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On the first eigenvalue of the Laplacian acting on p-forms

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§ 1. Introduction.

Let M be a compact and oriented Riemannian manifold isometrically immersed in a complete and simply connected space form of constant sectional curvature K. Let $\lambda_1^p(M)$ denote the first non-zero eignvalue of the Laplacian acting on p-forms on M. In this note we will be concerned with the next problem :

Estimate $\lambda_1^p(M)$ from above in terms of the quantities determined by M (the volume $Vol(M)$, the diameter $d(M)$, etc.) and the immersion (the mean curvature vector field η, the second fundamental form S, etc.). Furthermore, give the equality condition.

For this problem, in the case $K=0$, Bleecker-Weiner ([1]) obtained an estimate of $\lambda_1^0(M)$. Masal'cev ([4]) also obtained the same result by a different method but under the additional assumption that M is a hypersurface. Masal'cev ([5]) gave an estimate of $\lambda_1^p(M)$ for $1\leqslant p\leqslant\dim M-1$, when M is a hypersurface. In the case $K\neq 0$, Masal'cev ([6]) obtained an estimate of $\lambda_1^0(M)$ for a hypersurface without the equality condition. Now we can compare the first eigenvalue $\lambda_1^p(M)$ with that of the standard m-sphere $S^m(1)$ of constant curvature 1. The purpose of the present note is to present the following

THEOREM. *Let M be an $m(\geqslant 2)$-dimensional compact and oriented Riemannian manifold without boundary.*

(A) *If M is isometrically immersed in the n-dimensional Euclidean space E^n, then for $0\leqslant p\leqslant m$, we have*

$$\lambda_1^p(M)\leqslant\frac{\lambda_1^p(S^m(1))}{m\,Vol(M)}\int_M\|S\|^2dV_M. \tag{1.1}$$

Equality holds iff M is embedded as a geodesic sphere in some $(m+1)$-dimensional totally geodesic submanifold in E^n. Here $\|S\|$ denotes the length of S and dV_M the volume form of M.

(B) *If M is isometrically immersed in $S^n(1)$, then for $1\leqslant p\leqslant m-1$, we have*

$$(1.2)\qquad \lambda_1^p(M)\leqslant \lambda_1^p\left(S^m(1)\right)\left[\frac{S(M)}{\sqrt{m}}+\tan\left(\frac{\rho(M)}{2}\right)\right]^2.$$

Equality holds iff M is embedded as a geodesic sphere in some $(m+1)$-dimensional totally geodesic submanifold in $S^n(1)$. Here $S(M)$ denotes the maximum of $\|S\|$ and we put $\rho(M):=\pi-\max\{\rho(x, M);\ x\in S^n(1)\}$, where ρ is the distance function of $S^n(1)$.

(C) *If M is isometrically immersed in an n-dimensional complete and simply connected hyperbolic space form $H^n(-1)$ of constant curvature -1. then for $1\leqslant p\leqslant m-1$, we have*

$$(1.3)\qquad \lambda_1^p(M)\leqslant \lambda_1^p\left(S^m(1)\right)\left[\frac{S(M)}{\sqrt{m}}+\tanh\left(\frac{d(M)}{2}\right)\right]^2.$$

§ 2. Preliminaries.

Let R^n be the n-dimensional real number space and $(x^1, \cdots, x^n)$ its canonical coordinate system. Put

$$r(x):=\sqrt{\sum_{\alpha=1}^{n}(x^\alpha)^2}\qquad (x\in R^n).$$

For a real number K, we define a C^∞-function Φ on R^n by $\Phi:=1+\frac{K}{4}r^2$. Let $M^n(K)$ denote the Riemannian manifold equipped with the Riemannian metric $G:=\Phi^{-2}\sum_{\alpha=1}^{n}(dx^\alpha)^2$ whose underlying manifold is R^n for $K\geqslant 0$ and the open disc $\left\{x\in R^n;\ r(x)^2<-\frac{4}{K}\right\}$ for $K<0$, respectively. Then $M^n(K)$ is simply connected and has the constant sectional curvature K. If $K\leqslant 0$, then $M^n(K)$ is complete and if $K>0$, then for any point z on the standard n-sphere $S^n(K)$ of constant curvature K, $M^n(K)$ is isometric to $S^n(K)-\{z\}$ by the stereographic projection. Let $\tilde{\rho}$ denote the distance function of $M^n(K)$, then for any point $x\in M^n(K)$ we have

$$(2.1)\qquad r(x)=\begin{cases}\dfrac{2}{\sqrt{K}}\tan\left(\dfrac{\sqrt{K}}{2}\tilde{\rho}(0, x)\right) & (K>0),\\ \tilde{\rho}(0, x) & (K=0),\\ \dfrac{2}{\sqrt{-K}}\tanh\left(\dfrac{\sqrt{-K}}{2}\tilde{\rho}(0, x)\right) & (K<0).\end{cases}$$

The Christoffel symbol $\left\{{\alpha \atop \beta\ \gamma}\right\}$ of $M^n(K)$ with respect to (x^α) is given by

$$(2.2)\qquad \left\{{\alpha \atop \beta\ \gamma}\right\}=\frac{K}{2\Phi}(\delta_{\beta\gamma}x^\alpha-\delta_{\gamma\alpha}x^\beta-\delta_{\alpha\beta}x^\gamma).$$

