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Borel sets in non-separable Banach spaces*)

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1. INTRODUCTION Let E be a Banach space. I shall write $\mathscr{B}=\mathscr{B}(E)$ for the σ-algebra of norm-Borel sets of E and $\mathscr{C}=\mathscr{C}(E)$ for the "cylindrical σ-algebra" of E *i.e.* the smallest σ-algebra of subsets of E for which every element of the continuous dual E' of E is measurable. (By [2], Theorem 2.3 this is just the Baire σ-algebra for the weak topology $\mathscr{T}_s(E, E')$.) Of course $\mathscr{C}\subseteq\mathscr{B}$; if E is separable, $\mathscr{C}=\mathscr{B}$. The question I wish to address in this note is: if $\mathscr{C}=\mathscr{B}$, does it follow that E is separable? This question was suggested to me by S. Okada. I shall show here that the answer is "no".

2. DEFINITIONS (a) If E is a Banach space, the *density character* of E, $d(E)$, is the smallest cardinal of any dense subset of E. Note that if E is infinite-dimensional, then $d(E)$ is also the smallest algebraic dimension of any dense linear subspace of E — the "topological dimension" of E.

(b) If X and Y are any sets and Σ, T are σ-algebras of subsets of X and Y respectively, I shall write $\Sigma\otimes_\sigma T$ for the σ-algebra of subsets of $X\times Y$ generated by $\{A\times B: A\in\Sigma, B\in T\}$. I shall write $\mathscr{P}X$ for the power set of X.

(c) If X is any set. then $\ell^1(X)$ is the Banach space of all functions $x: X\rightarrow\boldsymbol{R}$ such that $\|x\|=\Sigma_{t\in X}|x(t)|<\infty$.

3. LEMMA *If X is an infinite set such that $\mathscr{P}(X\times X)=\mathscr{P}X\otimes_\sigma\mathscr{P}X$, then $\mathscr{B}(\ell^1(X))=\mathscr{C}(\ell^1(X))$.*

PROOF (a) I begin with two set-theoretic remarks. First, if Y is any set of the same cardinal as X, then $\mathscr{P}(X\times Y)=\mathscr{P}X\otimes_\sigma\mathscr{P}Y$. Consequently we can use induction to see that, for any $n\geqslant 0$, $\mathscr{P}(X^{n+1})$ is the σ-algebra of subsets of X^{n+1} generated by $\mathscr{R}_n=\{A_0\times\cdots\times A_n: A_i\subseteq X\,\forall\, i\leq n\}$. Secondly the diagonal $\{(t, t): t\in X\}$ belongs to $\mathscr{P}X\otimes_\sigma\mathscr{P}X$, and therefore belongs to the σ-algebra of subsets of $X\times X$ generated by some sequence $\langle A_n\times B_n\rangle_{n\in N}$. Let $\mathscr{E}$ be the countable subalgebra of $\mathscr{P}X$ generated by $\{A_n: n\in N\}\cup\{B_n:$

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$n \in \boldsymbol{N}\}$; then $\mathcal{E}$ separates the points of X. So if $\mathscr{X}$ is the (countable) family of all finite partitions of X into sets belonging to $\mathcal{E}$, we see that for every finite $I \subseteq X$ there is a $\mathscr{D} \in \mathscr{X}$ such that each element of $\mathscr{D}$ contains at most one point in I.

(b) The proof will aim to show that every norm-open set belongs to $\mathscr{C}$; I proceed by considering successively more complicated sets. First, balls centred at 0 belong to $\mathscr{C}$. To see this, write $\int_D x = \Sigma_{t \in D} x(t)$ for $D \subseteq X$ and $x \in \ell^1(X)$, so that $\int_D \in (\ell^1)'$ for each $D \subseteq X$, and consider, for any $\gamma \geq 0$,

$$B_\gamma = \left\{x : \|x\| \leq \gamma\right\},$$

$$C_\gamma = \bigcap_{\mathscr{D} \in \mathscr{X}} \left\{x : \Sigma_{D \in \mathscr{D}} \left|\int_D x\right| \leq \gamma\right\}.$$

