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On parabolic equations in n space variables and their solutions in regions with edges

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1. Introduction

In this paper we study the initial-Dirichlet problem for parabolic equations of the form

$$Lu=f,$$
$$L=a_{ik}(x)\frac{\partial^{2}}{\partial x_{i}\partial x_{k}}+a_{i}(x,t)\frac{\partial}{\partial x_{i}}+a(x,t)-\frac{\partial}{\partial t}. \tag{1.1}$$

Here f depends on $x=(x_{1},\cdots,x_{n})$ and t, and we use the *summation convention* (summations from 1 to n).

Equation (1.1) will be considered in a region $\Omega=G\times J\subset \boldsymbol{R}^{n+1}$, where $J=\{t|0<t\leqq T\}$ and $G\subset \boldsymbol{R}^{n}$ has edges satisfying conditions to be specified below. L is assumed to have $C^{\alpha}(\bar{\Omega})$-coefficients, where $0<\alpha<1$, and $f\in C^{\alpha}(\bar{\Omega})$, too.

We shall prove that, under these assumptions and suitable conditions concerning the initial and boundary data, for bounded solutions u of that problem we have $D_{x}u\in C^{\nu}(\bar{\Omega})$, where $0<\nu<1$ and D_{x} denotes partial differentiation with respect to any x_{i}, $i=1,\cdots,n$. Also $D_{x}^{2}u\in C^{0}(\bar{\Omega})$ under an additional assumption.

Our method is based on Schauder type estimates and barrier functions, and the results will extend those in [3] for $n=2$.

Furthermore, it is interesting to note that the method can be modified so that it yields similar results for bounded solutions of the Dirichlet problem for *elliptic* equations in n-dimensional regions with edges. This will be explained at the end of this paper.

We mention that an early paper on regions with edges was T. Carleman's thesis [4] for the n-dimensional Laplace equation. Mixed boundary value problems in two-dimensional regions with corners were also considered by N. M. Wigley [10]. Systems of the form $\Delta u=F(x,u,\text{grad }u)$ in such regions were recently studied by G. Dziuk [5], who obtained results on the smoothness of solutions. Publications on elliptic equations with $n=2$ in regions with corners are numerous (cf. the references in [7]), whereas for general n comparatively little is known, in particular with respect to parabolic equations. In the case of the Dirichlet problem for elliptic equations in n variables, a Sobolev space approach is due to V. A. Kondrat'ev [8].

The material in this paper is arranged as follows. Our main result (Theorem 1) is stated in Sec. 3. It will be obtained from Theorems 2-4 in Secs. 4-6. Theorem 2 in Sec. 4 concerns bounds for solutions u, and Theorem 3 in Sec. 5 gives bounds for partial derivatives of u. In both sections the region is of a special type, namely, a sector of a cylinder. In Theorem 4 (Sec. 6) we prove that $D_x u$ is of class C^ν in the closure of such a special region. In Sec. 7, Theorem 1 will then be obtained from Theorem 4. Finally, in Sec. 8 we shall be concerned with the Dirichlet problem for elliptic equations to which the present method, in a modified form, is also applicable.

We want to thank the referee of this paper for suggesting a modification of our barrier function, which entailed the additional result (3. 5 b).

2. Some notations

This section contains some general notations needed throughout the paper. Let $G\subset \boldsymbol{R}^n$ be any bounded domain, $J=\{t|0<t\leqq T\}$ with constant $T>0$, and set $\Omega=G\times J\subset \boldsymbol{R}^{n+1}$. In Ω we use the metric defined by

$$d(P, Q)=\left(|x-\tilde{x}|^2+|t-\tilde{t}|\right)^{1/2},$$

where P: (x, t), Q: $(\tilde{x}, \tilde{t})$ and

$$|x|^2=\sum_{i=1}^{n} x_i^2 \qquad x=(x_1, \cdots, x_n).$$

For a function u on Ω we define, as usual (cf. [6]),

$$\|u\|_0^\Omega=\sup_\Omega \left|u(x, t)\right|$$

$$H_\alpha^\Omega(u)=\sup_{\substack{P,Q\in\Omega\\ P\neq Q}} \frac{|u(x, t)-u(\tilde{x}, \tilde{t})|}{d(P, Q)^\alpha} \qquad (0<\alpha<1)$$

$$\|u\|_\alpha^\Omega=\|u\|_0^\Omega+H_\alpha^\Omega(u)$$

$$\|u\|_{2+\alpha}^\Omega=\|u\|_\alpha^\Omega+\sum\|D_x u\|_\alpha^\Omega+\sum\|D_x^2 u\|_\alpha^\Omega+\|D_t u\|_\alpha^\Omega$$

provided each expression in the right-hand sides exists and is finite. Here, $D_t=\partial/\partial t$ and D_x^j denotes any partial derivative of order j with respect to $x_1, \cdots, x_n$.

