, v)^T\in R^2$. Then $f(0)=0$, and $p(x)\to+\infty$ as $||x||\to+\infty$. Furthermore,

$$\begin{aligned} p\big(f(x)\big)-p(x) &= \left(\frac{\frac{1}{4}}{(1+2u^2)^2-1}\right)v^2+\left(\frac{\frac{1}{9}}{(1+3v^2)^2-1}\right)u^2 \\ &\leqslant\left(\frac{1}{4}-1\right)v^2+\left(\frac{1}{9}-1\right)u^2<0, \quad \text{for each} \quad x=(u, v)^T\neq 0. \end{aligned}$$

According to Theorem 8, $(u_k, v_k)^T \to 0$ for all $(u, v)^T = (u_0, v_0)^T \in R^2$.

5. Minimization.

This section is devoted primarily to the study of the problem of finding a fixed point for certain mappings in a metric space (or a normed linear space) M, which can be replaced by a problem of minimizing a real functional on M. We begin with the following rather sharp improvement of Theorem 3 concerning only the existence of a fixed point.

PROPOSITION 3. *Let $f: M \to 2^M$ be an arbitrary mapping and suppose that $f(M)$ is contained in a compact set $D \subset M$. Suppose that there exists a lower semi-continuous functional $p: M \to [0, \infty)$ such that for each x in M with $x \notin f(x)$ and some $y \in f(x)$, $p(y) < p(x)$. Then f has a fixed point.*

PROOF. Since p is lower semi-continuous and D is compact, there exists x^* in D such that

$$p(x^*) = \min \left\{ p(x) : x \in D \right\}.$$

If $x^* \notin f(x^*)$, then $p(y) < p(x^*)$ for some $y \in f(x^*) \subset D$. This gives a construction of x^*, so that $x^* \in f(x^*)$.

PROPOSITION 4. *Let E be a reflexive Banach space, K a nonempty weakly closed subset of E, and $f: K \to K$ an arbitrary mapping. Suppose that f is strictly p-nonexpansive with p being weakly lower semi-continuous and $p(x) \to +\infty$ as $\|x\| \to +\infty$. Then f has a fixed point in K.*

Let $\dot{p}(x) = p(f(x)) - p(x)$. Then $\dot{p}(x) < 0$ for each $x \neq f(x)$. Thus our assumption is closely related to a result of LaSalle and Lefschetz [25, p. 67].

COROLLARY 3. *Let E be a reflexive Banach space and K a nonempty weakly closed subset of E. Suppose that $f: K \to K$ is weakly continuous, $f(K)$ is bounded, and $\|f^2(x) - f(x)\| < \|f(x) - x\|$ for each x in K with $x \neq f(x)$. Then f has a fixed point in K.*

PROOF of COROLLARY 3. Let $p(x) = \|f(x) - x\|$. Since f is weakly continuous, p is weakly lower semi-continuous. Since

$$\left\| f(x) - x \right\| \geqslant \|x\| - \left\| f(x) \right\|$$

and $f(K)$ is bounded, $p(x) \to +\infty$ as $\|x\| \to +\infty$. By Proposition 4, f has a fixed point in K.

COROLLARY 4. *Let E be a reflexive Banach space and K a nonempty closed convex subset of E. Suppose that $f: K \to K$ is an arbitrary mapping and f is strictly p-nonexpansive with p, a lower semi-continuous quasi-*

convex functional on K (a functional p defined on a convex set K is said to be quasi-convex if $p(\alpha x+(1-\alpha)y)\leqslant \max\{p(x), p(y)\}$ for each x, y in K and every $\alpha\in(0,1)$) and $p(x)\to+\infty$ as $||x||\to+\infty$. Then f has a fixed point in K.

PROOF of COROLLARY 4. Since K is closed and convex, K is weakly closed. Since p is quasi-convex, each level sets $\Phi[p,\alpha]$ is convex. The conclusion follows immediately from the following fact and Proposition 4: A lower semi-continuous real functional having convex level sets is weakly lower semi-continuous.

