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Note on automorphisms in separable extension of non commutative ring

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Preliminaries

All definitions and terminologies in this paper are the same as those in the same author's papers [8], [11] and [13]. So Λ shall be a ring with an identity 1, Γ a subring of Λ which contains 1, C the center of Λ, C' the center of Γ and $\Delta=V_{\Lambda}(\Gamma)=\{x\in\Lambda | xr=rx$ for all $r\in\Gamma\}$. Λ is an H-separable extension of Γ if $\Lambda\otimes_{\Gamma}\Lambda$ is a $\Lambda-\Lambda$-direct summand of some finite direct sum of copies of Λ. In this case Λ is a separable extension of Γ, *i.e.*, map π of $\Lambda\otimes_{\Gamma}\Lambda$ to Λ such that $\pi(x\otimes y)=xy$, for x, $y\in\Lambda$, splits as $\Lambda-\Lambda$-map. As for the fundamental properties of H-separable extension, see [4], [5] and [12]. In [11] and [13] the author showed that in case Γ is a simple artinean ring, Λ is an H-separable extension of Γ if and only if Λ is an inner Galois extension of Γ. It is well known that in this case every automorphism of Λ which fixes all elements of Γ is an inner automorphism. In this paper we will generalize this theorem to the case of ordinal H-separable extensions (Theorem 2). We will also show that every G-Galois extension such that all elements of G are inner automorphisms is an H-separable extension (Theorem 3). For a two-sided Λ-module M, we denote C-submodule $\{m\in M| xm=mx$ for all $x\in\Lambda\}$ by M^{Λ}. Then, Λ is H-separable over Γ if and only if $\Delta\otimes_{C}M^{\Lambda}\cong M^{\Gamma}$ by $(d\otimes m\rightarrow dm)$ (see Theorem 1.2 [8]) for every two sided Λ-module M. We will use this theorem very often throughout this paper. For a ring Λ we denote the Jacobson radical of Λ by $J(\Lambda)$. We will also study in § 3 in what case $J(\Lambda)=\Lambda J(\Gamma)=J(\Gamma)\Lambda$ and $J(\Gamma)=J(\Lambda)\cap\Gamma$ holds when Λ is H-separable over Γ.

1. Automorphisms in H-separable extensions.

The first result is a supplement of Theorem 2 [5].

THEOREM 1. *Let Λ be an H-separable extension of Γ. Then every ring endomorphism of Λ which fixes all elements of Γ is an automorphism and fixes all elements of $V_{\Lambda}(V_{\Lambda}(\Gamma))$.*

PROOF. Let σ be an arbitrary ring endomorphism of Λ with $\sigma(r)=r$ for all $r\in\Gamma$. Then, $\sigma\in\mathrm{Hom}\,({}_{\Gamma}\Lambda_{\Gamma}, {}_{\Gamma}\Lambda_{\Gamma})\cong\Delta\otimes_C\Delta^0$ (see (1. 5) [12]). Hence there exists $\sum d_i\otimes e_i^0\in\Delta\otimes_C\Delta^0$ such that $\sigma(x)=\sum d_i x e_i$ for all $x\in\Lambda$. Then for any $r\in V_\Lambda(\Delta)$, $\sigma(r)=r\sum d_i e_i=r$, since $\sigma(1)=\sum d_i e_i=1$. Thus σ fixes all elements of $V_\Lambda(\Delta)$. Then σ fixes all elements of C, since $C\subset V_\Lambda(\Delta)$. Then by Theorem 2 (b) [5], σ is an automorphism.

THEOREM 2. *Let Λ be an H-separable extension of Γ, and let $\bar{\Lambda}=\Lambda/J(\Lambda)$, $\bar{\Gamma}=\Gamma/J(\Lambda)\cap\Gamma$ and $\bar{\Delta}=V_{\bar{\Lambda}}(\bar{\Gamma})$. Then if Λ is artinean, and if Δ is mapped onto $\bar{\Delta}$ by the natural map, every automorphism of Λ which fixes all elements of Γ is an inner automorphism.*

In order to prove this theorem we need the following

PROPOSITION 1. *Let Λ be a separable extension of Γ, and $\mathfrak{a}$ be an ideal of Λ which is contained in $J(\Lambda)$. Let σ be an automorphism of Λ which fixes all elements of Γ. Then if σ induces the identity automorphism of $\bar{\Lambda}$, $\bar{\sigma}(\bar{x})=\overline{\sigma(x)}$ for all $x\in\Lambda$, σ is an inner automorphism, where $\bar{\Lambda}=\Lambda/\mathfrak{a}$ and $\bar{x}=x+\mathfrak{a}$ in $\bar{\Lambda}$, for $x\in\Lambda$.*