Evidently $B_\gamma \subseteq C_\gamma$. On the other hand, if $x \in \ell^1 \backslash B_\gamma$, so that $\|x\| > \gamma$, there is a finite set $J \subseteq X$ such that

$$\Sigma_{t \in J} \left|x(t)\right| - \Sigma_{t \in X \backslash J} \left|x(t)\right| > \gamma .$$

Now there is a $\mathscr{D} \in \mathscr{X}$ such that each member of $\mathscr{D}$ contains at most one element of J, and it is easy to see that $\Sigma_{D \in \mathscr{D}} \left|\int_D x\right| > \gamma$, so that $x \notin C_\gamma$. Thus C_γ is actually equal to B_γ. But since each $\mathscr{D} \in \mathscr{X}$ is finite and $\mathscr{X}$ itself is countable, $C_\gamma \in \mathscr{C}$, so $B_\gamma \in \mathscr{C}$.

(c) For the next step, take any $A \subseteq X$, $\alpha \in \boldsymbol{R} \backslash \{0\}$, and $0 < \delta \leqslant |\alpha|$. Write

$$Q = \left\{x : \exists t \in A, \left|x(t) - \alpha\right| < \delta\right\}.$$

Then $x \in Q$ iff

$$(*) \quad \exists \delta_1 \in \boldsymbol{Q} \cap]0, \delta[\text{ such that}$$
$$\forall \mathscr{D} \in \mathscr{X} \; \exists D \in \mathscr{D},\ E \in \mathcal{E} \text{ such that}$$
$$E \subseteq D \quad \text{and} \quad \left|\int_{E \cap A} x - \alpha\right| \leqslant \delta_1 .$$

(Here $\boldsymbol{Q}$ is the set of rational numbers.) To see this, suppose first that $x \in Q$. Take $t \in A$ such that $|x(t) - \alpha| < \delta$; take $\delta_1 \in]|x(t) - \alpha|, \delta[\cap \boldsymbol{Q}$. Let $I \subseteq X$ be a finite set such that $t \in I$ and $\int_{X \backslash I} |x| \leqslant \delta_1 - |x(t) - \alpha|$. Let $E_0 \in \mathcal{E}$ be such that $E_0 \cap I = \{t\}$. Now let $\mathscr{D}$ be any partition belonging to $\mathscr{X}$. Let $D \in \mathscr{D}$ be such that $t \in D$ and set $E = D \cap E_0 \in \mathcal{E}$. We have $E \cap I = \{t\} \subseteq E \cap A$, so

$$\left|\int_{E\cap A} x-\alpha\right| \leqslant \left|x(t)-\alpha\right| + \int_{E\cap A\setminus\{t\}} |x|$$
$$\leqslant \left|x(t)-\alpha\right| + \int_{X\setminus I} |x| \leqslant \delta_1 .$$

Thus every $x \in Q$ satisfies the condition (*). On the other hand, suppose that $x \in \ell^1$ satisfies (*). Let $I \subseteq X$ be a finite set such that $\int_{X\setminus I}|x| < \delta - \delta_1$. Let $\mathscr{D} \in \mathscr{X}$ be such that each $D \in \mathscr{D}$ meets I in at most one point. By hypothesis, there is a $D \in \mathscr{D}$ and an $E \in \mathscr{E}$ such that $E \subseteq D$ and $\left|\int_{E\cap A} x-\alpha\right| \leqslant \delta_1$, so that

$$\left|\int_{E\cap A\cap I} x-\alpha\right| \leqslant \left|\int_{E\cap A} x-\alpha\right| + \int_{E\cap A\setminus I} |x|$$
$$< \delta_1 + \delta - \delta_1 = \delta .$$

As $\delta \leqslant |\alpha|$, $E\cap A\cap I \neq \emptyset$; but $E\cap A\cap I \subseteq D\cap I$, so $E\cap A\cap I$ must be a singleton $\{t\}$, where $t \in A$ and $|x(t)-\alpha| < \delta$. Thus $x \in Q$.