Let M be an $m(\geqslant 2)$-dimensional compact and oriented Riemannian manifold without boundary isometrically immersed in $M^n(K)$. We also denote by x^α, r, Φ the restriction of x^α, r, Φ to M respectively. For a local coordinate system (U, u^j) on M, we put $x_j^\alpha := \frac{\partial x^\alpha}{\partial u^j}$, $r_j^2 := \frac{\partial r^2}{\partial u^j}$. Then we have

$$(2.3)^{*)} \qquad \sum_\alpha x_i^\alpha x_j^\alpha = \Phi^2 g_{ij}\,.$$

Here g is the Riemannian metric of M and $g_{ij} := g\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right)$. Let $\{\xi_t = (\xi_t^1, \cdots, \xi_t^n)\,;\ t = m+1, \cdots, n\}$ be orthonormal vector fields defined along U and perpendicular to M. Then we have

$$(2.4) \qquad \sum_\alpha \xi_t^\alpha x_j^\alpha = 0 \ ({}^\forall t, {}^\forall j)\,, \qquad \sum_\alpha \xi_t^\alpha \xi_{\bar t}^\alpha = \Phi^2 \delta_{t\bar t}\,.$$

The α-component of the second fundamental form $S\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right)$ is given by $\nabla_i x_j^\alpha + \sum_{\beta,\gamma} \left\{ {\alpha \atop \beta\ \gamma} \right\} x_i^\beta x_j^\gamma$ (cf. Spivak [8] p. 20). Here ∇ denotes the Levi-Civita connection with respect to g. Thus putting $S_{ij}^t := G\left(S\left(\frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j}\right), \xi_t\right)$, we get

$$(2.5) \qquad \nabla_i x_j^\alpha = \sum_t S_{ij}^t \xi_t^\alpha - \sum_{\beta,\gamma} \left\{ {\alpha \atop \beta\ \gamma} \right\} x_i^\beta x_j^\gamma\,.$$

The mean curvature vector field η is expressed on U as

$$\eta = \frac{1}{m} \sum_t \Big(\sum_{i,j} g^{ij} S_{ij}^t \Big) \xi_t\,,$$

where we put $(g^{ij}) := (g_{ij})^{-1}$. If (u^j) is a normal coordinate system centered at a point $x_0 \in M$, Then we have at x_0

$$(2.6) \qquad \|\eta\|^2 = \frac{1}{m^2} \sum_t \Big(\sum_j S_{jj}^t \Big)^2, \qquad \|S\|^2 = \sum_{t;i,j} (S_{ij}^t)^2\,.$$

From (2.6) we see easily that the inequality $m\|\eta\|^2 \leqslant \|S\|^2$ holds on M.

For each $p \in \{0, 1, \cdots, m\}$, $A^p(M)$ denotes the space of all differential p-forms on M and $\lambda_1^p(M)$ the first non-zero eignvalue of the Laplacian $\Delta = d\delta + \delta d$ acting on $A^p(M)$. For $\omega, \bar\omega \in A^p(M)$, we put

*) We will use the following convention on the range of indices unless otherwise stated: $\alpha, \beta, \gamma, \bar\alpha, \bar\beta, \bar\gamma \in \{1, \cdots, n\}$; $i, j, k, l \in \{1, \cdots, m\}$; $t, \bar t \in \{m+1, \cdots, n\}$. For a fixed $p \in \{1, \cdots, m\}$, $\mu, \nu, \bar\mu, \bar\nu \in \{1, \cdots, p\}$; σ and $\bar\sigma$ run over all permutations of $\{1, \cdots, p\}$ and $\varepsilon(\sigma)$, $\varepsilon(\bar\sigma)$ denote their signs. Repeated indices under a summation sign without indication are summed over the respective ranges.

$$(\omega|\bar{\omega}) := \sum_{j_1<\cdots<j_p} \omega_{j_1,\cdots,j_p} \bar{\omega}^{j_1,\cdots,j_p},$$

$$\langle\omega, \bar{\omega}\rangle := \int_M (\omega|\bar{\omega})\, dV_M \quad \text{and} \quad ||\omega|| := \sqrt{\langle\omega, \omega\rangle}.$$

Then we have the so-called minimum principle, namely,

$$\lambda_1^p(M) = \inf\left\{\frac{\langle\Delta\omega, \omega\rangle}{||\omega||^2};\ \omega\in A^p(M),\ \omega\neq 0,\ \langle\omega, \bar{\omega}\rangle = 0 \text{ for any harmonic } p\text{-form } \bar{\omega}\right\}.$$

For each $\alpha_1, \cdots, \alpha_p \in \{1, \cdots, n\}$ we define a p-form $\omega^{\alpha_1,\cdots,\alpha_p}$ on M by

$$\omega^{\alpha_1,\cdots,\alpha_p} := dx^{\alpha_1}\wedge\cdots\wedge dx^{\alpha_p}|M.$$

Since $\omega^{\alpha_1,\cdots,\alpha_p}$ is an exact form, we have $\langle\omega^{\alpha_1,\cdots,\alpha_p}, \bar{\omega}\rangle = 0$ for any harmonic p-form $\bar{\omega}$ (The Hodge Decomposition Theorem) and

$$\langle\Delta\omega^{\alpha_1,\cdots,\alpha_p}, \omega^{\alpha_1,\cdots,\alpha_p}\rangle = ||\delta\omega^{\alpha_1,\cdots,\alpha_p}||^2.$$

Hence by means of the minimum principle, we get

$$(2.7) \qquad \lambda_1^p(M) \sum_{\alpha_1,\cdots,\alpha_p} ||\omega^{\alpha_1,\cdots,\alpha_p}||^2 \leqslant \sum_{\alpha_1,\cdots,\alpha_p} ||\delta\omega^{\alpha_1,\cdots,\alpha_p}||^2.$$

LEMMA 1. *For each* $p\in\{1, \cdots, m\}$, *we have*

$$(2.8) \qquad \sum_{\alpha_1,\cdots,\alpha_p} ||\omega^{\alpha_1,\cdots,\alpha_p}||^2 = \frac{m!}{(m-p)!}\int_M \Phi^{2p}\, dV_M.$$