Let $G_0=G\times\{t|t=0\}$ and $S=\partial G\times J$. Then

$$\phi \in C^{2+\alpha}(A), \qquad A=\bar{G}_0 \cup S$$

means that ϕ is defined on A and there exists a function $\Psi \in C^{2+\alpha}(\bar{\Omega})$ such that $\Psi|_A = \phi$. We then define

$$\|\phi\|_{2+\alpha} = \inf_{\Psi} \|\Psi\|_{2+\alpha},$$

the infimum being taken over all those functions Ψ.

3. Main result

We shall now state our main theorem (Theorem 1, below), which is concerned with the smoothness of solutions of parabolic equations in regions with edges. The proof will be based on results to be obtained in the next three sections and will be given in Sec. 7.

We start from a bounded domain $G \subset \boldsymbol{R}^n$, $n \geqq 2$, whose boundary ∂G consists of hypersurfaces $\Gamma_1, \cdots, \Gamma_m$ of class $C^{2+\alpha}$, where $0<\alpha<1$. We assume that Γ_i intersects only with Γ_{i-1} and Γ_{i+1}, these intersections being $(n-2)$-dimensional edges E_{i-1} and E_i, respectively. (Here Γ_{m+1} means Γ_1). We now introduce $\Omega = G \times J$, where $J=\{t|0<t \leqq T\}$. In Ω we consider the initial-Dirichlet problem

$$Lu = f \tag{3.1}$$

$$u(x, 0) = 0, \qquad x \in \bar{G} \tag{3.2a}$$

$$u\Big|_{\partial G \times \bar{J}} = \phi(x, t), \tag{3.2b}$$

assuming that

(i) $a_{ik} \in C^{\alpha}(\bar{G})$

(ii) a_i, a, $f \in C^{\alpha}(\bar{\Omega})$

(iii) $\phi(x, 0)=0$ and

$$\phi \in C^{2+\alpha}\Big[(\partial G \setminus \bigcup_i E_i) \times \bar{J}\Big] \cap C^0(\partial G \times \bar{J})$$

It is known (cf. [6]) that any solution of (3.1), (3.2) satisfying (i)–(iii) is of class

$$C^{2+\alpha}\Big[(\bar{G} \setminus \bigcup_i E_i) \times \bar{J}\Big] \cap C^0(\bar{\Omega}).$$

The presence of the edges E_i affects only the smoothness of the functions $D_x u$ and $D_x^2 u$. In Theorem 1 we give sufficient conditions for $D_x u$ to be Hölder continuous. This needs a short preparation, as follows.

Let P: x^0 be any point on E_i. Let π_1 and π_2 be the two hyperplanes which touch Γ_i and Γ_{i+1} at P making there an angle $\gamma(P)$. We now transform the equation

$$a_{ik}(x^0)\, u^*_{x_i x_k} = 0 \tag{3. 3}$$

to canonical form. Note that this is an equation with constant coefficients, since P is fixed. That transformation maps π_1 and π_2 onto two hyperplanes which at the image of P make an angle $\omega(P)$. It is this angle that we need for stating Theorem 1. In fact, the theorem shows that $\omega(P)$ plays a significant role in determining the smoothness of the solution u of our problem near $\cup E_i \times \bar{J}$.

THEOREM 1. *Let u be a solution of* (3. 1), (3. 2) *in $\Omega = G \times J$, and assume* (i)-(iii) *to be satisfied. Suppose further that $\omega(P) < \pi$ for every $P \in \bigcup_i E_i$. Then there exist numbers ν, κ, χ, $0 < \nu, \kappa, \chi < 1$, such that*

$$D_x u \in C^{\nu}(\bar{\Omega}) \tag{3. 4}$$

and

$$\tilde{u} \in C^{\chi}(\bar{\Omega}) \qquad \textit{where} \qquad \tilde{u}(x, t) = \delta^{\kappa} D_x^2 u(x, t) \tag{3. 5a}$$

$\chi = \min(\alpha, \kappa+\nu-1)$ and δ is the distance from (x, t) to $\bigcup_i E_i \times \bar{J}$. Furthermore, if $\omega(P) < \pi/2$ for every $P \in \bigcup_i E_i$, then

$$D_x^2 u \in C^0(\bar{\Omega})\,. \tag{3. 5b}$$

4. Bounds for solutions

As indicated in the Introduction, in this and the next two sections we shall obtain results for special regions, from which the theorem on the smoothness of solutions in a general region with edges will then follow in Sec. 7.