Accordingly we have

$$Q = \bigcup_{\delta_1 \in \boldsymbol{Q}\cap]0,\delta[} \bigcap_{\mathscr{D}\in\mathscr{X}} \bigcup_{D\in\mathscr{D}, E\in\mathscr{E}, E\subseteq D} \left\{x : \left|\int_{E\cap A} x-\alpha\right| \leqslant \delta_1\right\}$$

which belongs to $\mathscr{C}$ because $\boldsymbol{Q}$, $\mathscr{X}$ and $\mathscr{E}$ are countable.

(d) For the third step, let $\alpha_0, \cdots, \alpha_n \in \boldsymbol{R}\setminus\{0\}$ and suppose that $0 < \delta < \min_{i\leq n}|\alpha_i|$. If $A \subseteq X^{n+1}$, write

$$Q(A) = \left\{x : \exists \langle t_i\rangle_{i\leq n} \in A \text{ such that } \left|x(t_i)-\alpha_i\right| < \delta \,\forall\, i \leq n\right\} .$$

Write $\mathscr{A} = \{A : A \subseteq X^{n+1}, Q(A) \in \mathscr{C}\}$.

If $A = \prod_{i\leqslant n} A_i \in \mathscr{R}_n$, then

$$Q(A) = \bigcap_{i\leq n}\left\{x : t \in A_i, \left|x(t)-\alpha_i\right| < \delta\right\}$$

belongs to $\mathscr{C}$ by (c) above, so $\mathscr{R}_n \subseteq \mathscr{A}$. Next, $\mathscr{A}$ is closed under countable unions because $Q(\bigcup_{m\in N} A_n) = \bigcup_{m\in N} Q(A_m)$ for any sequence $\langle A_m\rangle_{m\in N}$ in $\mathscr{P}(X^{n+1})$. Thirdly, if $\langle A_m\rangle_{m\in N}$ is a decreasing sequence in $\mathscr{P}(X^{n+1})$, $Q(\bigcap_{m\in N} A_m) = \bigcap_{m\in N} Q(A_m)$; for if $x \in \bigcap_{m\in N} Q(A_m)$, then the set

$$J = \left\{\langle t_i\rangle_{i\leq n} : \left|x(t_i)-\alpha_i\right| < \delta \,\forall\, i \leq n\right\}$$

is finite (in fact $\#(J).(\min_{i\leq n}|\alpha_i| - \delta) \leq \|x\|$) and meets every A_m, so must meet $\bigcap_{m\in N} A_m$.

Thus $\mathscr{A}$ includes $\mathscr{R}_n$ and is closed under arbitrary countable unions and monotonic countable intersections. It follows that $\mathscr{A}$ includes the subalgebra of $\mathscr{P}(X^{n+1})$ generated by $\mathscr{R}_n$ and the therefore the σ-subalgebra of $\mathscr{P}(X^{n+1})$ generated by $\mathscr{R}_n$, which is $\mathscr{P}(X^{n+1})$ itself, by (a). So we see that $Q(A)\in\mathscr{C}$ for every $A\subseteq X^{n+1}$.

(e) Observe next that every non-empty norm-open set $G\subseteq\ell^1(X)\setminus\{0\}$ is expressible as a union of sets of the form

$$\left\{x: \left|x(t_i)-\alpha_i\right|<\delta\,\forall\, i\leqslant n,\ ||x||\leqslant\gamma\right\}$$

where $t_0,\cdots,t_n\in X$, $\alpha_0,\cdots,\alpha_n\in\boldsymbol{Q}\setminus\{0\}$, γ, $\delta\in\boldsymbol{Q}$, and $0<\delta<\min_{i\leq n}|\alpha_i|$. As there are only countably many choices of the parameters $\alpha_0,\cdots,\alpha_n$, γ, δ, G is expressible as a countable union of sets of the form $Q\cap B_\gamma$, where Q is of the type discussed in (d). Putting (b) and (d) together, we see that $G\in\mathscr{C}$. As G is arbitrary, every norm-open set belongs to $\mathscr{C}$ (because $\{0\}=B_0\in\mathscr{C}$), and $\mathscr{B}\subseteq\mathscr{C}$.