PROOF. Let $\delta(x)=\sigma(x)-x$ for $x\in\Lambda$. Then δ is a Γ-derivation of Λ to a $\Lambda-\Lambda$-module $\mathfrak{a}$, where the right Λ-module structure of $\mathfrak{a}$ is defined by $a\cdot x=a\sigma(x)$, for $a\in\mathfrak{a}$ and $x\in\Lambda$. Then by Satz 4. 2 [2], δ is an inner derivation, and there exists $a\in\mathfrak{a}$ such that $\sigma(x)-x=xa-a\sigma(x)$ $(=\delta(x))$, for all $x\in\Lambda$. Hence $(1+a)\,\sigma(x)=x(1+a)$. But since $a\in J(\Lambda)$, $1+a$ is a unit. Therefore σ is an inner automorphism.

PROPOSITION 2. *Let Λ be a two sided simple ring (not necessarily artinean) and an H-separable extension of some subring Γ. Then every automorphism of Λ which fixes all elements of Γ is an inner automorphism.*

PROOF. Let σ an automorphism of Λ with $\sigma(x)=x$ for all $x\in\Lambda$. Let Λ_σ be a $\Lambda-\Lambda$-bimodule defined by the following way; $\Lambda_\sigma=\Lambda$ as left Λ-module, but right Λ-module structure of Λ_σ is defined by $x\cdot y=x\sigma(y)$, for $x, y\in\Lambda$. Let $J_\sigma=\{a\in\Lambda|\,xa=a\sigma(x)$ for any $x\in\Lambda\}$. Then clearly $(\Lambda_\sigma)^\Lambda=J_\sigma\subseteq\Delta$ and $(\Lambda_\sigma)^\Gamma=\Delta$. On the other hand $\Delta=(\Lambda_\sigma)^\Gamma\cong\Delta\otimes_C(\Lambda_\sigma)^\Lambda=\Delta\otimes_C J_\sigma$, since Λ is an H-separable extension of Γ. Then, since C is a field, $[J_\sigma:C]=1$, and $J_\sigma=Cu_\sigma$ for some $0\neq u_\sigma\in J_\sigma$. Then, clearly Λu_σ is a two sided ideal of a simple ring Λ. Therefore, $\Lambda u_\sigma=\Lambda$, and we see that u_σ is an unit of Λ. Since $u_\sigma\in J_\sigma$, we have $\sigma(x)=u_\sigma^{-1}xu_\sigma$, for all $x\in\Lambda$.

PROOF of THEOREM 2. Let $\bar{\Gamma}'=V_{\bar{\Lambda}}(\bar{\Delta})$ and $\bar{C}$ be the center of $\bar{\Lambda}$. By Proposition 3. 2 [13] and Theorem 1. 3′ [8], $\bar{\Lambda}$ is an H-separable extension of both $\bar{\Gamma}$ and $\bar{\Gamma}'$. Since $\sigma(J(\Lambda))=J(\Lambda)$, σ induces an automorphism $\bar{\sigma}$ of $\bar{\Lambda}$ which fixes all elements of $\bar{\Gamma}$. Then $\bar{\sigma}$ fixes all elements of $\bar{\Gamma}'$ by Theorem