PROOF. Fix any point $x_0\in M$ and let (u^j) be a normal coordinate system on M centered at x_0. For each $j_1, \cdots, j_p \in \{1, \cdots, m\}$, the $(j_1, \cdots, j_p)$-component of $\omega^{\alpha_1,\cdots,\alpha_p}$ is given by

$$(\omega^{\alpha_1,\cdots,\alpha_p})_{j_1,\cdots,j_p} = \sum_\sigma \varepsilon(\sigma)\, x^{\alpha_1}_{j_{\sigma(1)}} \cdots x^{\alpha_p}_{j_{\sigma(p)}}.$$

Thus by means of (2.3) we have at x_0

$$\begin{aligned}
&\sum_{\alpha_1,\cdots,\alpha_p} (\omega^{\alpha_1,\cdots,\alpha_p}|\omega^{\alpha_1,\cdots\alpha_p})(x_0) \\
&= \sum_{\substack{j_1<\cdots<j_p \\ ;\,\alpha_1,\cdots,\alpha_p}} \sum_{\sigma,\bar{\sigma}} \varepsilon(\sigma)\,\varepsilon(\bar{\sigma})\, x^{\alpha_1}_{j_{\sigma(1)}} x^{\alpha_1}_{j_{\bar{\sigma}(1)}} \cdots x^{\alpha_p}_{j_{\sigma(p)}} x^{\alpha_p}_{j_{\bar{\sigma}(p)}} \\
&= \sum_{j_1<\cdots<j_p} p!\ \Phi^{2p} = \frac{m!}{(m-p)!}\,\Phi^{2p}.
\end{aligned}$$

By integrating both sides we get (2.8). Q. E. D.

Now we assume the following:

[*] Fix any point $x_0\in M$ and a normal coordinate system (u^j) on M centered at x_0.

For $\alpha_1, \cdots, \alpha_p \in \{1, \cdots, n\}$ and $j_2, \cdots, j_p \in \{1, \cdots, m\}$ with $j_2 < \cdots < j_p$, we have at x_0

$$
\begin{aligned}
(\delta\omega^{\alpha_1,\cdots,\alpha_p})_{j_2,\cdots,j_p}(x_0) &= -\sum_k \nabla_k (\omega^{\alpha_1,\cdots,\alpha_p})_{k,j_2,\cdots,j_p} \\
&= -\sum_{k;\mu;\sigma(\mu)=1} \varepsilon(\sigma) \sum_{\nu \neq \mu} x^{\alpha_1}_{j_{\sigma(1)}} \cdots \widehat{x^{\alpha_\nu}_{j_{\sigma(\nu)}}} \cdots x^{\alpha_\mu}_k \cdots x^{\alpha_p}_{j_{\sigma(p)}} (\nabla_k x^{\alpha_\nu}_{j_{\sigma(\nu)}}) \\
&\quad - \sum_{k;\mu;\sigma(\mu)=1} \varepsilon(\sigma)\, x^{\alpha_1}_{j_{\sigma(1)}} \cdots \widehat{x^{\alpha_\mu}_k} \cdots x^{\alpha_p}_{j_{\sigma(p)}} (\nabla_k x^{\alpha_\mu}_k)
\end{aligned}
$$

(cf. de Rham [7]). Here $\wedge$ over $x^{\alpha_\nu}_{j_{\sigma(\nu)}}$ indicates that it is omitted. Substituting (2.5), we get

$$
\begin{aligned}
&(\delta\omega^{\alpha_1,\cdots,\alpha_p})_{j_2,\cdots,j_p}(x_0) \\
&= -\sum_{k;\mu;\sigma(\mu)=1} \varepsilon(\sigma) \sum_{\nu \neq \mu} x^{\alpha_1}_{j_{\sigma(1)}} \cdots \widehat{x^{\alpha_\nu}_{j_{\sigma(\nu)}}} \cdots x^{\alpha_\mu}_k \cdots x^{\alpha_p}_{j_{\sigma(p)}} \sum_t S^t_{k j_{\sigma(\nu)}} \xi^{\alpha_\nu}_t \\
&\quad - \sum_{k;\mu;\sigma(\mu)=1} \varepsilon(\sigma)\, x^{\alpha_1}_{j_{\sigma(1)}} \cdots \widehat{x^{\alpha_\mu}_k} \cdots x^{\alpha_p}_{j_{\sigma(p)}} \sum_t S^t_{kk} \xi^{\alpha_\mu}_t \\
&\quad + \sum_{k;\mu;\sigma(\mu)=1} \varepsilon(\sigma) \sum_{\nu \neq \mu} x^{\alpha_1}_{j_{\sigma(1)}} \cdots \widehat{x^{\alpha_\nu}_{j_{\sigma(\nu)}}} \cdots x^{\alpha_\mu}_k \cdots x^{\alpha_p}_{j_{\sigma(p)}} \sum_{\beta,\gamma} \left\{ {\alpha_\nu \atop \beta\ \gamma} \right\} x^\beta_k x^\gamma_{j_{\sigma(\nu)}} \\
&\quad + \sum_{k;\mu;\sigma(\mu)=1} \varepsilon(\sigma)\, x^{\alpha_1}_{j_{\sigma(1)}} \cdots \widehat{x^{\alpha_\mu}_k} \cdots x^{\alpha_p}_{j_{\sigma(p)}} \sum_{\beta,\gamma} \left\{ {\alpha_\mu \atop \beta\ \gamma} \right\} x^\beta_k x^\gamma_k \\
&=: X^{(1)}_{(\alpha)(j)} + X^{(2)}_{(\alpha)(j)} + X^{(3)}_{(\alpha)(j)} + X^{(4)}_{(\alpha)(j)}\,.
\end{aligned}
$$

Here we put $(\alpha) := (\alpha_1, \cdots, \alpha_p)$, $(j) := (j_2, \cdots, j_p)$ for simplicity. Hence

$$
\begin{aligned}
&\sum_{\alpha_1,\cdots,\alpha_p} (\delta\omega^{\alpha_1,\cdots,\alpha_p} | \delta\omega^{\alpha_1,\cdots,\alpha_p})(x_0) \\
&\qquad = \sum_{\substack{j_2<\cdots<j_p \\ ;\alpha_1,\cdots,\alpha_p}} \sum_{a,b=1}^{4} X^{(a)}_{(\alpha)(j)} X^{(b)}_{(\alpha)(j)}\,.
\end{aligned}
$$

Then we have the following lemma which we will prove in the next Section.