4. Theorem *Let κ be an infinite cardinal. Then the following are equivalent*:

(i) *There is a Banach space E such that $d(E)=\kappa$ and $\mathscr{B}(E)=\mathscr{C}(E)$*;

(ii) $\mathscr{B}(\ell^1(\kappa))=\mathscr{C}(\ell^1(\kappa))$;

(iii) $\mathscr{P}(\kappa\times\kappa)=\mathscr{P}\kappa\otimes_\sigma\mathscr{P}\kappa$;

(iv) *if E is any Banach space with $d(E)\leqslant\kappa$, the* map $(\alpha, x, y)\mapsto\alpha x+y$: $\boldsymbol{R}\times E\times E\to E$ *is measurable for the σ-algebras $\mathscr{B}(\boldsymbol{R})\otimes_\sigma\mathscr{B}(E)\otimes_\sigma\mathscr{B}(E)$ and $\mathscr{B}(E)$.*

Proof (iii)$\Rightarrow$(ii) is Lemma 3 above, and (ii)$\Rightarrow$(i) is obvious. For the equivalence of (i), (iii) and (iv), we can use Theorem 1 of [5], which shows that for any Banach space E the map $(\alpha, x, y)\mapsto\alpha x+y$ is measurable iff $\mathscr{P}(d(E)\times d(E))=\mathscr{P}(d(E))\otimes_\sigma\mathscr{P}(d(E))$. But of course the map $(\alpha, x, y)\mapsto\alpha x+y$ is always measurable for $\mathscr{B}(R)\otimes_\sigma\mathscr{B}(E)\otimes_\sigma\mathscr{B}(E)$ and $\mathscr{C}(E)$, so if $\mathscr{B}(E)=\mathscr{C}(E)$ then $\mathscr{P}(d(E)\times d(E))=\mathscr{P}(d(E))\otimes_\sigma\mathscr{P}(d(E))$.

5. Okada's problem Okada's question therefore becomes: is there an uncountable κ satisfying the conditions of Theorem 4? Now condition (iii) has been extensively studied; see [3] for a recent survey of the known results. For our present enquiry the following are the most relevant:

(i) $\mathscr{P}(\aleph_1\times\aleph_1)=\mathscr{P}(\aleph_1)\otimes_\sigma\mathscr{P}(\aleph_1)$;

(ii) if Martin's Axiom is true, $\mathscr{P}(\kappa\times\kappa)=\mathscr{P}\kappa\otimes_\sigma\mathscr{P}\kappa$ for every $\kappa\leq\mathbf{c}$;

(iii) assuming that the continuum hypothesis is false, it is still undecidable whether $\mathscr{P}(\mathbf{c}\times\mathbf{c})=\mathscr{P}\mathbf{c}\otimes_\sigma\mathscr{P}\mathbf{c}$.

Of these we need only (i) to settle Okada's question; $\mathscr{B}(\ell^1(\aleph_1))=\mathscr{C}(\ell^1(\aleph_1))$ and $\ell^1(\aleph_1)$ is not separable. (This was conjectured by G. A. Edgar.)

6. Problem We can ask a similar question concerning $\mathscr{B}_s$, the algebra

of Borel sets for the weak topology $\mathscr{T}_s(E, E')$. G. A. Edgar ([2], Theorem 1. 1) has given an important class of spaces for which $\mathscr{B}_s=\mathscr{B}$; there is no restriction on their density character. (For instance, all uniformly convex spaces have this property.) However, we can ask: are there spaces of large density character for which $\mathscr{B}_s=\mathscr{C}$? in particular, is this always possible with $d(E)=\mathbf{c}$, even if $\mathscr{P}(\mathbf{c}\times\mathbf{c})\neq\mathscr{P}\mathbf{c}\hat{\otimes}_\sigma\mathscr{P}\mathbf{c}$?

It is perhaps worth repeating here a simple observation due to Okada. If $\mathscr{B}_s(E)=\mathscr{C}(E)$, then $\{0\}\in\mathscr{C}(E)$, so that E' is $\mathscr{T}_s(E', E)$-separable. If E is reflexive, it follows at once that E is separable. Of course, with $E=\ell^\infty(\boldsymbol{N})$, we have E' $\mathscr{T}_s(E', E)$-separable but $\mathscr{B}(E)\neq\mathscr{B}_s(E)\neq\mathscr{C}(E)$ (see [4]).

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