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On a special type of Galois extensions

Dedicated to Professor G. Azumaya on his 60th birthday

By Kozo SUGANO

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1. In order to generalize the notion of Azumaya algebra, we researched on a special type of separable extension, called H-separable extension, and have found that many properties which hold in Azumaya algebras hold also in H-separable extensions (see for example [5], [2], [7] and [8]). In this paper we shall study relations between Galois extensions and H-separable extensions, and shall obtain some necessary and suficient conditions for Galois extensions to be H-separable extensions. By the definition of Galois extension and by Cor. 1.1 [5], we can easily see that in the case of algebras over a commutative ring R, H-separable Galois extensions of R is same as central Galois extension of R. Throughout this paper Λ shall always be a ring with 1, Γ a subring of Λ which contains same 1, C the center of Λ and $\Delta=V_{\Lambda}(\Gamma)=\Lambda^{\Gamma}$, and $M^{\Lambda}=\{m\in M|xm=mx$ for all $x\in\Lambda\}$ for any $\Lambda-\Lambda$-module M.

2. First, we shall recall definitions.

DEFINITION 1. Λ is called an H-separable extension of Γ when Λ and Γ satisfy one of the following equivalent conditions;

(a) $\Lambda\otimes_{\Gamma}\Lambda$ is isomorphic to a direct summand of $\Lambda\oplus\Lambda\oplus\cdots\oplus\Lambda$ (finite direct sum) as $\Lambda-\Lambda$-module.

(b) Δ is C-finitely generated projective, and the following map η is an isomorphism

$$\eta:\ \Lambda\otimes_{\Gamma}\Lambda\rightarrow\mathrm{Hom}\,({}_{C}\Delta,{}_{C}\Lambda)\quad \eta(x\otimes y)(d)=xyd\ \ (x,y\in\Lambda,\ d\in\Delta)$$

(c) For any $\Lambda-\Lambda$-module M, the following map g_M is an isomorphism

$$g_M:\ \Delta\otimes_{C}M^{\Lambda}\rightarrow M^{\Gamma}\quad g_M(d\otimes m)=dm\ \ (d\in\Delta,\ m\in M^{\Lambda})$$

(d) $1\otimes 1\in\Delta(\Lambda\otimes_{\Gamma}\Lambda)^{\Lambda}$

As for the proof of equivalence of (a)$\sim$(d), see Theorem 1.2 [5], Prop. 1 [6] or (1.3) [7]. Note that Azumaya algebra always satisfies these conditions.

Next, let $\mathfrak{G}$ be a finite group of automorphisms of Λ which fix all elements of Γ. We can make $\Sigma_{\sigma\in\mathfrak{G}}\Lambda U_{\sigma}$ a ring by $(xU_{\sigma})(yU_{\tau})=x\sigma(y)\,U_{\sigma\tau}$ $(\sigma,\tau\in\mathfrak{G})$, where $\{U_{\sigma}\}_{\sigma\in\mathfrak{G}}$ is a free Λ-basis. We denote this ring by $\Delta(\Lambda:\mathfrak{G})$. Then we can always have a ring homomorphism j of $\Delta(\Lambda:\mathfrak{G})$ to $\mathrm{Hom}\,(\Lambda_{\Gamma},\Lambda_{\Gamma})$ such that $j(xU_{\sigma})(y)=x\sigma(y)$ $(x,y\in\Lambda,\ \sigma\in\mathfrak{G})$.

DEFINITION 2. We say that Λ is a $\mathfrak{G}$-Galois extension of Γ when Λ

and Γ satisfy the following three conditions; (1) Λ is right Γ-finitely generated projective (2) j is an isomorphism (3) $\Lambda^{\mathfrak{G}}=\Gamma$, where $\Lambda^{\mathfrak{G}}=\{x\in\Lambda|\sigma(x)=x$ for all $\sigma\in\mathfrak{G}\}$.