LEMMA 2. *In the above situation* [*] *we have at* x_0

$$
\begin{aligned}
X^{(1)} &:= \sum_{\substack{j_2<\cdots<j_p \\ ;\alpha_1,\cdots,\alpha_p}} \sum_{a,b=1}^{2} X^{(a)}_{(\alpha)(j)} X^{(b)}_{(\alpha)(j)} \\
&= \frac{p\cdot(m-2)!}{(m-p)!} \left[m^2(m-p)\|\eta\|^2 + (p-1)\|S\|^2 \right] \Phi^{2p}\,,
\end{aligned} \tag{2.9}
$$

$$
\begin{aligned}
X^{(2)} &:= 2 \sum_{\substack{j_2<\cdots<j_p \\ ;\alpha_1,\cdots,\alpha_p}} \sum_{a=1}^{2} \sum_{b=3}^{4} X^{(a)}_{(\alpha)(j)} X^{(b)}_{(\alpha)(j)} \\
&= -Kp(m-p+1) \frac{m!}{(m-p)!} G(\eta, x_0)\, \Phi^{2p+1}\,,
\end{aligned} \tag{2.10}
$$

$$
X^{(3)} := \sum_{\substack{j_2<\cdots<j_p \\ ;\alpha_1,\cdots,\alpha_p}} \sum_{a,b=3}^{4} X^{(a)}_{(\alpha)(j)} X^{(b)}_{(\alpha)(j)} \tag{2.11}
$$

$$=\frac{K^2p(m-p+1)\cdot m!}{4\cdot(m-p)!}r^2\Phi^{2p}$$
$$-\frac{K^2p[(3p+1)m-4p^2]\cdot(m-1)!}{16\cdot(m-p)!}(dr^2|dr^2)\,\Phi^{2(p-1)}.$$

§ 3. Proof of Lemma 2.

By means of (2. 2) we have

$$\sum_{\beta,\gamma}\left\{{\alpha\atop \beta\ \gamma}\right\}x_i^\beta x_j^\gamma=\frac{K}{2\Phi}\left(\Phi^2x^\alpha g_{ij}-\frac{1}{2}r_i^2x_j^\alpha-\frac{1}{2}r_j^2x_i^\alpha\right).$$

Thus we get

$$\sum_{\alpha,\beta,\gamma}\xi_t^\alpha\left\{{\alpha\atop \beta\ \gamma}\right\}x_i^\beta x_j^\gamma=\frac{K}{2}\Phi g_{ij}\sum_\alpha\xi_t^\alpha x^\alpha, \tag{3. 1}$$

$$\sum_{\beta,\gamma,\bar\beta,\bar\gamma}\left\{{\alpha\atop \beta\ \gamma}\right\}\left\{{\bar\alpha\atop \bar\beta\ \bar\gamma}\right\}x_i^\beta x_j^\gamma x_k^{\bar\beta}x_l^{\bar\gamma} \tag{3. 2}$$
$$=\frac{K^2}{4}\Phi^2x^\alpha x^{\bar\alpha}g_{ij}g_{kl}-\frac{K^2}{8}x^\alpha(r_k^2x_l^{\bar\alpha}+r_l^2x_k^{\bar\alpha})\,g_{ij}$$
$$-\frac{K^2}{8}x^{\bar\alpha}(r_i^2x_j^\alpha+r_j^2x_i^\alpha)\,g_{kl}$$
$$+\frac{K^2}{16\Phi}(r_i^2x_j^\alpha+r_j^2x_i^\alpha)(r_k^2x_l^{\bar\alpha}+r_l^2x_k^{\bar\alpha}).$$

We can prove easily the next

LEMMA 3. *Let $i\mapsto\psi(i)$ and $(i,j)\mapsto\theta(i,j)$ be functions of $i,j\in\{1,\cdots,m\}$ such that $\theta(i,j)=\theta(j,i)$. Then we have*

$$\sum_{j_1,\cdots,j_p\,;\,j_\mu\neq j_\nu(\mu\neq\nu)}\Big[\sum_\nu\psi(j_\nu)\Big]=\frac{p\cdot(m-1)!}{(m-p)!}\sum_j\psi(j), \tag{3. 3}$$

$$\sum_{j_1,\cdots,j_p\,;\,j_\mu\neq j_\nu(\mu\neq\nu)}\Big[\sum_\nu\psi(j_\nu)\Big]^2=p(m-p)\frac{(m-2)!}{(m-p)!}\sum_j\psi(j)^2 \tag{3. 4}$$
$$+p(p-1)\frac{(m-2)!}{(m-p)!}\Big[\sum_j\psi(j)\Big]^2,$$

$$\sum_{j_1,\cdots,j_p\,;\,j_\mu\neq j_\nu(\mu\neq\nu)}\Big[\sum_{\mu,\nu}\theta(j_\mu,j_\nu)\Big] \tag{3. 5}$$
$$=p(m-p)\frac{(m-2)!}{(m-p)!}\sum_j\theta(j,j)+p(p-1)\frac{(m-2)!}{(m-p)!}\sum_{i,j}\theta(i,j).$$