Let $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $x' = (x_3, \cdots, x_n)$ and

$$B_\sigma = \left\{(r, \theta, x') \middle| r < \sigma, \beta < \theta < \beta + \omega, |x_i| < \sigma, i > 2\right\}$$

where $\sigma > 0$, $0 < \omega < \pi$ and $\beta = (\pi - \omega)/2$. Let Π_1 and Π_2 denote the two portions of the hyperplanes

$$x_2 = x_1 \tan\beta$$

$$x_2 = x_1 \tan(\beta + \omega)$$

by which the sector B_σ is bounded laterally. Furthermore, let

$$N_c = \left\{x \middle| x \in B_\sigma, |x| < c\right\} \qquad (c > 0)\,.$$

Denote by S_c the portion of the boundary of N_c which lies on $\Pi_1 \cup \Pi_2$.

Let $E_c = \Pi_1 \cap \Pi_2 \cap \bar{N}_c$, so that E_c is the portion of the edge of B_σ in $\bar{N}_c$.

In $N_c \times J$ we consider the problem

$$Lu = f \tag{4.1}$$

$$u\Big|_{t=0} = 0 \tag{4.2a}$$

$$u\Big|_{S_c} = \phi(x, t) \tag{4.2b}$$

with parabolic L as in (1.1) under the following assumptions.

(i) $a_{ik} \in C^0(\bar{N}_c)$ and $a_{ik}(0) = \delta_{ik}$ when $i, k = 1, 2$.

(ii) a_{ik} $(i>2$ or $k>2)$, a_i, a and f are bounded in $\bar{N}_c \times J$.

(iii) $\phi \in C^2((S_c \backslash E_c) \times J) \cap C^0(S_c \times J)$, $\phi(x, 0) = 0$.

(iv) On E_c the function ϕ is zero together with its first partial derivatives in the directions perpendicular to E_c and such that $\theta = \beta$ or $\theta = \beta + \omega$, and its second derivatives in those directions are bounded.

Note that, by (ii), in this and the next two sections the coefficients a_{ik} with $i>2$ or $k>2$ may also depend on t.

In the present section we shall obtain bounds for solutions of the problem (4.1), (4.2) and in the next section bounds for partial derivatives of these solutions with respect to the x_i' s.

THEOREM 2. *Let u be a bounded solution of the problem* (4.1), (4.2) *in $N_c \times J$. Suppose that the assumptions* (i)-(iv) *are satisfied. Then there exists a number $c_1 < c/3$ such that in $\bar{N}_{c_1} \times J$ we have*

$$|u(x, t)| \leqq K r^\mu$$

where $K>0$ is constant, $r^2 = x_1^2 + x_2^2$, and

$$\mu = \begin{cases} 2 & \text{if } \omega < \pi/2 \\ \dfrac{\pi}{\omega} - \varepsilon & \text{if } \omega \geqq \pi/2 \end{cases}$$

with arbitrarily small $\varepsilon > 0$.

PROOF. Let $\xi \in C^3(\bar{N}_{3c_1})$ with c_1 to be determined later and

$$\xi(|x|) = \begin{cases} 1 & \text{if } 0 \leqq |x| \leqq c_1 \\ 0 & \text{if } 2c_1 \leqq |x| \leqq 3c_1 . \end{cases}$$

Then $w = \xi u$ is defined in the region $N_{3c_1} \times J$, which in this proof will be simply denoted by $\mathcal{N}$. In $\mathcal{N}$ the function w satisfies the equation

$$Lw = F \tag{4.3}$$

where

$$F=\xi f-2a_{ik}\xi_{x_i}u_{x_k}-a_{ik}\xi_{x_ix_k}u-a_i\xi_{x_i}u\,.$$

Furthermore, the function $\psi_1=w|_{S_c}$ satisfies the above conditions (iii) and (iv).

We now introduce the function

$$v(x)=-Kr^{\mu}\cos\lambda\left(\theta-\frac{\pi}{2}\right)$$

defined in N_{3c_1}, where $K>0$ will be specified later and

$$1<\mu<\lambda=\frac{\pi-2\eta}{\omega}$$

with $0<\eta<\pi/2$. Here, if $\pi/\omega\leqq 2$, we take $\eta>0$ arbitrarily small, whereas if $\pi/\omega>2$, we take $\eta<\pi/2-\omega$ and $\mu=2$.

We can rewrite Lw in the form

$$Lw=w_{x_1x_1}+w_{x_2x_2}+\tilde{a}_{ik}w_{x_ix_k}+a_iw_{x_i}+aw-w_t$$

where

$$\tilde{a}_{ik}=\begin{cases}a_{ik}-\delta_{ik} & \text{if } i,\ k=1,2\\ a_{ik} & \text{otherwise.}\end{cases}$$

In particular,

$$Lv(x)=K(\lambda^2-\mu^2)\,r^{\mu-2}\cos\lambda\left(\theta-\frac{\pi}{2}\right)$$
$$+K\tilde{a}_{ik}(x)\,h_{ik}(x)\,r^{\mu-2}+Kh_1(x,t)\,r^{\mu-1}+Kh_2(x,t)\,r^{\mu}\,,$$

where h_{ik} and h_i are bounded functions, say,

$$\sum_{i=1}^{n}\sum_{k=1}^{n}\left|h_{ik}(x)\right|+\sum_{i=1}^{n}\left|h_i(x,t)\right|\leqq K_0\qquad(x,t)\in\mathcal{N}\,.$$