By Prop. 2.4 [3], Λ is a $\mathfrak{G}$-Galois extension of Γ if and only if $\Lambda^{\mathfrak{G}}=\Gamma$, and there exist x_i, y_i $(i=1, 2, \cdots, n)$ in Λ such that $\Sigma x_i\sigma(y_i)=\delta_{1,\sigma}$.

3. For any ring Λ, we denote the opposite ring of Λ by Λ^0, and for any subset X of Λ, we put $X^0=\{x^0|x\in X\}$. Let σ be any ring automorphism of Λ, then we shall put $J_\sigma=\{a\in\Lambda|xa=a\sigma(x)$ for any $x\in\Lambda\}$. From Λ we can obtain a new $\Lambda-\Lambda$-module Λ_σ as follows; $\Lambda_\sigma=\Lambda$ as left Λ-module, but as right module $x\cdot y=x\sigma(y)$ $(x, y\in\Lambda)$. Then, we have $J_\sigma=(\Lambda_\sigma)^\Lambda$ and $\Delta=(\Lambda_\sigma)^\Gamma$.

Next, note that, if Λ is an H-separable extension of Γ, we have a ring isomorphism $\bar{\eta}$ of $\Lambda\otimes_C\Delta^0$ to $\mathrm{Hom}(\Lambda_\Gamma, \Lambda_\Gamma)$ such that $\bar{\eta}(x\otimes d^0)(y)=xyd$ $(x, y\in\Lambda, d\in\Delta)$ (See 1.5 [7]).

PROPOSITION 1. *Let Λ be an H-separable extension of Γ and $\mathfrak{G}$ a finite group of ring automorphisms of Λ which fix all elements of Γ. Then, we have*

(1) *j is an isomorphism if and only if $\Delta=\Sigma^{\oplus}_{\sigma\in\mathfrak{G}}J_\sigma$ (direct sum). In case these conditions are satisfied, we have $\Lambda^{\mathfrak{G}}=V_\Lambda(V_\Lambda(\Gamma))$ $(=\Gamma')$.*

(2) *In case Λ is, furthermore, right Γ-finitely generated projective, Λ is $\mathfrak{G}$-Galois extension of Γ' if and only if $\Delta=\Sigma^{\oplus}_{\sigma\in\mathfrak{G}}J_\sigma$.*

PROOF. (1). Suppose $\Delta=\Sigma^{\oplus}_{\sigma\in\mathfrak{G}}J_\sigma$. Since Λ is H-separable over Γ, $\Delta\otimes_C J_\sigma\cong\Delta$ by $d\otimes d_\sigma\to dd_\sigma(d\in\Delta, d_\sigma\in J_\sigma)$, for each $\sigma\in\mathfrak{G}$. Hence we have $\Delta J_\sigma=\Delta$ and $\Lambda J_\sigma=\Lambda$. Now consider a map g_σ of $\Lambda\otimes_C J^0_\sigma$ to ΛU_σ defined by $g_\sigma(x\otimes d^0_\sigma)=xd_\sigma U_\sigma(x\in\Lambda, d_\sigma\in J_\sigma)$, for each $\sigma\in G$. Since $\Lambda J_\sigma=\Lambda$, each g_σ is a left Λ-epimorphism. Then we obtain a left Λ-epimorphism g of $\Lambda\otimes_C\Lambda^0$ $(=\Sigma^{\oplus}_{\sigma\in\mathfrak{G}}J^0_\sigma)$ to $\Delta(\Lambda;\mathfrak{G})$ $(=\Sigma^{\oplus}_{\sigma\in\mathfrak{G}}\Lambda U_\sigma)$ with $g=\Sigma g_\sigma$. It is easy to compute that $jg=\bar{\eta}$. Then since g is an epimorphism and $\bar{\eta}$ is an isomorphism, j is an isomorphism. Next, for any $x\in\Lambda^{\mathfrak{G}}$, $d\in\Delta$, we have that $d=\Sigma d_\sigma(d_\sigma\in J_\sigma)$ and $xd=\Sigma xd_\sigma=\Sigma d_\sigma\sigma(x)=\Sigma d_\sigma x=dx$. Hence we have $\Lambda^{\mathfrak{G}}\subseteq V_\Lambda(\Delta)$. $\Lambda^{\mathfrak{G}}\supseteq V_\Lambda(\Delta)$ is obvious from $\mathfrak{G}\subseteq\mathrm{Hom}(\Lambda_\Gamma, \Lambda_\Gamma)\cong\Lambda\otimes_C\Delta^0$. Thus we have $\Lambda^{\mathfrak{G}}=V_\Lambda(\Delta)=\Gamma'$. Conversely suppose that j is an isomorphism. Then, $\Delta\cong\mathrm{Hom}({}_\Lambda\Lambda_\Gamma, {}_\Lambda\Lambda_\Gamma)\cong[\Delta(\Lambda;\mathfrak{G})]^\Lambda=[\Sigma^{\oplus}_{\sigma\in\mathfrak{G}}(\Lambda U_\sigma)]^\Lambda=\Sigma^{\oplus}_{\sigma\in\mathfrak{G}}J_\sigma U_\sigma$. Then it follows that $\Delta=\Sigma^{\oplus}_{\sigma\in\mathfrak{G}}J_\sigma$. (2). By Theorem 1.3′ [5], Λ is also H-separable over Γ', and $\mathrm{Hom}(\Lambda_{\Gamma'}, \Lambda_{\Gamma'})=\mathrm{Hom}(\Lambda_\Gamma, \Lambda_\Gamma)$ $(\cong\Lambda\otimes_C\Delta^0)$. Hence Λ is also Γ'-finitely generated projective by Morita Theorem. Therefore, (2) follows form (1) and the definition of Galois extension.