1). Proof of (2. 9). Using (2. 3) and (2. 4), we have

$$\sum_{\alpha_1,\cdots,\alpha_p}\sum_{a,b=1}^{2} X^{(a)}_{(\alpha)(j)} X^{(b)}_{(\alpha)(j)}$$

$$=p!\ \Phi^{2p}\Big[\sum_{i;\,t}\sum_{\nu=2}^{p}(S^t_{ij_\nu})^2-\sum_{t}\sum_{\mu,\nu=2}^{p}(S^t_{j_\mu j_\nu})^2$$

$$+\sum_t\Big(\sum_{\nu=2}^{p}S^t_{j_\nu j_\nu}\Big)^2-2\sum_t\Big(\sum_{\nu=2}^{p}S^t_{j_\nu j_\nu}\Big)\Big(\sum_j S^t_{jj}\Big)+\sum_t\Big(\sum_j S^t_{jj}\Big)^2\Big].$$

By means of (2. 6) and Lemma 3, we get (2. 9). Q. E. D.

2). Proof of (2. 10). Using (2. 3), (2. 4) and (3. 1), we have

$$2\sum_{\alpha_1,\cdots,\alpha_p}\sum_{a=1}^{2}\sum_{b=3}^{4} X^{(a)}_{(\alpha)(j)} X^{(b)}_{(\alpha)(j)}$$

$$=K\cdot p!\,(m-p+1)\,\Phi^{2p-1}\sum_t\Big(\sum_{\nu=2}^{p}S^t_{j_\nu j_\nu}-\sum_i S^t_{ii}\Big)\Big(\sum_\alpha \xi^\alpha_t x^\alpha_0\Big).$$

By means of (2. 6) and (3. 3), we get (2. 10). Q. E. D.

3). Proof of (2. 11). Using (2. 3), we have

$$\sum_{\alpha_1,\cdots,\alpha_p}(X^{(3)}_{(\alpha)(j)})^2$$

$$=\Phi^{2(p-1)}\sum_{\mu;\sigma(\mu)=1}\Big[\sum_{\nu\neq\mu}\begin{Bmatrix}\alpha_\nu\\ \beta\ \gamma\end{Bmatrix}\begin{Bmatrix}\alpha_\nu\\ \bar\beta\ \bar\gamma\end{Bmatrix}x^\beta_i x^\gamma_{j_{\sigma(\nu)}}x^{\bar\beta}_i x^{\bar\gamma}_{j_{\sigma(\nu)}}$$

$$-\sum_{\nu\neq\mu;\bar\nu\neq\mu;\nu\neq\bar\nu}\begin{Bmatrix}\alpha_\nu\\ \beta\ \gamma\end{Bmatrix}\begin{Bmatrix}\alpha_\nu\\ \bar\beta\ \bar\gamma\end{Bmatrix}x^\beta_{j_{\sigma(\bar\nu)}}x^\gamma_{j_{\sigma(\nu)}}x^{\bar\beta}_{j_{\sigma(\bar\nu)}}x^{\bar\gamma}_{j_{\sigma(\nu)}}$$

$$+\sum_{\nu\neq\mu;\bar\nu\neq\mu;\nu\neq\bar\nu}\begin{Bmatrix}\alpha_\nu\\ \beta\ \gamma\end{Bmatrix}\begin{Bmatrix}\alpha_\nu\\ \bar\beta\ \bar\gamma\end{Bmatrix}x^\beta_{j_{\sigma(\nu)}}x^\gamma_{j_{\sigma(\nu)}}x^{\bar\beta}_{j_{\sigma(\bar\nu)}}x^{\bar\gamma}_{j_{\sigma(\bar\nu)}}\Big]$$

$$+\Phi^{2(p-2)}\sum_{\mu;\,\sigma(\mu)=1}\Big[-\sum_{\nu\neq\mu}x^{\alpha_\nu}_i x^{\alpha_\mu}_k\begin{Bmatrix}\alpha_\nu\\ \beta\ \gamma\end{Bmatrix}\begin{Bmatrix}\alpha_\mu\\ \bar\beta\ \bar\gamma\end{Bmatrix}x^\beta_k x^\gamma_{j_{\sigma(\nu)}}x^{\bar\beta}_i x^{\bar\gamma}_{j_{\sigma(\nu)}}$$

$$+\sum_{\nu\neq\mu;\bar\nu\neq\mu;\nu\neq\bar\nu}x^{\alpha_\nu}_{j_{\sigma(\nu)}}x^{\alpha_{\bar\nu}}_{j_{\sigma(\bar\nu)}}\begin{Bmatrix}\alpha_\nu\\ \beta\ \gamma\end{Bmatrix}\begin{Bmatrix}\alpha_{\bar\nu}\\ \bar\beta\ \bar\gamma\end{Bmatrix}x^\beta_i x^\gamma_{j_{\sigma(\nu)}}x^{\bar\beta}_i x^{\bar\gamma}_{j_{\sigma(\bar\nu)}}$$

$$-\sum_{\nu\neq\mu;\bar\nu\neq\mu;\nu\neq\bar\nu}x^{\alpha_\nu}_{j_{\sigma(\bar\nu)}}x^{\alpha_{\bar\nu}}_{j_{\sigma(\bar\nu)}}\begin{Bmatrix}\alpha_\nu\\ \beta\ \gamma\end{Bmatrix}\begin{Bmatrix}\alpha_{\bar\nu}\\ \bar\beta\ \bar\gamma\end{Bmatrix}x^\beta_i x^\gamma_{j_{\sigma(\nu)}}x^{\bar\beta}_i x^{\bar\gamma}_{j_{\sigma(\nu)}}$$