Since the $\tilde{a}_{ik}$ are continuous in $\bar{N}_{3c_1}$ and zero at $x=0$, we can find a positive c_1 so small that

$$\left|\tilde{a}_{ik}(x)\right|<\varepsilon/4K_0\qquad\text{when}\quad |x|<3c_1\,.$$

Also, for $\beta\leqq\theta\leqq\beta+\omega$ we have $\cos\lambda\left(\theta-\frac{\pi}{2}\right)\geqq\sin\eta$. Hence in $\mathcal{N}$ we obtain

$$Lv(x)\geqq K\left[(\lambda^2-\mu^2)\sin\eta-\varepsilon-K_0r-K_0r^2\right]r^{\mu-2}\,.\tag{4. 4}$$

We now take $\varepsilon<(\lambda^2-\mu^2)\sin\eta$ and then c_1 so small that the expression in the brackets $[\cdots]$ in (4. 4) is positive. Since $\mu\leqq 2$, by taking c_1 sufficiently small and K sufficiently large we get

$$Lv(x) \geqq F(x, t)$$

in $\mathscr{N}$. It follows that

$$L(w-v)(x, t) \leqq 0 \text{ in } \mathscr{N} .$$

We now show that $w-v$ can be made nonnegative on $\partial N_{3c_1} \times J$ by taking K sufficiently large. We first consider $S_{3c_1} \times J$, which in this proof we simply denote by $\mathscr{S}$. On $\mathscr{S}$ we have $w=\psi_1$, where

$$\psi_1(0, 0, x', t) = D_\beta \psi_1(0, 0, x', t) = D_{\beta+\omega} \psi_1(0, 0, x', t) = 0 ; \qquad (4.5)$$

here $D_\beta \psi_1$ and $D_{\beta+\omega} \psi_1$ are the derivatives of ψ_1 in the directions perpendicular to E_c and such that $\theta=\beta$ and $\theta=\beta+\omega$ respectively. Hence at any point $(x, t) \in \mathscr{S}$ we have

$$D_l \psi_1(x, t) = \int_{(0,0,x',t)}^{(x_1,x_2,x',t)} D_l^2 \psi_1(\tilde{x}_1, \tilde{x}_2, x', t)\, d\tilde{r}, \qquad l = \beta,\ \beta+\omega \qquad (4.6)$$

Now $D_l^2 \psi_1$ is bounded, say,

$$\left| D_l^2 \psi_1(x, t) \right| \leqq 2\hat{K}_1 \quad \text{on} \quad \mathscr{S} . \qquad (4.7\text{a})$$

Hence from (4.6) we obtain

$$\left| D_l \psi_1(x, t) \right| \leqq 2\hat{K}_1 r \quad \text{on} \quad \mathscr{S} \qquad (4.7\text{b})$$

and in a similar fashion

$$\left| \psi_1(x, t) \right| \leqq \hat{K}_1 r^2 \quad \text{on} \quad \mathscr{S} . \qquad (4.7\text{c})$$

On $\mathscr{S}$ we thus have

$$w(x, t) - v(x) = \psi_1(x, t) + K r^\mu \sin \eta \geqq -\hat{K}_1 r^2 + K r^\mu \sin \eta$$

which can be made nonnegative by taking K sufficiently large. On the remaining part of $\partial N_{3c_1} \times J$ we have $w=0$ and thus

$$w(x, t) - v(x) \geqq K r^\mu \sin \eta \geqq 0 .$$

Our intermediate result is $L(w-v)(x, t) \leqq 0$ in $\mathscr{N}$, whereas $w(x, t) - v(x) \geqq 0$ on $\partial N_{3c_1} \times J$. Applying the maximum principle for parabolic equations (cf. [9], pp. 174-175) we conclude that $w(x, t) - v(x) \geqq 0$ in $\mathscr{N}$. Hence

$$w(x, t) \geqq -K r^\mu \operatorname{oos} \lambda\left(\theta - \frac{\pi}{2}\right) \geqq -K r^\mu .$$

By a similar argument we can show that in $\mathscr{N}$,

$$w(x, t) \leqq Kr^{\mu},$$

provided K is taken sufficiently large and $c_1>0$ sufficiently small. Together,

$$\left|w(x, t)\right| \leqq Kr^{\mu}.$$

Since $w=u$ in $N_{c_1}\times J$, we obtain the desired result, and the proof is complete.

5. Bounds for $D_x u$ and $D_x^2 u$

Having estimated bounded solutions u of the problem (4. 1), (4. 2), we now estimate their derivatives u_{x_i} and $u_{x_i x_k}$.