1. Since $\bar{\Gamma}' \supseteq \bar{C}$, all central idempotents of $\bar{\Lambda}$ are also central idempotents of $\bar{\Gamma}'$. Hence if $\bar{\Lambda} = \bar{\Lambda}_1 \oplus \bar{\Lambda}_2 \oplus \cdots \oplus \bar{\Lambda}_2$ is a decomposition of $\bar{\Lambda}$ into simple rings, and if $\bar{1} = \bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_n$, $\bar{\Lambda}_i = \bar{\Lambda}\bar{e}_i$ with $\bar{e}_i$ primitive idempotents of the center of $\bar{\Lambda}$, then $\bar{\Gamma}'_1 \oplus \bar{\Gamma}'_2 \oplus \cdots \oplus \bar{\Gamma}'_n = \bar{\Gamma}'$ with $\bar{\Gamma}'_i = \bar{\Gamma}'\bar{e}_i$ as ring, and $\bar{\Gamma}'_i$ is a subring of $\bar{\Lambda}_i$ for each i. Then clearly each $\bar{\Lambda}_i$ is an H-separable extension of $\bar{\Gamma}'_i$ and $\bar{\sigma}_i$, the restriction of $\bar{\sigma}$ to $\bar{\Lambda}_i$, is an automorphism of $\bar{\Lambda}_i$ which fixes all elements of $\bar{\Gamma}_i$, because $\bar{\sigma}(\bar{e}_i) = \bar{e}_i$ for each i. Therefore, each $\bar{\sigma}_i$ is an inner automorphism of $\bar{\Lambda}_i$ induced by a unit of $V_{\bar{\Lambda}_i}(\bar{\Gamma}'_i)$. Then $\bar{\sigma}$ is an inner automorphism of $\bar{\Lambda}$ induced by a unit of $\bar{\Delta} = \sum^{\oplus} V_{\bar{\Lambda}_i}(\bar{\Gamma}'_i)$. Let $\bar{d}$ be such a unit of $\bar{\Delta}$, *i.e.*, $\bar{\sigma}(\bar{x}) = \bar{d}^{-1}\bar{x}\bar{d}$, for all $\bar{x} \in \bar{\Lambda}$, and d be a representative of $\bar{d}$ in Λ. By assumption we can choose d from Δ. Since $\bar{d}$ is a unit in $\bar{\Lambda}$, we have $dd' = 1 + m$ for some $d' \in \Delta$ and $m \in J(\Lambda)$. But $1+m$ is a unit. Hence d is also a unit in Δ. Let τ be an automorphism of Λ defined by $\tau(x) = d\sigma(x)d^{-1}$ for all $x \in \Lambda$. Then τ fixes all elements of Γ since $d \in \Delta$, and we see that $\bar{\tau}(\bar{x}) =$ identity on $\bar{\Lambda}$. Then by Prop. 1, τ is an inner automorphism, and σ is also an inner automorphism of Λ.

2. Relation with Galois extensions

Let Λ be a ring and G a finite group of automorphisms of Λ, $\Lambda^G = \{x \in \Lambda \mid \sigma(x) = x \text{ for all } \sigma \in G\}$. Let $S = \Delta(\Lambda : G)$ be the trivial crossed product of Λ and G, that is, $S = \sum_{\sigma \in G}^{\oplus} \Lambda U_\sigma$, a free Λ-module with a free basis $\{U_\sigma\}_{\sigma \in G}$, where the product is defied by $\lambda U_\sigma \gamma U_\tau = \lambda\sigma(\gamma) U_{\sigma\tau}$ for $\lambda, \gamma \in \Lambda$, $\sigma, \tau \in G$. Then there exists a ring homomorphism

$$j: \Delta(\Lambda : G) \longrightarrow \mathrm{Hom}(\Lambda_\Gamma, \Lambda_\Gamma) \qquad j(\lambda U_\sigma)(x) = \lambda\sigma(x)$$

for $\lambda, x \in \Lambda$, $\sigma \in G$. Now, following T. Kanzaki [6], we say that Λ is a G-Galois extension of Γ, if (1) $\Gamma = \Lambda^G$ (2) Λ is right Γ-finitely generated projective, and (3) map j is an isomorphism.

LEMMA 1. *Let Λ be a G-Galois extension of Γ. Then, we have*

(1) *There exists $c \in C$ with $t_G(c) = 1$ if and only if ${}_\Gamma\Gamma_\Gamma < \oplus {}_\Gamma\Lambda_\Gamma$, where $t_G(x) = \sum_{\sigma \in G} \sigma(x)$ for $x \in \Lambda$.*

(2) *Suppose furthermore $C \subseteq \Gamma$, then $|G| = n$ is a unit in C if and only if ${}_\Gamma\Gamma_\Gamma < \oplus {}_\Gamma\Lambda_\Gamma$.*