$$-2\sum_{\nu\neq\mu;\bar\nu\neq\mu;\nu\neq\bar\nu}x^{\alpha_\nu}_i x^{\alpha_{\bar\nu}}_{j_{\sigma(\bar\nu)}}\begin{Bmatrix}\alpha_\nu\\ \beta\ \gamma\end{Bmatrix}\begin{Bmatrix}\alpha_{\bar\nu}\\ \bar\beta\ \bar\gamma\end{Bmatrix}x^\beta_{j_{\sigma(\nu)}}x^\gamma_{j_{\sigma(\nu)}}x^{\bar\beta}_i x^{\bar\gamma}_{j_{\sigma(\bar\nu)}}$$

$$+2\sum_{\nu\neq\mu;\bar\nu\neq\mu;\nu\neq\bar\nu}x^{\alpha_\nu}_i x^{\alpha_{\bar\nu}}_{j_{\sigma(\bar\nu)}}\begin{Bmatrix}\alpha_\nu\\ \beta\ \gamma\end{Bmatrix}\begin{Bmatrix}\alpha_{\bar\nu}\\ \bar\beta\ \bar\gamma\end{Bmatrix}x^\beta_{j_{\sigma(\bar\nu)}}x^\gamma_{j_{\sigma(\nu)}}x^{\bar\beta}_i x^{\bar\gamma}_{j_{\sigma(\nu)}}$$

$$+\sum_{\substack{\bar\mu\neq\mu;\ \nu\neq\mu,\bar\mu;\\ \bar\nu\neq\mu,\bar\mu;\ \nu\neq\bar\nu}}\Big\{-x^{\alpha_\nu}_{j_{\sigma(\nu)}}x^{\alpha_{\bar\nu}}_{j_{\sigma(\bar\nu)}}\begin{Bmatrix}\alpha_\nu\\ \beta\ \gamma\end{Bmatrix}\begin{Bmatrix}\alpha_{\bar\nu}\\ \bar\beta\ \bar\gamma\end{Bmatrix}x^\beta_{j_{\sigma(\bar\mu)}}x^\gamma_{j_{\sigma(\nu)}}x^{\bar\beta}_{j_{\sigma(\bar\mu)}}x^{\bar\gamma}_{j_{\sigma(\bar\nu)}}$$

$$+x^{\alpha_\nu}_{j_{\sigma(\nu)}}x^{\alpha_{\bar\nu}}_{j_{\sigma(\bar\nu)}}\begin{Bmatrix}\alpha_\nu\\ \beta\ \gamma\end{Bmatrix}\begin{Bmatrix}\alpha_{\bar\nu}\\ \bar\beta\ \bar\gamma\end{Bmatrix}x^\beta_{j_{\sigma(\bar\nu)}}x^\gamma_{j_{\sigma(\nu)}}x^{\bar\beta}_{j_{\sigma(\bar\mu)}}x^{\bar\gamma}_{j_{\sigma(\bar\mu)}}$$

$$+x^{\alpha_\nu}_{j_{\sigma(\bar\nu)}}x^{\alpha_{\bar\nu}}_{j_{\sigma(\bar\nu)}}\begin{Bmatrix}\alpha_\nu\\ \beta\ \gamma\end{Bmatrix}\begin{Bmatrix}\alpha_{\bar\nu}\\ \bar\beta\ \bar\gamma\end{Bmatrix}x^\beta_{j_{\sigma(\bar\mu)}}x^\gamma_{j_{\sigma(\nu)}}x^{\bar\beta}_{j_{\sigma(\bar\mu)}}x^{\bar\gamma}_{j_{\sigma(\nu)}}$$

$$- x^{\alpha_\nu}_{j_{\sigma(\bar\nu)}} x^{\alpha_{\bar\nu}}_{j_{\sigma(\bar\nu)}} \left\{ \begin{matrix} \alpha_\nu \\ \beta \ \gamma \end{matrix} \right\} \left\{ \begin{matrix} \alpha_{\bar\nu} \\ \bar\beta \ \bar\gamma \end{matrix} \right\} x^{\beta}_{j_{\sigma(\nu)}} x^{\gamma}_{j_{\sigma(\nu)}} x^{\bar\beta}_{j_{\sigma(\bar\mu)}} x^{\bar\gamma}_{j_{\sigma(\bar\mu)}}$$

$$- x^{\alpha_\nu}_{j_{\sigma(\bar\mu)}} x^{\alpha_{\bar\nu}}_{j_{\sigma(\bar\nu)}} \left\{ \begin{matrix} \alpha_\nu \\ \beta \ \gamma \end{matrix} \right\} \left\{ \begin{matrix} \alpha_{\bar\nu} \\ \bar\beta \ \bar\gamma \end{matrix} \right\} x^{\beta}_{j_{\sigma(\bar\nu)}} x^{\gamma}_{j_{\sigma(\nu)}} x^{\bar\beta}_{j_{\sigma(\bar\mu)}} x^{\bar\gamma}_{j_{\sigma(\nu)}}$$

$$+ x^{\alpha_\nu}_{j_{\sigma(\bar\mu)}} x^{\alpha_{\bar\nu}}_{j_{\sigma(\bar\nu)}} \left\{ \begin{matrix} \alpha_\nu \\ \beta \ \gamma \end{matrix} \right\} \left\{ \begin{matrix} \alpha_{\bar\nu} \\ \bar\beta \ \bar\gamma \end{matrix} \right\} x^{\beta}_{j_{\sigma(\nu)}} x^{\gamma}_{j_{\sigma(\nu)}} x^{\bar\beta}_{j_{\sigma(\bar\mu)}} x^{\bar\gamma}_{j_{\sigma(\bar\nu)}} \Big\} \Big] ,$$

$$2 \sum_{\alpha_1, \cdots, \alpha_p} X^{(3)}_{(\alpha)(j)} X^{(4)}_{(\alpha)(j)}$$