For this purpose we shall need the following subregions of N_{c_1}:

$$R_s=\left\{x\middle|2^{-s-2}c_2 \leqq r \leqq 2^{-s-1}c_2, |x_i| \leqq 2^{-s}c_2 \text{ for } i>2\right\}\cap N_{c_1};$$

here $s=-1, 0, 1, \cdots$. We also define

$$\tilde{R}_s=R_{s-1}\cup R_s\cup R_{s+1} \qquad s=0, 1, \cdots$$

Here $c_2>0$ is assumed sufficiently small so that $\tilde{R}_0\subset N_{c_1/2}$.

Instead of assumptions (i)-(iv) we now make the following ones.

(i*) $a_{ik}\in C^{\alpha}(\bar{N}_c)$ and $a_{ik}(0)=\delta_{ik}$ for $i, k=1, 2$

(ii*) $a_{ik}(i>2$ or $k>2)$, a_i, a, $f\in C^{\alpha}(\bar{N}_c\times J)$.

(iii*) $\phi\in C^{2+\alpha}((S_c\backslash E_c)\times J)\cap C^0(S_c\times J)$, $\phi(x, 0)=0$.

(iv*) Same as (iv) in Sec. 4.

THEOREM 3. *Let u be a bounded solution of the problem* (4. 1), (4. 2) *in $N_c\times J$. Suppose that the assumptions* (i*)-(iv*) *are satisfied. Then if $\omega<\pi$ in B_σ, in $N_{c_1/2}\times J$ we have*

$$\left|D_x^j u(x, t)\right| \leqq K_j r^{\mu-j} \qquad j=1, 2,$$

with μ as in Theorem 2.

PROOF. Let $\tilde{\Gamma}_s$ denote the portion of the boundary of $\tilde{R}_s$ which lies on $\Pi_1\cup\Pi_2$, and let us use in this proof the notations

$$\mathscr{R}_s=R_s\times J,\ \tilde{\mathscr{R}}_s=\tilde{R}_s\times J,\ \tilde{\mathscr{G}}_s=\tilde{\Gamma}_s\times J.$$

We now apply the transformation

$$x=2^{-s}y \qquad y=(y_1, \cdots, y_n) \tag{5. 1}$$

which maps R_s, $\tilde{R}_s$ and $\tilde{\Gamma}_s$ onto R_0, $\tilde{R}_0$ and $\tilde{\Gamma}_0$, respectively. In $\tilde{\mathscr{R}}_0$ the function $U(y, t)=u(2^{-s}y, t)$ is defined and satisfies the parabolic equation

$$b_{ik}U_{y_i y_k}+2^{-s}b_i U_{y_i}+2^{-2s}bU-2^{-2s}U_t=2^{-2s}g \tag{5. 2}$$

where b_{ik}, b_i and b are the coefficients of L in the new variables, and $g(y, t)=f(x, t)$. Clearly $U(y, 0)=0$. The boundary value ψ_2 of U on $\tilde{\mathscr{G}}_0$ is given by

$$\psi_2(y, t)=\psi(2^{-s}y, t) \tag{5.3}$$

We now apply a Schauder type estimate for parabolic equations (cf. [6]) to the solution U of (5.2) in $\mathscr{R}_0$ and $\tilde{\mathscr{R}}_0$, finding

$$||U||_{2+\alpha}^{\mathscr{R}_0} \leqq A_1\left(||U||_0^{\tilde{\mathscr{R}}_0}+2^{-2s}||g||_\alpha^{\tilde{\mathscr{R}}_0}+||\psi_2||_{2+\alpha}^{\tilde{\mathscr{G}}_0}\right) \tag{5.4}$$

We estimate each term on the right-hand side of (5.4). In $\tilde{\mathscr{R}}_s$ we have (cf. Theorem 2)

$$\left|u(x, t)\right| \leqq Kr^\mu$$

and thus in $\tilde{\mathscr{R}}_0$,

$$\left|U(y, t)\right| \leqq A_2 2^{-s\mu}$$

and

$$||U||_0^{\tilde{\mathscr{R}}_0} \leqq A_3 2^{-s\mu} \tag{5.5 a}$$

Since $f\in C^\alpha(\tilde{\mathscr{R}}_s)$, it follows that $||g||_\alpha^{\tilde{\mathscr{R}}_0}$ in the next term is indeed finite. From (5.3) we have

$$\frac{\partial\psi_2}{\partial y_i}=2^{-s}\frac{\partial\psi}{\partial x_i}, \qquad \frac{\partial^2\psi_2}{\partial y_i\partial y_k}=2^{-2s}\frac{\partial^2\psi}{\partial x_i\partial x_k}.$$

Similarly,

$$H_\alpha^{\tilde{\mathscr{G}}_0}(D_y{}^2\psi_2)=2^{-s(2+\alpha)}H_\alpha^{\tilde{\mathscr{G}}_s}(D_x{}^2\psi).$$