PROOF. (1). If there exists $c \in C$ with $t_G(c) = 1$, we obtain a $\Lambda - \Lambda$-map f of Λ to Λ defined by $f(x) = xc$ for $x \in \Lambda$. Then we have $(t_G \circ f)(r) = t_G(rc) = rt_G(c) = r$ for all $r \in \Gamma$. Therefore ${}_\Gamma\Gamma_\Gamma < \oplus {}_\Gamma\Lambda_\Gamma$. Conversely suppose ${}_\Gamma\Gamma_\Gamma < \oplus {}_\Gamma\Lambda_\Gamma$. Then since Λ is right Γ-finitely generated projective, $\mathrm{Hom}(\Lambda_\Gamma, \Lambda_\Gamma)$ is a separable extension of Λ by Theorem 7 [10]. Then $S = \Delta(\Lambda, G)$ is a separable extension of Λ. Hence there exists $\sum\alpha_i\otimes\beta_i\in(S\otimes_\Lambda S)^S$ with $\sum\alpha_i\beta_i=1$. We can put $\sum\alpha_i\otimes\beta_i=\sum x_{\sigma,\tau}U_\sigma\otimes U_\tau=\sum x_{\sigma,\sigma^{-1}\tau}U_\sigma\otimes U_{\sigma^{-1}\tau}$, where $x_{\sigma,\tau}\in\Lambda$ and σ, $\tau\in G$. $\sum\alpha_i\beta_i=1$ implies $\sum x_{\sigma,\sigma^{-1}\tau}U_\tau=U_1$. Hence we have $\sum x_{\sigma,\sigma^{-1}}=1$ and $\sum x_{\sigma,\sigma^{-1}\tau}=0$ $(\tau\neq1)$. On the other hand, $\sum U_\rho\alpha_i\otimes\beta_i=\sum\alpha_i\otimes\beta_i U_\rho$, for all $\rho\in G$, implies that $\rho(x_{\sigma,\tau})=x_{\rho\sigma,\tau\rho^{-1}}$ for all σ, τ, $\rho\in G$, and $\sum x\alpha_i\otimes\beta_i=\sum\alpha_i\otimes\beta_i x$ for all $x\in\Lambda$, implies that $x_{\sigma,\tau}\in J_{\sigma\tau}$ for all σ, $\tau\in G$. Especially we have $\rho(x_{1,1})=x_{\rho,\rho^{-1}}$ and $x_{1,1}\in J_1=C$. Hence we have $1=\sum x_{\sigma,\sigma^{-1}}=\sum\sigma(x_{1,1})$. (2) follows from (1), since $t_G(n^{-1})=\sum n^{-1}\sigma(1)=n^{-1}n=1$, and $x_{1,1}\in C\subseteq\Gamma$ implies that $t_G(x_{1,1})=\sum\sigma(x_{1,1})=nx_{1,1}=1$.

PROPOSITION 3. *Let Λ be an H-separable and G-Galois extension of Γ. Then we have*

(1) $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$.

(2) *Δ is a rank n projective module, where $n=|G|$.*

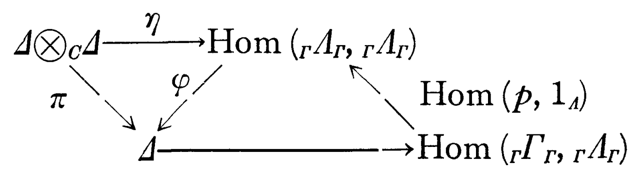
(3) *Following three conditions are equivalent*

(i) *$n(=|G|)$ is a unit.*

(ii) *Γ is a $\Gamma-\Gamma$-direct summand of Λ.*

(iii) *Δ is a separable C-algebra.*

PROOF. (1). By Prop. 1, we see that every element σ of G fixes all element of $\Gamma'(=V_\Lambda(\Delta))$. Thus we have $\Gamma'\subseteq\Lambda^G=\Gamma$. The converse inclusion is obvious. (2). By (1. 3) (4) [12], $\Delta=(\Lambda_\sigma)^\Gamma\cong\Delta\otimes_C(\Lambda_\sigma)^\Lambda=\Delta\otimes_C J_\sigma$. It is already known that Δ is C-finitely generated projective, and C is a C-direct summand of Δ. Hence J_σ is rank one projective C-module. On the other hand, $\Delta\cong$ Hom $({}_\Lambda\Lambda_\Gamma, {}_\Lambda\Lambda_\Gamma)\cong\Delta(\Delta: G)^\Lambda=(\sum^\oplus\Lambda U_\sigma)^\Lambda=\sum^\oplus_{\sigma\in G}J_\sigma$. Thus Δ is rank n projective C-module. (3). Since $C\subseteq V_\Lambda(\Delta))=\Gamma$ by (1), the equivalence of (i) and (ii) follows from Lemma 1. The equivalence of (ii) and (iii) follows from Prop. 4. 7 [3] and Corollary 1. 2 [9], since $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$. But the author will repeat the proof here for the convenience to readers. Suppose (ii), and let p be the $\Gamma-\Gamma$-map of Λ to Γ with $p(1)=1$. Then we have a commutative diagram of $\Delta-\Delta$-maps



