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Some remarks on p-blocks of finite groups

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In this paper we are concerned with modular representations of finite groups. Let G be a finite group and p a fixed rational prime. Let K be a complete p-adic field of characteristic 0 and R the ring of p-local integers in K with the principal maximal ideal (π) and the residue class field $\bar{R}=R/(\pi)$ of characteristic p. We assume throughout the paper that fields K and $\bar{R}$ are both splitting fields for all subgroups of the given group G. We mention here [2] and [3] as general references for the modular representation theory of finite groups.

1. In this section we shall give some necessary and sufficient condition for G to be p-nilpotent. If B is a p-block of G, then let $Irr(B)$ denote the set of irreducible K-characters of G in B. For a class function θ of G we put $\theta_{B}=\sum_{\chi\in Irr(B)}(\theta, \chi)\chi$. Let $B_{0}(G)$ denote the principal p-block of G.

We prove the following.

THEOREM 1. *Let H be a subgroup of G which contains a Sylow p-subgroup P of G. If $1_{H}{}^{G}{}_{B_{0}(G)}(x)=1$ for any p-element $x\neq 1$ in G, then H controls the fusion of elements of P.*

To prove the theorem we use the following elementary lemma which follows from Brauer's Second Main Theorem.

LEMMA. *Let θ be a class function of G, x a p-element of G and B a p-block of G. Then $\theta_{B}(x)=\sum\theta_{|C_{G}(x)b}(x)$ where b ranges over the set of p-blocks of $C_{G}(x)$ with $b^{G}=B$.*

PROOF OF THEOREM 1. Let $x\neq 1$ be an element of P, $C=C_{G}(x)$, $B=B_{0}(G)$ and $b=B_{0}(C)$. By Mackey decomposition we have $1_{H}{}^{G}{}_{|C}=\sum(1_{H^{y}\cap C})^{C}$ where y ranges over a complete set of representatives of (H, C)-double cosets in G. Thus the above lemma and the result of Brauer (Theorem 65. 4 [2]) show that $1_{H}{}^{G}{}_{B}(x)=\sum(1_{H^{y}\cap C})^{C}{}_{b}(x)$. If $x\in H^{y}\cap C$, then $(1_{H^{y}\cap C})^{C}{}_{b}(x)=(1_{H^{y}\cap C})^{C}{}_{b}(1)$. and if $x\notin H^{y}\cap C$, then $(1_{H^{y}\cap C})^{C}{}_{b}(x)=0$ by (6. 3) IV in [3]. As $1_{H}{}^{G}{}_{B}(x)=1$ by our assumption, $x\in H^{y}\cap C$ if and only if $y\in HC$. Therefore if $x^{y}\in H$ for some element y, then there exists an element h in H such that $x^{y}=x^{h}$ and therefore the theorem is proved.

As an easy corollary of Theorem 1 we have the following.

COROLLARY 2. *Let P be a Sylow p-subgroup of G. Then G is p-nilpotent if and only if $1_P{}^G{}_{B_0(G)}(x)=1$ for any p-element $x\neq 1$ of G.*

PROOF. If G is p-nilpotent, then it is easy to show that $1_P{}^G{}_{B_0(G)}=1_G$ and therefore $1_P{}^G{}_{B_0(G)}(x)=1$ for any element x of G. Conversely assume that $1_P{}^G{}_{B_0(G)}=1$ on p-elements $\neq 1$ of G. Then Theorem 1 shows that two elements of P are conjugate in G if and only if in P. Thus the corollary follows from the well-known result on Transfer Theory.

2. If B is a p-block of G then for an $R[G]$-module V we define $V_B=Ve$ where e is the centrally primitive idempotent of $R[G]$ corresponding to B (in this paper modules will always be right unital). Let $L_0(G)$ denote the trivial R-free $R[G]$-module of R-rank 1. If V is an $R[G]$-module then let $\bar{V}=V/V(\pi)$ which is an $\bar{R}[G]$-module.

Let P be a Sylow p-subgroup of G. If $P\lhd G$, then $L_0(P)^G$ is completely reducible and every irreducible $R[G]$-module has a vertex P. In this connection we have the following.

THEOREM 3. *Let G be a finite group and B a p-block of G with defect group D. Let $N=N_G(D)$ and b a p-block of N with $b^G=B$. then the following are equivalent.*

(1) $G\rhd D\,\mathrm{Ker}\,B$ *where* $\mathrm{Ker}\,B=\bigcap_{\chi\in_9 Irr(B)}\ker\chi$.

(2) *For every irreducible $R[G]$-module L in B L_N is also irreducible.*

(3) *$\overline{L_0(D)^G{}_B}$ is completely reducible and every irreducible $R[G]$-module in B has a vertex D.*

PROOF. (1)→(3). Since $G\rhd D\,\mathrm{Ker}\,B$, $\overline{L_0(D)^G{}_B}$ is considered as an $\bar{R}[G/D]$-module. Every irreducible $R[G]$-module in B has kernel containing D and is projective as an $R[G/D]$-module. Thus the result follows.