$$= -2\Phi^{2(p-1)} \sum_{\mu;\, \sigma(\mu)=1;\, \nu\neq\mu} \left\{ \begin{matrix} \alpha_\nu \\ \beta \ \gamma \end{matrix} \right\} \left\{ \begin{matrix} \alpha_\nu \\ \bar\beta \ \bar\gamma \end{matrix} \right\} x^{\beta}_{j_{\sigma(\nu)}} x^{\gamma}_{j_{\sigma(\nu)}} x^{\bar\beta}_{i} x^{\bar\gamma}_{i}$$

$$+ \Phi^{2(p-2)} \sum_{\mu;\, \sigma(\mu)=1} \Big[2 \sum_{\nu\neq\mu} x^{\alpha_\nu}_{j_{\sigma(\nu)}} x^{\alpha_\mu}_{k} \left\{ \begin{matrix} \alpha_\nu \\ \beta \ \gamma \end{matrix} \right\} \left\{ \begin{matrix} \alpha_\mu \\ \bar\beta \ \bar\gamma \end{matrix} \right\} x^{\beta}_{k} x^{\gamma}_{j_{\sigma(\nu)}} x^{\bar\beta}_{i} x^{\bar\gamma}_{i}$$

$$-2 \sum_{\nu\neq\mu;\, \bar\nu\neq\mu;\, \nu\neq\bar\nu} x^{\alpha_\nu}_{j_{\sigma(\nu)}} x^{\alpha_{\bar\nu}}_{j_{\sigma(\bar\nu)}} \left\{ \begin{matrix} \alpha_\nu \\ \beta \ \gamma \end{matrix} \right\} \left\{ \begin{matrix} \alpha_{\bar\nu} \\ \bar\beta \ \bar\gamma \end{matrix} \right\} x^{\beta}_{j_{\sigma(\bar\nu)}} x^{\gamma}_{j_{\sigma(\nu)}} x^{\bar\beta}_{i} x^{\bar\gamma}_{i}$$

$$+2 \sum_{\nu\neq\mu;\, \bar\nu\neq\mu;\, \nu\neq\bar\nu} x^{\alpha_\nu}_{j_{\sigma(\bar\nu)}} x^{\alpha_{\bar\nu}}_{j_{\sigma(\bar\nu)}} \left\{ \begin{matrix} \alpha_\nu \\ \beta \ \gamma \end{matrix} \right\} \left\{ \begin{matrix} \alpha_{\bar\nu} \\ \bar\beta \ \bar\gamma \end{matrix} \right\} x^{\beta}_{j_{\sigma(\nu)}} x^{\gamma}_{j_{\sigma(\nu)}} x^{\bar\beta}_{i} x^{\bar\gamma}_{i} \Big] ,$$

$$\sum_{\alpha_1, \cdots, \alpha_p} (X^{(4)}_{(\alpha)(j)})^2$$

$$= p\,!\, \Phi^{2(p-1)} \sum_{\alpha} \Big(\sum_{i;\beta,\gamma} \left\{ \begin{matrix} \alpha \\ \beta \ \gamma \end{matrix} \right\} x^{\beta}_{i} x^{\gamma}_{i} \Big)^2$$

$$- \Phi^{2(p-2)} \sum_{\mu;\sigma(\mu)=1;\, \nu\neq\mu} x^{\alpha_\nu}_{j_{\sigma(\nu)}} x^{\alpha_\mu}_{j_{\sigma(\nu)}} \left\{ \begin{matrix} \alpha_\nu \\ \beta \ \gamma \end{matrix} \right\} \left\{ \begin{matrix} \alpha_\mu \\ \bar\beta \ \bar\gamma \end{matrix} \right\} x^{\beta}_{i} x^{\gamma}_{i} x^{\bar\beta}_{k} x^{\bar\gamma}_{k} .$$

Substituting (3. 2), we get

$$\sum_{\alpha_1, \cdots, \alpha_p} \sum_{a,b=3}^{4} X^{(a)}_{(\alpha)(j)} X^{(b)}_{(\alpha)(j)}$$

$$= \frac{K^2 \cdot p\,!\,(m-p+1)^2}{4} r^2 \Phi^{2p}$$

$$- \frac{K^2 \cdot p\,!}{16} (2m+2mp-3p^2-2p+1)\,(dr^2|dr^2)\, \Phi^{2(p-1)}$$

$$- \frac{K^2 \cdot p\,!}{16} (m^2-4mp+4p^2) \Big(\sum_{\nu=2}^{p} r^2_{j_\nu} r^2_{j_\nu} \Big) \Phi^{2(p-1)} .$$

By means of (3. 3) we obtain (2. 11). Q. E. D.

§ 4. Proof of Theorem.

Since the duality $\lambda_1^p(M) = \lambda_1^{m-p}(M)$ holds, it is sufficient to prove our theorem for $p \leqslant \left[\frac{m}{2} \right]$. Here $\left[\frac{m}{2} \right]$ is the integral part of $\frac{m}{2}$. We recall that for the standard m-sphere $S^m(K)$ of constant curvature $K>0$ we have

$$\lambda_1^p\left(S^m(K)\right)=\begin{cases} mK & (p=0), \\ p(m-p+1)\,K & \left(1\leqslant p\leqslant\left[\frac{m}{2}\right]\right) \end{cases}$$

(see Ikeda-Taniguchi [3]).