where $\eta(d\otimes e)(x)=dxe$, $\pi(d\otimes e)=de$, for d, $e\in\Delta$ and $x\in\Lambda$, $\varphi(f)=f(1)$, for $f\in$ Hom $({}_\Gamma\Lambda_\Gamma, {}_\Gamma\Lambda_\Gamma)$, and $n(d)=dr$ $(=rd)$, for $d\in\Delta$, $r\in\Gamma$. η and n are isomorphisms (see (1. 5) [12]). Thus π splits as $\Delta-\Delta$-map. Suppose (iii). Then there exists $\sum d_i\otimes e_i\in(\Delta\otimes_C\Delta)^\Delta$ with $\sum d_i e_i=1$. Hence we obtain map p of Λ to Γ' $(=V_\Lambda(\Delta))$ such that $p(x)=\sum d_i x e_i$ for all $x\in\Lambda$. p is a $\Gamma'-\Gamma'$-map with $p(r)=r$ for all $r\in\Gamma$. Thus we have (ii).

As an example of H-separable G-Galois extensionst we have

THEOREM 3. *Let Λ be a G-Galois extension of Γ. Then if all elements of G are inner automorphisms of Λ, then Λ is an H-separable extension of Γ, and Δ is a free C-module of rank n, where $n=|G|$.*

PROOFS For each $\sigma\in G$, let γ_σ be a unit of Δ such that $\sigma(x)=\gamma_\sigma^{-1}x\gamma_\sigma$ for all $x\in\Lambda$s Note that each ΛU_σ is a $\Lambda-\Lambda$-module with formulae $U_\sigma\lambda=\sigma(\lambda)U_\sigma$ for each $\lambda\in\Lambda$, and that j is a $\Lambda-\Lambda$-isomorphisms Then for each $\sigma\in G$, define a map f_σ of ΛU_σ to Λ by $f_\sigma(\lambda U_\sigma)=\lambda\gamma_\sigma^{-1}$ for each $\lambda\in\Lambda$. Then since $f_\sigma(U_\sigma\lambda)=f_\sigma(\sigma(\lambda)U_\sigma)=\sigma(\lambda)\gamma_\sigma^{-1}=\gamma_\sigma^{-1}\lambda=f_\sigma(U_\sigma)\lambda$ for each $\lambda\in\Lambda$, f_σ is a $\Lambda-\Lambda$-isomorphism. Hence we have $\mathrm{Hom}(\Lambda_\Gamma,\Lambda_\Gamma)=\Lambda\oplus\Lambda\oplus\cdots\oplus\Lambda$ (n folds) as $\Lambda-\Lambda$-module. Then,

$$\Delta\cong\mathrm{Hom}({}_\Lambda\Lambda_\Gamma,{}_\Lambda\Lambda_\Gamma)=\left[\mathrm{Hom}(\Lambda_\Gamma,\Lambda_\Gamma)\right]^\Lambda\cong\left[\Lambda\oplus\Lambda\oplus\cdots\oplus\Lambda\right]^\Lambda=C\oplus C\oplus\cdots\oplus C$$

Hence Δ is a free C-module of rank n. On the other hand, since Λ is right Γ-finitely generated projective, we have

$$\Lambda\otimes_\Gamma\Lambda\cong\Lambda\otimes_\Gamma\mathrm{Hom}({}_\Lambda\Lambda,{}_\Lambda\Lambda)\cong\mathrm{Hom}\left({}_\Lambda\mathrm{Hom}(\Lambda_\Gamma,\Lambda_\Gamma),{}_\Lambda\Lambda\right)$$
$$\cong\mathrm{Hom}\left({}_\Lambda(\Lambda\oplus\Lambda\oplus\cdots\oplus\Lambda),{}_\Lambda\Lambda\right)\cong\Lambda\oplus\Lambda\oplus\cdots\oplus\Lambda$$

as $\Lambda-\Lambda$-module. Thus Λ is an H-separable extension of Γ.