REMARK. Note that the 'only if' part of (1) holds without the condition that Λ is H-separable over Γ (see Prop. 1 [4]).

THFOREM 2. *Let Λ be a $\mathfrak{G}$-Galois extension of Γ. Then the following conditions are equivalent;*

(i) *Λ is an H-separable extension of Γ.*

(ii) *For each $\sigma\in\mathfrak{G}$, map g_σ of $\Delta\otimes_C J_\sigma$ to Δ defined by $g_\sigma(d\otimes d_\sigma)=dd_\sigma$ $(d\in\Delta, d_\sigma\in J_\sigma)$ is an isomorphism.*

(iii) *$J_\sigma J_{\sigma^{-1}}=C$ for every $\sigma\in\mathfrak{G}$.*

In case these conditions are satisfied, for any σ, $\tau\in\mathfrak{G}$, we have $J_\sigma J_\tau=J_{\tau\sigma}$, and each J_σ is a C-progenerator of rank 1.

PROOF. (i)$\Rightarrow$(ii). This follows from Definition 1 (c). (ii)$\Rightarrow$(iii). Since $\Delta=\Sigma^{\oplus}_{\sigma\in\mathfrak{G}}J_\sigma$ by the above remark, $\Delta\otimes_C J_\sigma\cong\Delta$ implies that $\Delta=\Sigma^{\oplus}_{\tau\in\mathfrak{G}}J_\tau J_\sigma\subseteq\Sigma^{\oplus}_{\sigma\in\mathfrak{G}}J_{\sigma\tau}=\Delta$. Therefore we have $J_\tau J_\sigma=J_{\sigma\tau}$ for any τ, $\sigma\in\mathfrak{G}$. Especially $J_\sigma J_{\sigma^{-1}}=C$. (iii)$\Rightarrow$(i). Each element of $J_{\sigma^{-1}}$ can be regarded as an element of $\mathrm{Hom}_C(J_\sigma, C)$, since $J_{\sigma^{-1}}J_\sigma=C$. Let $1=\Sigma x_i y_i$ with $x_i\in J_\sigma$ and $y_i\in J_{\sigma^{-1}}$. Then $\{x_i, y_i\}$ forms a dual basis of J_σ over C, since $x=\Sigma x_i y_i x$ for every $x\in J_\sigma$. Also it can easily be shown that $J_{\sigma^{-1}}=\mathrm{Hom}_C(J_\sigma, C)$. and $C\cong\mathrm{Hom}_C(J_\sigma, J_\sigma)$. Thus J_σ is a C-progenerator of rank 1. Next, for each $\sigma\in\mathfrak{G}$, consider a left Λ-homomorphism g_σ of $\Lambda\otimes_C J_\sigma$ to ΛU_σ defined by $g_\sigma(x\otimes d_\sigma)=xd_\sigma U_\sigma$ $(x\in\Lambda$, $d_\sigma\in J_\sigma)$. Of course $\Lambda\otimes_C J_\sigma$ and ΛU_σ are $\Lambda-\Lambda$-modules. Pick