Consequently, by (4.7) we obtain

$$||\psi_2||_{2+\alpha}^{\tilde{\mathscr{G}}_0} \leqq A_4 2^{-2s} \tag{5.5 b}$$

Hence (5.4) and (5.5) now yield

$$||U||_{2+\alpha}^{\mathscr{R}_0} \leqq A_5 2^{-s\mu} \tag{5.6}$$

Remembering that $u(2^{-s}y, t)=U(y, t)$, we see that

$$\frac{\partial U}{\partial y_i}=2^{-s}\frac{\partial u}{\partial x_i}, \qquad \frac{\partial^2 U}{\partial y_i\partial y_k}=2^{-2s}\frac{\partial^2 u}{\partial x_i\partial x_k}. \tag{5.7}$$

Furthermore, in $\mathscr{R}_0$ we have

$$\left|\frac{\partial U}{\partial y_i}\right| \leqq ||U||_{2+\alpha}^{\mathscr{R}_0} \tag{5.8a}$$

as well as

$$\left|\frac{\partial^2 U}{\partial y_i \partial y_k}\right| \leqq \|U\|_{2+\alpha}^{\mathscr{R}_0} \tag{5.8b}$$

From (5.6)-(5.8) we obtain the estimates for the derivatives u_{x_i} and $u_{x_i x_k}$, i, $k=1, \cdots, n$, stated in the theorem, and the proof is complete.

6. Smoothness of solutions

We shall now obtain a theorem on the smoothness of first derivatives of bounded solutions u of the problem (4.1), (4.2), namely, that the $D_x u$ are Hölder continuous, as well as on the smoothness of second derivatives multiplied by a suitable factor r^κ, $0<\kappa<1$, which products are proved to be Hölder continuous. We also show that for small angles, the $D_x^2 u$ are continuous. In the present section this will be proved for a cylindrical sector (as considered in Secs. 4 and 5) and in the next section for a general region $\Omega=G\times J$.

THEOREM 4. *Under the assumptions of Theorem* 3 *there exist constants* $c_3>0$ *and* κ, $\chi\in(0,1)$ *such that*

$$D_x u\in C^\nu(\bar{N}_{c_3}\times J)\,, \qquad \nu=\mu-1$$

and

$$r^\kappa D_x^2 u\in C^\chi(\bar{N}_{c_3}\times J)\,, \qquad \chi=\min(\alpha, \kappa+\nu-1)\,.$$

If $\omega<\pi/2$, *then*

$$D_x^2 u\in C^0(\bar{N}_{c_3}\times J)\,.$$

PROOF. We take $c_3>0$ sufficiently small so that, by Theorem 3, in $\bar{N}_{c_3}\times J$ we have

$$\left|D_x^j u(x,t)\right| \leqq K_j r^{1+\nu-j}\,, \qquad j=1,2\,. \tag{6.1}$$

In that region we consider any two points P and Q, whose distance from $E_{c_3}\times J$ we denote by r_1 and r_2, respectively, assuming that $0\leqq r_2\leqq r_1\leqq c_3$, without restriction. If $r_2\leqq r_1/2$, then $d(P,Q)\geqq r_1/2$, so that in this case we have

$$\frac{|D_x u(P)-D_x u(Q)|}{d(P,Q)^\nu} \leqq \frac{2K_1 r_1^\nu}{(r_1/2)^\nu}=K_3\,. \tag{6.2}$$

We show that in the other case, $r_2>r_1/2$, we also have such an inequality. Let P: $(x_1^0, x_2^0, \cdots, x_n^0, t)$. Let

$$G_P=\left\{x \middle| x\in N_{c_3},\ r_1/2\leqq r\leqq r_1,\ |x_i-x_i^0|\leqq r_1/2,\ i>2\right\},$$

where $r^2=x_1^2+x_2^2$, as before. We now apply the transformation

$$\begin{aligned} x_i &= Mz_i & &\text{if } i=1,2, \\ x_i-x_i^0 &= M(z_i-x_i^0) & &\text{if } i>2, \end{aligned} \qquad M=2r_1/c_3. \tag{6.3}$$

This transformation maps G_P onto the region

$$G'_P=\left\{z \middle| c_3/4\leqq\rho\leqq c_3/2,\ |z_i-x_i^0|\leqq c_3/4,\ i>2\right\}$$

where $\rho^2=z_1^2+z_2^2$. In $G_P^1\times J$ the function $W(z,t)=u(x,t)$ satisfies the parabolic equation

$$B_{ik}W_{z_iz_k}+MB_iW_{z_i}+M^2BW-M^2W_t=M^2F, \tag{6.4}$$

where B_{ik}, B_i, B and F are the coefficients in $Lu=f$, represented in terms of the new variables. We also consider the region

$$G''_P=\left\{z \middle| c_3/8\leqq\rho\leqq c_3,\ |z_i-x_i^0|\leqq c_3/4,\ i>2\right\}.$$