1). Proof of Theorem (A). For $p=0$ the inequality (1.1) is nothing but one obtained by Bleecker-Weiner. So we will prove (1.1) for $1\leqslant p\leqslant\left[\frac{m}{2}\right]$. By integrating the both sides of (2.9) we get

$$\begin{aligned}\sum_{\alpha_1,\cdots,\alpha_p}&||\delta\omega^{\alpha_1,\cdots,\alpha_p}||^2\\ &=\frac{p\cdot(m-2)\,!}{(m-p)\,!}\int_M\left[m^2(m-p)||\eta||^2+(p-1)||S||^2\right]dV_M\\ &\leqslant\frac{p(m-p+1)\cdot(m-1)\,!}{(m-p)\,!}\int_M||S||^2dV_M,\text{ because of } m||\eta||^2\leqslant||S||^2.\end{aligned}$$

Thus from (2.7) and (2.8) we obtain (1.1). If the equality holds in (1.1), then we have $m||\eta||^2=||S||^2$ on M. This implies that M is a totally umbilical submanifold in E^n. Hence M is embedded as a geodesic sphere in some $(m+1)$-dimensional totally geodesic submanifold in E^n (see Chen [2] p. 50). Conversely, if M is embedded as a geodesic sphere in some $(m+1)$-dimensional totally geodesic submanifold in E^n, then we see easily that the equality holds in (1.1). Q. E. D.

REMARK 1. From the above argument we have the inequality

$$\lambda_1^p(M)\leqslant\frac{p}{m\,(m-1)\,Vol(M)}\int_M\left[m^2(m-p)||\eta||^2+(p-1)||S||^2\right]dV_M$$

$(1\leqslant p\leqslant m)$ under the assumption of our theorem (A).

2). Proof of Theorem (B). Choose a point $z_0\in S^n(1)$ such that $\rho(z_0, M)\geqslant\rho(z, M)$ for any point $z\in S^n(1)$. Then we have $z_0\notin M$. The antipodal point $\tilde{z}_0$ of z_0 satisfies $\rho(M)=\max\{\rho(\tilde{z}_0, x);\ x\in M\}$. We see that $0<\rho(M)<\pi$. We may identify $M^n(1)$ defined in Section 2 with the tangent space $T_{\tilde{z}_0}S^n(1)$ to $S^n(1)$ at $\tilde{z}_0$ with the Riemannian metric which is induced by the stereographic projection $\zeta: S^n(1)-\{z_0\}\to T_{\tilde{z}_0}S^n(1)$. Since $z_0\notin M$, we may regard $M\subset S^n(1)$ as $\zeta(M)\subset M^n(1)$. Thus we arrive at the same situation as Section 2. In (2.10) we have

$$-G(\eta, x_0)\leqslant\left|G(\eta, x_0)\right|\leqslant||\eta||\,||x_0||=||\eta||\,r(x_0)\,\Phi^{-1}\leqslant\frac{S(M)}{\sqrt{m}}r(M)\,\Phi^{-1},$$

where we put $r(M):=\max\{r(x);\ x\in M\}$. From (2.9) and (2.10) we get

$$X^{(1)}+X^{(2)}\leqslant p(m-p+1)\left[\frac{S(M)^2}{m}+\frac{S(M)}{\sqrt{m}}r(M)\right]\frac{m!}{(m-p)!}\Phi^{2p}\,. \tag{4.1}$$

By means of $p\leqslant\frac{m}{2}$ we have $(3p+1)m-4p^2>0$ in (2.11) and we get

$$X^{(3)}\leqslant p(m-p+1)\frac{r(M)^2}{4}\frac{m!}{(m-p)!}\Phi^{2p}\,. \tag{4.2}$$

From (4.1) and (4.2) we have

$$\sum_{\alpha_1,\cdots,\alpha_p}||\delta\omega^{\alpha_1,\cdots,\alpha_p}||^2 \leqslant p(m-p+1)\left[\frac{S(M)}{\sqrt{m}}+\frac{r(M)}{2}\right]^2\frac{m!}{(m-p)!}\int_M\Phi^{2p}dV_M\,.$$

The choice of z_0 and (2.1) imply that $r(M)=2\tan\left(\frac{\rho(M)}{2}\right)$. Hence from (2.7) and (2.8) we obtain (1.2). If the equality holds in (1.2), then we have $m||\eta||^2=||S||^2$ on M. This implies that M is a totally umbilical submanifold in $S^n(1)$. Therefore M is embedded as a geodesic sphere in some $(m+1)$-dimensional totally geodesic submanifold in $S^n(1)$ (see Chen [2] p. 50). Conversely, if M is embedded as a geodesic sphere in some $(m+1)$-dimensional totally geodesic submanifold in $S^n(1)$, then $||S||^2$ is constant and equal to $m\left(\frac{\cos\rho(M)}{\sin\rho(M)}\right)^2$. On the other hand, since M is isometric to $S^m\left(\frac{1}{\sin^2\rho(M)}\right)$, we have $\lambda_1^p(M)=p(m-p+1)\frac{1}{\sin^2\rho(M)}$. Therefore the equality holds in (1.2). Q. E. D.

REMARK 2. By means of $S^n(1)\subset E^{n+1}$, using the inequality (1.1), we have also the inequality

$$\lambda_1^p(M)\leqslant\lambda_1^p\left(S^m(1)\right)\left[\frac{1}{m\,Vol(M)}\int_M||S||^2dV_M+1\right]$$

under the assumprion of our theorem (B).

3). Proof of Theorem (C). Since $M^n(-1)\cong H^n(-1)$ is homogeneous, we may assume that $0\in M\subset M^n(-1)$. By an argument similar to that of the proof of theorem (B), we get

$$\lambda_1^p(M)\leqslant p(m-p+1)\left[\frac{S(M)}{\sqrt{m}}+\frac{r(M)}{2}\right]^2.$$

If ρ_M denotes the distance function of M, then we have $\tilde{\rho}(0,x)\leqslant\rho_M(0,x)\leqslant d(M)$ for any point $x\in M$. From (2.1) we have $r(M)\leqslant2\tanh\left(\frac{d(M)}{2}\right)$ to obtain the inequality (1.3). Q. E. D.

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