any $x, y\in\Lambda$ and $d_\sigma\in J_\sigma$. Then, $g_\sigma((x\otimes d_\sigma)y)=g_\sigma(xy\otimes d_\sigma)=xyd_\sigma U_\sigma=xd_\sigma\sigma(y)U_\sigma=xd_\sigma U_\sigma y=g_\sigma(x\otimes d_\sigma)y$. Therefore g_σ is a $\Lambda-\Lambda$-homomorphism. Since $J_{\sigma^{-1}}J_\sigma=C$, we can put $1=\Sigma u_j v_j(u_j\in J_{\sigma^{-1}}, v_j\in J_\sigma)$. Then for any $x\in\Lambda$, $xU_\sigma=g_\sigma(\Sigma xu_j\otimes v_j)$. Thus g_σ is an epimorphism. If $\Sigma x_i\otimes d_i\in\mathrm{Ker}\,g_\sigma(x_i\in\Lambda, d_i\in J_\sigma)$, then $\Sigma x_i\otimes d_i=\Sigma x_i\otimes d_i u_j v_j=\Sigma x_i d_i u_j\otimes v_j=0$, since $\Sigma x_i d_i=0$ and $d_i u_j\in J_\sigma J_{\sigma^{-1}}=C$. Hence $\mathrm{Ker}\,g_\sigma=0$. Thus g_σ is a $\Lambda-\Lambda$-isomorphism. Then since J_σ is C-finitely generated projective, $\Lambda\otimes_C J_\sigma<\oplus(\Lambda\oplus\Lambda\oplus\cdots\oplus\Lambda)$ as $\Lambda-\Lambda$-module. Therefore, $\mathrm{Hom}(\Lambda_\Gamma, \Lambda_\Gamma)\cong\Sigma^{\oplus}_{\sigma\in\mathfrak{G}}\Lambda U_\sigma<\oplus(\Lambda\oplus\Lambda\oplus\cdots\oplus\Lambda)$ as $\Lambda-\Lambda$-module. Then, we have $\Lambda\otimes_\Gamma\Lambda\cong\Lambda\otimes_\Gamma\mathrm{Hom}({}_\Lambda\Lambda, {}_\Lambda\Lambda)\cong\mathrm{Hom}({}_\Lambda\mathrm{Hom}(\Lambda_\Gamma, \Lambda_\Gamma), {}_\Lambda\Lambda)<\oplus(\Lambda\oplus\Lambda\oplus\cdots\oplus\Lambda)$, as $\Lambda-\Lambda$-module.

COROLLARY 3. *Let Λ be a $\mathfrak{G}$-Galois extension of Γ. Then if all elements of $\mathfrak{G}$ are inner-automorphisms of Λ, Λ is an H-separable extension of Γ.*

PROOF. For each $\sigma\in\mathfrak{G}$, let r_σ be a unit of Δ such that $\sigma(x)=r_{\sigma^{-1}}xr_\sigma$ for all $x\in\Lambda$. Then $r_\sigma\in J_\sigma$, and $r_\sigma r_{\sigma^{-1}}$ is a unit in C. Since $J_\sigma J_{\sigma^{-1}}$ is an ideal of C, and $r_\sigma r_{\sigma^{-1}}\in J_\sigma J_{\sigma^{-1}}$, we have $C=J_\sigma J_{\sigma^{-1}}$.