We again apply the Schauder-type estimate, writing $\Lambda'=G'_P\times J$, $\Lambda''=G''_P\times J$ and $\Lambda^*=\Gamma''_P\times J$, for simplicity; here Γ''_P is the part of the boundary of G''_P which lies on the hyperplanes Π_1 and Π_2. The estimate is

$$||W||_{2+\alpha}^{\Lambda'}\leqq A_6\left[||W||_0^{\Lambda''}+M^2||F||_\alpha^{\Lambda''}+||\psi_3||_{2+\alpha}^{\Lambda^*}\right].$$

Using an idea similar to that in the proof of Theorem 3, we conclude that

$$||W||_{2+\alpha}^{\Lambda'}\leqq A_7r_1^\mu, \qquad \mu=1+\nu.$$

Furthermore,

$$||D_zW||_\nu^{\Lambda'}\leqq||W||_{2+\alpha}^{\Lambda'}$$

and

$$||D_z^2W||_\chi^{\Lambda'}\leqq||W||_{2+\alpha}^{\Lambda'} \qquad (0<\chi\leqq\alpha).$$

Also

$$D_z^kW=M^kD_x^ku \qquad (k=1,2)$$

and

$$H_\nu^{\Lambda'}(D_zW)=M^\mu H_\nu^{\tilde\Lambda}(D_xu), \qquad \tilde\Lambda=G_P\times J.$$

Consequently,

$$H_\nu^{\tilde\Lambda}(D_xu)\leqq K_4$$

as well as

$$H_{\chi}^{\tilde{A}}(D_x^2 u) \leqq K_5 r^{\mu-2-\chi}.$$

To obtain (6. 2) in the case when $r_2 > r_1/2$, besides P and Q we also consider the point P_1 defined as follows. If

$$P: (r_1 \cos\theta_1, \ r_1 \sin\theta_1, \ x^{(1)\prime}, \ t)$$

and

$$Q: (r_2 \cos\theta_2, \ r_2 \sin\theta_2, \ x^{(2)\prime}, \ t)$$

then P_1 has the coordinates

$$P_1: (r_1 \cos\theta_2, \ r_1 \sin\theta_2, \ x^{(2)\prime}, \ t).$$

If $d(P, P_1) \leqq r_1/2$, then $Q \in \tilde{A}$, where $D_x u \in C^{\nu}(\tilde{A})$. If $d(P, P_1) > r_1/2$, then $d(P, Q) \geqq d(P, P_1) > r_1/2$ and $d(P, Q) \geqq d(P_1, Q)$. Hence in this case,

$$\begin{aligned} \frac{|D_x u(P) - D_x u(Q)|}{d(P, Q)^{\nu}} &\leqq \frac{|D_x u(P) - D_x u(P_1)|}{d(P, P_1)^{\nu}} \\ &+ \frac{|D_x u(P_1) - D_x u(Q)|}{d(P_1, Q)^{\nu}}. \end{aligned} \tag{6.5}$$

Since

$$\frac{|D_x u(P) - D_x u(P_1)|}{d(P, P_1)^{\nu}} \leqq \frac{2K_1 r_1^{\nu}}{(r_1/2)^{\nu}} = K$$

and $Q \in G_{P_1} \times J$, we conclude that the right-hand side of (6. 5) is bounded. This proves the first statement of the theorem. The other statements are obtained in a similar fashion.

7. Proof of Theorem 1

Without loss of generality we can take $m=2$. Then $\partial G = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = E_1 = E$ is the single edge. It suffices to prove the theorem in a neighborhood of an arbitrary point $P: (x_1^0, \cdots, x_n^0, t) \in E \times \bar{J}$. To P there corresponds $P_0: (x_1^0, \cdots, x_n^0) \in E$. Suppose that in a neighborhood of P_0 the two hypersurfaces intersecting at E have the representations

$$x_1 = h_1(x_2, x') \quad \text{and} \quad x_2 = h_2(x_1, x'),$$

respectively, where h_1 and h_2 are of class $C^{2+\alpha}$. The transformation

$$y_i = \begin{cases} x_i - h_i & \text{if } i = 1, 2 \\ x_i - x_i^0 & \text{if } i > 2 \end{cases} \tag{7.1}$$

maps P_0 onto the origin of the y-coordinate system, and transforms those two hypersurfaces into the hyperplanes $y_1=0$ and $y_2=0$. Furthermore, from (3. 1) we obtain another parabolic equation with the y_i as the independent variables. The latter equation we transform again by a linear transformation such that afterwards the coefficients A_{ik} of the resulting principal part have the property

$$A_{ik}(0)=\delta_{ik} \qquad i, k=1, \cdots, n$$

and the hyperplanes $y_1=0$ and $y_2=0$ are mapped onto two hyperplanes making an angle $\omega=\omega(P_0)<\pi$. Such a transformation exists, and its Jacobian is not zero. We finally choose $\beta=(\pi-\omega)/2$ and apply a rotation such that afterwards the hyperplanes have the representations

$$z_2=z_1 \tan \beta \quad \text{and} \quad z_2=z_1 \tan (\beta+\omega) . \tag{7. 2}$$