(3)→(2). By our assumprion $\overline{L_0(D)^G{}_B}=\sum\oplus n_iL_i$ where L_i's are irreducible $\bar{R}[G]$-modules in B. By Nakayama Relation (see [3], p 141) $n_i=\dim_{\bar{R}}U_i/|D|$ where U_i is the principal indecomposable $\bar{R}[G]$-module corresponding to L_i. Then assertion (1)→(3) shows that $\overline{L_0(D)^N{}_b}=\sum\oplus m_jM_j$ where M_j's are irreducible $\bar{R}[N]$-modules in b. By the same reason as the above $m_j=\dim_{\bar{R}}V_j/|D|$ where V_j is the principal indecomposable $R[N]$-module corresponding to M_j. Then by Green Correspondence with respect to (G, D, N) the numbers of L_i's and M_j's are equal and after suitable rearrangement $n_i=m_i$ and L_i corresponds to M_i. Then by Nakayama Relation $U_{iN}\cong V_i$ and therefore $L_{iN}=M_i$.

(2)→(1). This is proved by the similar argument in [4] (Theorem 4). By our assumption $\bigcap\mathrm{Ker}\,L\supseteq D\,\mathrm{Ker}\,B$ where L ranges over the set of all irreducible $\bar{R}[G]$-modules in B. Thus $D\,\mathrm{Ker}\,B=\bigcap\mathrm{Ker}\,L\lhd G$.

REMARK 1. *The equivalence of* (1) *and* (2) *in case* $B=B_0(G)$ *is the result of Isaacs and Smith* (*Theorem* 4, [4]).

As a corollary of this theorem we have the following.

COROLLARY 4. *Let* G *be a finite group and* P *a Sylow p-subgroup of* G. *Then* G *has p-length* 1 *if and only if* $\overline{L_0(P)^G}_{B_0(G)}$ *is completely reducible and every irreducible* $\bar{R}[G]$*-module in* $B_0(G)$ *has a vertex* P.

REMARK 2. *The condition that every irreducible* $\bar{R}[G]$*-module in* $B_0(G)$ *in the above can not be dropped as the group* S_4, *the symmetric group of degree* 4 *shows.*

3. In this section we shall prove some results related to the result of Brauer (Theorem 2, [1]). In [1] Brauer has proved the following.

THEOREM (Brauer, [1]). *Let* G *be a finite group and* P *a Sylow p-subgroup of* G. *If* B *is a p-block of* G *with defect group* $D\subseteq P$, *then* $\dim_{\bar{R}}\overline{L_0(P)^G}_B=|P:D|v$, *where* $(p, v)=1$.

If $\dim_{\bar{R}} L_0(P)^G_B$ is a power of p, then Brauer's Theorem implies that B has the unique irreducible $\bar{R}[G]$-module in it. In particular, G is p-nilpotent if and only if $\dim_{\bar{R}}\overline{L_0(P)^G}_{B_0(G)}$ is a power of p. This is the result of Brauer (Corollary 2, [1]). Furthermore we have the following.

COROLLARY 5. *Let* G, P, B *and* D *be as in the above. Assume furthermore* $P\trianglerighteq D$. *If* $\dim_{\bar{R}}\overline{L_0(P)^G}_B$ *is a power of p, then* $G\trianglerighteq D\,\mathrm{Ker}\,B$ *and* $[G, D]\subseteq[P, D]\,\mathrm{Ker}\,B$.

PROOF. Let $N=N_G(D)$. By the theorem of Brauer $\overline{L_0(P)^G}_B=L$ is the unique irreducible $\bar{R}[G]$-module in B and has dimension $|P:D|$. Since $N\supseteq P$, L_N is irreducible and therefore $G\trianglerighteq D\,\mathrm{Ker}\,B$ by Theorem 3. In order to prove the second statement we may assume $\mathrm{Ker}\,B=1$ and $D\triangleleft G$. Let V be an arbitrary R-free $R[D]$-module of R-rank 1 with $\mathrm{Ker}\,V\supseteq[P, D]$. Let H be the inertia group of V in G. By ([3], p 163) $V^G{}_{B|D}=n\sum\oplus V^x$ for some positive integer n where x ranges over a complete set of representatives of right H-cosets in G. Since $\bar{V}=\overline{L_0(D)}$, $\dim_{\bar{R}}\bar{V}^G{}_B$ is a power of p and therefore so is $|G:H|$. As $\mathrm{Ker}\,V\supseteq[P, D]$ it follows that $H\supseteq P$ and we have $G=H$. Thus $x^{-1}x^y\in\mathrm{Ker}\,V$ for elements $x\in D$ and $y\in G$ and therefore the result follows.

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