PROOF of PROPOSITION 4. Let $\bar{x}\in K$. By Lemma 3, the level set $\Phi[p, p(\bar{x})]$ is weakly compact. Since p is weakly lower semi-continuous and has a weakly compact level set, there exists $x^*\in K$ such that

$$p(x^*)=\min\left\{p(x):\ x\in K\right\}.$$

Hence x^* is a fixed point of f.

EXAMPLE 5. Let l_2 be the real sequence space of square summable sequances $x=(x_k)$ with norm

$$||x||=\left(\sum_{k=1}^{\infty} x_k^2\right)^{\frac{1}{2}}.$$

Let $K=\{x=(x_k):\ x_1=2x_2=\cdots=2^{k-1}x_k=\cdots\}$ be a hyperplane of l_2. Then K is a weakly closed subset of l_2. For each $x=(x_k)\in K$, define $f\colon K\to K$ by

$$f(x)=\begin{cases}(1, 1/2, \cdots, 1/2^{k-1}, \cdots) & \text{if } x_1=3/2\,,\\ (0, 0, \cdots, 0, \cdots) & \text{if } x_1\neq 3/2\,.\end{cases}$$

Let $p(x)=||x||=\left(\frac{4}{3}\right)^{\frac{1}{2}}|x_1|$ for $x\in K$. Then

$$p\big(f(x)\big)=\begin{cases}4/3 & \text{if } x_1=3/2\,,\\ 0 & \text{if } x_1\neq 3/2\,.\end{cases}$$

So that f is strictly p-nonexpansive on K. Since the norm is weakly lower semi-continuous, p is weakly lower semi-continuous. Furthermore, $p(x)\to+\infty$ as $||x||\to+\infty$. Therefore, f satisfies all the conditions of Proposition 4. To see the advantage of Proposition 4, we note that f is discontinuous and $||f^2(x)-f(x)||\geqslant||f(x)-x||$ for some x in K with $x\neq f(x)$. For instance, let $x=(3/2, 3/4, \cdots, 3/2^{k-1}, \cdots)$. Then $f(x)=(1, 1/2, \cdots, 1/2^{k-1}, \cdots)$, $f^2(x)=(0, 0, \cdots, 0, \cdots)$. Therefore, $||f^2(x)-f(x)||=4/3>||f(x)-x||=1/3$.

6. The Browder-Göhde-Kirk Theorem.

Let K be a nonempty weakly compact convex subset of a Banach space E and suppose K possesses normal structure. (A nonempty convex $K \subset E$ is said to have normal structure [3] if each convex bounded subset S of K with positive diameter d contains a point x such that $\sup\{||x-y|| : y \in S\} < d$.) The well-known Browder-Göhde-Kirk fixed point theorem [6], [18], [21] asserts that each nonexpansive mapping $f: K \to K$ has a fixed point. Perhaps the most natural problem in link with the results of Browder-Göhde-Kirk and Caristi-Kirk is the following sample: Let K be a nonempty weakly compact convex subset of a Banach space E and let K have normal structure and $f: K \to K$. Suppose that f is p-nonexpansive with p being non-constant lower semi-continuous. Does f have a fixed point in K? The answer is in the negative as can be seen from the following result.

THEOREM 9. *Let K be a nonempty weakly compact convex subset of a separable infinite-dimensional Hilbert space E. Then there exists a p-nonexpansive mapping $f: K \to K$ with p being weakly non-constant continuous on K such that f does not have a fixed point in K.*

Note here that each nonempty closed bounded convex subset of a uniformly convex Banach space has normal structure [28].