EXAMPLE 1. Let R be an arbitrary ring with 1, $\Lambda=(R)_2$, 2×2-full matrix ring over R, and $\Gamma=\left\{\begin{bmatrix} a & b \\ -b & a\end{bmatrix}\middle| a, b\in R\right\}$. Put $I=\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix}$, and let E be the identity matrix in Λ. Let $\iota(A)=A$ and $\sigma(A)=I^{-1}AI$, for every $A\in\Lambda$. Then Λ is a $\mathfrak{G}$-Galois extension of Γ with $\mathfrak{G}=\{\iota, \sigma\}$. Consequently, Λ is an H-separable extension of Γ. Because, for the matrix units $e_{i,j}(i, j=1, 2)$ of Λ, we have $\Sigma e_{i,1}e_{1,i}=E$ and $\Sigma e_{i,1}\sigma(e_{1,i})=e_{1,1}e_{2,2}+e_{2,1}(-e_{2,1})=0$. By direct computations, we have that $\sigma^2=\iota$ and $\Lambda^{\mathfrak{G}}=\Gamma$.

EXAMPLE 2. Let R, Λ, E, I, ι and σ be as in Example 1, and add $J=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$ $K=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, and Put $\tau(A)=J^{-1}AJ$ and $\rho(A)=K^{-1}AK$ for all A in Λ. Furthermore suppose that 2 is a unit in R. Then Λ is a Galois extension of R relative to an Abelian group $\mathscr{H}=\{\iota, \sigma, \tau, \rho\}$. Because, by direct computations, we see that $\sigma\tau=\tau\sigma=\rho$, $\rho\tau=\tau\rho=\sigma$, $\sigma\rho=\rho\sigma=\tau$ and $\Lambda^{\mathscr{H}}=R$. On the other hand, $2^{-1}\in R$ implies that $\{E, I, J, K\}$ is a free C-basis of $V_\Lambda(R)=(C)_2$. It is also easy to see that $J_\sigma=CI$, $J_\tau=CJ$ and $J_\rho=CK$. Thus we have $V_\Lambda(R)=\Sigma^{\oplus}_{\chi\in\mathscr{H}}J_\chi$. Hence Λ is $\mathscr{H}$-Galois extension of R by Prop. 1.

REMARK. In Example 1, if we put R=ring of rational integers, 2 is not a unit in R. Thus orders of Galois groups are not always units in H-separable Galois extensions, though they are always units in central Galois algebras (see Cor. 3 [4]). In case 2 is unit in R, we have (1) Γ is $\Gamma-\Gamma$-direct summand of Λ (2) $V_\Lambda(\Gamma)=C'$ (the center of Γ) (3) $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$, by next proposition which is proved in [9];

PROPOSITION 4. *Let Λ be an H-separable $\mathfrak{G}$-Galois extension of Γ. Then the following assertions hold*

(1) $V_\Lambda(V_\Lambda(\Gamma))=\Gamma$.

(2) *Δ is C-finitely generated projective of rank n, where $n=|\mathfrak{G}|$.*

(3) *The following conditions are equivalent;*

(i) $n^{-1}\in R$ (ii) ${}_\Gamma\Gamma_\Gamma<\oplus{}_\Gamma\Lambda_\Gamma$ (iii) *Δ is a separable C-algebra*

PROOF. See Prop. 3 [9].

REMARK. From (1) it follows that, in case Λ is H-separable and $\mathfrak{G}$-galois over Γ, the center of Γ is equal to the center of Δ, which we shall denote by C'.

REMARK. Suppose that Λ is a $\mathfrak{G}$-Galois extension of Γ, and put $\mathfrak{K}=\{\sigma\in\mathfrak{G}\,|\,\sigma|\Delta=\text{identity}\}$. Then, $\mathfrak{K}$ is contained in the center of $\mathfrak{G}$. Because, for any $\sigma\in\mathfrak{K}$ and $\tau\in\mathfrak{G}$, $J_\tau=\sigma(J_\tau)\subseteq J_{\sigma\tau\sigma^{-1}}$. But $\Sigma_{\tau\in\mathfrak{G}}J_\tau$ is a direct sum. Hence $J_\tau=J_{\sigma\tau\sigma^{-1}}$, consequently, $\sigma\tau=\tau\sigma$.