REMARK. In the proof of Theorem 3, we see that the $\Lambda-\Lambda$-isomorphism of $\Lambda\oplus\Lambda\oplus\cdots\oplus\Lambda$ to $\Delta(\Lambda:G)$ is given by; $(\lambda_\rho,\lambda_\sigma,\cdots,\lambda_\tau)\to\sum_{\sigma\in G}\lambda_\sigma\gamma_\sigma U_\sigma$. On the other hand, the isomorphism $\mathrm{Hom}({}_\Lambda\Lambda_\Gamma,{}_\Lambda\Lambda_\Gamma)\to\Delta$ is given by; $f\to f(1)$ for $f\in\mathrm{Hom}({}_\Lambda\Lambda_\Gamma,{}_\Lambda\Lambda_\Gamma)$. Therefore, $C\oplus C\oplus\cdots\oplus C$ is mapped onto $j(\sum_{\sigma\in G}^{\oplus}C\gamma_\sigma U_\sigma)=\sum_{\sigma\in G}^{\oplus}C\gamma_\sigma$. Thus we have $V_\Lambda(\Gamma)=\sum_{\sigma\in G}^{\oplus}C\gamma_\sigma$.

REMARK. Λ is a G-Galois extension of Γ if and only if there exist $x_i, y_i\in\Lambda$ $(i=1,2,\cdots,n)$ such that $\sum x_i\sigma(y_i)=\sigma_{1,\sigma}$ by Prop. 2. 4 [6]. Then, under the condition of Theorem 3, it can be directly computed that $1\otimes 1=\sum_{\sigma\in G}\gamma_\sigma(\sum x_i\otimes\sigma(y_i)\gamma_\sigma^{-1})$ in $\Lambda\otimes_\Gamma\Lambda$, with $\gamma_\sigma\in\Delta$ and $\sum x_i\otimes\sigma(y_i)\gamma_\sigma^{-1}\in(\Lambda\otimes_\Gamma\Lambda)^\Lambda$. We call these $\{\gamma_\sigma,\sum x_i\otimes\sigma(y_i)\gamma_\sigma^{-1}\}_{\sigma\in G}$ an H-system for $\Lambda|\Gamma$ (see [5]).

3. On radicals in *H*-separable extensions.

PROPOSITION 4. *Let Λ be an H-separable extension of Γ with ${}_\Gamma\Gamma_\Gamma<\oplus\ {}_\Gamma\Lambda_\Gamma$. Then if $\Lambda/J(\Lambda)$ is artinean, we have $J(\Lambda)=\Lambda J(\Gamma)=J(\Gamma)\Lambda$ and $J(\Gamma)=J(\Lambda)\cap\Gamma$.*

Proof. By Theorem 4.1 (2) [13], $J(\Lambda)=\Lambda(J(\Lambda)\cap\Gamma)=(J(\Lambda)\cap\Gamma)\Lambda$. Hence we need only to show that $J(\Gamma)=J(\Lambda)\cap\Gamma$. Since $\Gamma=V_{\Lambda}(\Delta)$, every element of $J(\Lambda)\cap\Gamma$ has its quasi-inverse in Γ. Therefore $J(\Lambda)\cap\Gamma\subseteq J(\Gamma)$. Let $\bar{\Lambda}=\Lambda/J(\Lambda)$ and $\bar{\Gamma}=\Gamma/J(\Lambda)\cap\Gamma$. Then $\bar{\Lambda}$ is an H-separable extension of $\bar{\Gamma}$, and ${}_{\bar{\Gamma}}\bar{\Gamma}_{\bar{\Gamma}}<\oplus_{\bar{\Gamma}}\bar{\Lambda}_{\bar{\Gamma}}$, by Prop. 3.4 (1) [13]. Let $\bar{\Lambda}=\bar{\Gamma}\oplus M$ as $\bar{\Gamma}-\bar{\Gamma}$-module and $\mathfrak{l}$ be an arbitrary left ideal of $\bar{\Gamma}$. Then $\bar{\Lambda}=\bar{\Lambda}\mathfrak{l}\oplus L$ as left $\bar{\Lambda}$-module. Then $\bar{\Lambda}=(\mathfrak{l}\oplus M\mathfrak{l})\oplus L$ and $\bar{\Gamma}=\mathfrak{l}\oplus(M\mathfrak{l}+L)\cap\bar{\Gamma}$ as left $\bar{\Gamma}$-module. Thus every left ideal of $\bar{\Gamma}$ is a $\bar{\Gamma}$-direct summand of $\bar{\Gamma}$, and we see that $\bar{\Gamma}$ is a semisimle ring. Then, $J(\bar{\Gamma})=0$, and $J(\Gamma)\subseteq J(\Lambda)\cap\Gamma$. Therefore, we have $J(\Gamma)=J(\Lambda)\cap\Gamma$.