PROOF of THEOREM 2. We may take the Hilbert space E to be the real sequence space l_2. Let

$$K = \left\{x = (x_k) \in l_2 : \ x_1 = 2x_2 = \cdots = 2^{k-1} x_k = \cdots, \ x_k \in [-1, 1], k = 1, 2, \cdots\right\}.$$

Then K is nonempty weakly compact convex subset of l_2. For each $x = (x_k) \in K$, define $f: K \to K$ by

$$f(x) = \begin{cases} (1, 1/2, 1/4, \cdots, 1/2^{k-1}, \cdots) & \text{if } \ x_1 \in [-1, 1)\,, \\ (-1, -1/2, -1/4, \cdots, -1/2^{k-1}, \cdots) & \text{if } \ x_1 = 1\,, \end{cases}$$

and define $p: K \to [0, \infty)$ by

$$p(x) = 2 - (4/3)^{\frac{1}{2}} |x_1|\,.$$

Then $p(f(x)) = 2 - (4/3)^{\frac{1}{2}} \leqslant 2 - (4/3)^{\frac{1}{2}} |x_1| = p(x)$. Hence f is p-nonexpansive. It remains to verify that p is weakly continuous on K. If $x^n = (x_k^n) \in K$ converges weakly to $\bar{x} = (\bar{x}_k) \in K$, then $x_1^n \to \bar{x}_1$ (see, [1, p. 236]). Therefore,

$$2 - (4/3)^{\frac{1}{2}} |x^n| \longrightarrow 2 - (4/3)^{\frac{1}{2}} |x_1|\,.$$

This implies that $p(x^n) \to p(\bar{x})$. Consequently, p is weakly continuous. However, f has no fixed point in K.

It should be remarked that f does not possess a fixed point even if we impose certain regular conditions on f in the problem, for instance, f is continuous and $\inf_{x\in K}||x-f(x)||=0$, as the following example shows.

EXAMPLE 6. Let $K=\{x=(x_k)\in l_2 : ||x||\leqslant 1\}$. Then K is convex, weakly compact and has normal structure. Define f: $K\to K$ by

$$f(x)=\left((1-||x||^2)^{\frac{1}{2}}, x_1, x_2, \cdots\right), \qquad \text{for each} \quad x=(x_k)\in K .$$

Then $||f(x)||=1$ and f is continuous in the strong topology. Define p: $K\to[0, \infty)$ by $p(x)=1-||x||$. Then p is continuous and $p(f(x))=1-||f(x)||=0\leqslant 1-||x||=p(x)$. Let $x^n=(x_k^n)\in K$ be such that

$$x_1^n=\cdots=x_{2^n}^n=1/2^{\sqrt{n}} \quad \text{and} \quad x_{2^n+m}^n=0 \qquad \text{for} \quad m=1, 2, \cdots .$$

Then $f(x^n)=(0, x_1^n, x_2^n, \cdots, x_{2^n}^n, 0, \cdots)$. It follows that

$$\left\|x^n-f(x^n)\right\|=(2/2^n)^{\frac{1}{2}}\longrightarrow 0 \quad \text{as} \quad n\to\infty .$$

Therefore, we see that $\inf_{x\in K}||x-f(x)||=0$. Suppose there exists $u=(u_k)\in K$ such that $f(u)=u$. Since $||f(u)||=1$, $||u||=1$. Thus

$$f(u)=(0, u_1, u_2, \cdots)=(u_1, u_2, u_3, \cdots) .$$

Consequently, $u=(0, 0, \cdots)$, a contradiction to $||u||=1$. Therefore, f has no fixed point in K.

From Theorem 9 and this example, we conclude that the Browder-Göhde-Kirk fixed point theorem cannot be extended in such a "natural way."

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Added in proof. A simple proof of Theorem 3 runs as follows: Since p is decreasing on $\{x_k\}$, $\lim_{k\to\infty} p(x_k)=\delta\geqslant 0$. Suppose $\{x_{n(k)}\}$ is a subsequence of $\{x_k\}$ which converges to $\zeta\in M$. Since p is continuous, $\delta=p(\zeta)$. By upper semi-continuity of f at ζ, $\operatorname{dist}(x_{n(k)+1}, f(\zeta))\to 0$, and since $f(\zeta)$ is compact, some subsequence of $\{x_{n(k)+1}\}$ must converge to a point $\eta\in f(\zeta)$. It follows that $\delta=p(\eta)=p(\zeta)$; thus $\zeta\in f(\zeta)$.

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