Let $N_{P_0,c_1}\subset G$ be the intersection of G and a ball of radius c_1 about P_0. In z-space this intersection corresponds to a domain N_{0,c_2} bounded by the two hyperplanes and a surface having a distance $c_2>0$ from the origin of the z-coordinate system. Let N_c, $c<c_2$, denote the intersection of N_{0,c_2} and a ball of radius $c>0$ about the origin. In $N_c\times J$ the function $V(z, t)=u(x, t)$ satisfies a parabolic equation of the form (4. 1) with coefficients such that (i*) and (ii*) (cf. Sec. 5, Theorem 3) hold. Clearly, $V(z, 0)=0$, and the boundary function $\Phi(z, t)$ satisfies (iii*) on $S_c\times J$ with S_c as in Sec. 4. We shall now determine a function $q(z)\in C^{2+\alpha}(\bar{N}_c)$ such that $U=V-q$ and its boundary value $\psi(z, t)$ on $S_c\times J$ satisfy all the conditions in Theorem 4.

Indeed, consider the function

$$\begin{aligned} q(z, t) = & \Phi(0, 0, z', t)+(z_1 \cos \beta+z_2 \sin \beta)\, \Phi_\beta(0, 0, z', t) \\ & +\operatorname{cosec} \omega(-z_1 \sin \beta+z_2 \cos \beta)\Big[\Phi_{\omega+\beta}(0, 0, z', t) \\ & -\Phi_\beta(0, 0, z', t) \cos \omega\Big] \end{aligned}$$

where $z'=(z_3, \cdots, z_n)$ and Φ_β and $\Phi_{\omega+\beta}$ are the first derivatives of Φ in the directions $\theta=\beta$ and $\theta=\beta+\omega$, respectively, and perpendicular to E_c. Since $\psi(z, t)$ now satisfies (iii*) and (iv*) (cf. Sec. 5), and $U=V-q$ is a solution of an equation of the form (4. 1), all the conditions (4. 2) and (i*)-(iv*) hold true. Hence from Theorem 4 it follows that there exist constants $c_3>0$ and ν, κ, χ, $0<\nu$, κ, $\chi<1$, such that

$$D_z U\in C^\nu(\bar{N}_{c_3}\times \bar{J})$$

and

$$\rho^\kappa D_z^2 U \in C^\chi(\bar{N}_{c_3} \times \bar{J})$$

where $\rho^2 = z_1^2 + z_2^2$. We return to the x-space. Since q is of class $C^{2+\alpha}$ and each of the above transformations is of that class and has a nonzero Jacobian, we conclude that in a neighborhood $N = \bar{N}_{P_0, c_4} \times \bar{J}$, $0 < c_4 < c$, we have

$$D_x u \in C^\nu(N) \quad \text{as well as} \quad \delta^\kappa D_x^2 u \in C^\chi(N)\,.$$

Furthermore, if $\omega(P_0) < \pi/2$ for all $P_0 \in \bigcup_i E_i$, we can take $\mu = 2$ and obtain $D_x^2 u \in C^0(N)$. Theorem 1 is proved.

8. Dirichlet problem for elliptic equations

As it was mentioned in the Introduction, the present method can also be applied to the Dirichlet problem

$$L_0 u = f \qquad \text{in} \quad G \tag{8.1}$$

$$u|_{\partial G} = \phi \tag{8.2}$$

where L_0 is an elliptic operator defined by

$$L_0 = a_{ik}(x) \frac{\partial^2}{\partial x_i \partial x_k} + a_i(x) \frac{\partial}{\partial x_i} + a(x)\,.$$

Here, $x = (x_1, \cdots, x_n)$, and G is a domain with edges as in Sec. 3 (and the other notations used below are as in Sec. 3, too).

Indeed, by suitable modifications of the present method the following theorem can be obtained (see A. Azzam [2]).

THEOREM 5. *Let u be a bounded solution of* (8.1), (8.2), *where a_{ik}, a_i, a, $f \in C^\alpha(\bar{G})$ and*

$$\phi \in C^{2+\alpha}(\partial G \setminus \cup E_i) \cap C^0(\partial G)\,.$$

For every $P \in \cup E_i$, let $\omega(P) < \pi$. Then there exist constants ν, κ, χ, $0 < \nu$, κ, $\chi < 1$, such that

$$u \in C^{1+\nu}(\bar{G})$$

and

$$w \in C^\chi(\bar{G}) \quad \textit{where} \quad w(x) = \rho^\kappa D^2 u(x)$$

and ρ is the distance from x to $\bigcup_i E_i$.

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