Remark. In general $J(\Lambda)=\Lambda J(\Gamma)=\Lambda J(\Gamma)$ and $J(\Lambda)\cap\Gamma=J(\Gamma)$ do not hold in H-separable extensions. Let D be a division ring and Λ be the $n\times n$-full matrix ring over D, and Γ the lower triangular matrix subring of Λ. Let $e_{i,j}$ be the matrix units of Λ. Then it is easily proved that $\sum e_{i,1}\otimes e_{1,i}\in(\Lambda\otimes_C\Lambda)^{\Lambda}$, $\sum e_{i,1}e_{1,i}=1$. But since $e_{i,1}\in\Gamma$, for each i, $\sum e_{i,1}\otimes e_{1,i}\in(\Gamma\otimes_D\Lambda)^{\Gamma}$. Hence map π of $\Gamma\otimes_D\Lambda$ to Λ defined by $\pi(r\otimes x)=rx(r\in\Gamma, x\in\Lambda)$, splits as $\Gamma-\Lambda$-map. Then by Prop. 2.2 [9], Λ is an H-separable extension of Γ. It is also clear that Λ is left Γ-finitely generated projective. But $J(\Lambda)=0$ and $J(\Gamma)\neq 0$.

Before explaining some examples in which the conditions of Theorem 4 holds, we need some preperations. The next two propositions are supplements of results which have been obtained in [13].

Proposition 5. *Let Λ be an H-separable extension of Γ such that ${}_{\Gamma}\Gamma<\oplus_{\Gamma}\Lambda$. Then* Hom$({}_{\Gamma}\Lambda, {}_{\Gamma}\Gamma)$ *is a left Λ-progenerator.*

Proof. Since ${}_{\Gamma}\Gamma<\oplus_{\Gamma}\Lambda$, Hom$({}_{\Gamma}\Lambda, {}_{\Gamma}\Gamma)$ is a left Λ-direct summand of Hom$({}_{\Gamma}\Lambda, {}_{\Gamma}\Lambda)$. But Hom$({}_{\Gamma}\Lambda, {}_{\Gamma}\Lambda)=\Delta\otimes_C\Lambda<\oplus\Lambda\oplus\Lambda\oplus\cdots\oplus\Lambda$ as $\Lambda-\Lambda$-module, since Δ is C-finitely generated projective. Hence Hom$({}_{\Gamma}\Lambda, {}_{\Gamma}\Gamma)$ is left Λ-finitely generated projective. On the other hand, in Prop. 1.1 (1) [13], we have already shown that Hom$({}_{\Gamma}\Lambda, {}_{\Gamma}\Gamma)$ is a left Λ-generator.

Proposition 6. *Let Λ be an H-separable extension of Γ. Then,*

(1) *If Γ is left Γ-cogenerator, then Λ is a left Λ-cogenerator*

(3) *If Γ is a left PF-ring, then Λ is a left PF-ring.*

(3) *If Γ is left self injective, then Λ is left self-injective*

(4) *If Γ is a quasi-Frobenius ring, then Λ is a quasi-Frobenius ring.*

Proof. (3) and (4) are shown in [13]. Hence we need only to show (1). But this follows from Korollar 1 [15], since Hom$({}_{\Gamma}\Lambda, {}_{\Gamma}\Gamma)\subseteq$Hom$({}_{\Gamma}\Lambda, {}_{\Gamma}\Lambda)<\oplus\Lambda\oplus\Lambda\oplus\cdots\oplus\Lambda$ as left Λ-module. Since left PF-ring is a ring which is left self injective and a left cogenerator (see [1]), (2) follows from (1) and (3).

It is well known that if Λ is a left (or right) PF-ring, $\Lambda/J(\Lambda)$ is artinean. Therefore we have

COROLLARY 1. *Let Γ be a left (or right) PF-ring, and Λ an H-separable extension of Γ. Then if Γ is a $\Gamma-\Gamma$-direct summand of Λ, $J(\Lambda)=\Lambda J(\Gamma)=J(\Gamma)\Lambda$ and $J(\Gamma)=J(\Lambda)\cap\Gamma$.*

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