PROPOSITION 5. *Let Λ be an H-separable extension of Γ and σ be any automorphism of Λ which fixes all elements of Γ. Then $\sigma|\Delta=$identity if and only if $J_\sigma\subseteq V_\Delta(\Delta)$, the center of Δ.*

PROOF. If $\sigma|\Delta=$identity, then for any $a\in J_\sigma$ and $d\in\Delta$, $da=a\sigma(d)=ad$. Hence $J_\sigma\subseteq V_\Delta(\Delta)$. On the other hand, by Definition 1 (c), $\Delta J_\sigma=\Delta$. Then, $\Delta J_\sigma J_{\sigma^{-1}}=\Delta J_{\sigma^{-1}}=\Delta$. But $J_\sigma J_{\sigma^{-1}}$ is an ideal of C and $C_C<\oplus\Delta_C$. Hence $J_\sigma J_{\sigma^{-1}}=C$. Similarly $J_{\sigma^{-1}}J_\sigma=C$. Put $1=\Sigma y_i x_i (y_i\in J_{\sigma^{-1}}, x_i\in J_\sigma)$, and suppose $J_\sigma\subseteq V_\Delta(\Delta)$. Then for any $d\in\Delta$, $\sigma^{-1}(d)=\Sigma y_i x_i\sigma^{-1}(d)=\Sigma y_i\sigma^{-1}(d)\,x_i=\Sigma dy_i x_i=d$. Hence $\sigma(d)=d$.

THFOREM 6. *Let Λ be an H-separable and $\mathfrak{G}$-Galois extension of Γ and $\mathfrak{K}$ as in the remark above. Then if $|\mathfrak{G}|$ is a unit, we have;*

(1) *Λ is an H-separable $\mathfrak{K}$-Galois extension of $\Lambda^{\mathfrak{K}}$.*
(2) *$\Lambda^{\mathfrak{K}}$ is an H-separable $\mathfrak{G}/\mathfrak{K}$-Galois extension of Γ.*
(3) *The following conditions are equivalent;*
(i) $C'=\Sigma^{\oplus}_{\sigma\in\mathfrak{K}}J_{\sigma}$ (ii) $\Lambda^{\mathfrak{K}}=\Gamma\Delta$ (iii) *Δ is a central $\mathfrak{G}/\mathfrak{K}$-Galois extension of C'.*

PROOF. Put $|\mathfrak{G}|=n$. Since $n^{-1}\in C$, $t_{\mathfrak{G}}(n^{-1})=\Sigma_{\sigma\in\mathfrak{G}}\sigma(n^{-1})=1$. Therefore, in this case Λ is a Galois extension of Γ relative to $\mathfrak{G}$ in the sense of Y. Takeuchi [10], which is under stronger conditions than ours. Therefore, by Theorem 1 [10], Λ is a $\mathfrak{K}$-Galois extension of $\Lambda^{\mathfrak{K}}$, and $\Lambda^{\mathfrak{K}}$ is a $\mathfrak{G}/\mathfrak{K}$-Galois extension of Γ in our sense, too. Then by Prop. 1 [4], $V_{\Lambda}(\Lambda^{\mathfrak{K}})=\Sigma^{\oplus}_{\sigma\in\mathfrak{K}}J_{\sigma}$. Hence, $J_{\sigma}J_{\sigma^{-1}}=C$ for all $\sigma\in\mathfrak{K}\subseteq\mathfrak{G}$, by Theorem 2. Then Λ is H-separable over $\Lambda^{\mathfrak{K}}$ by Theorem 2. Furthermore put $\bar{J}_{\bar{\sigma}}=\{a\in\Lambda^{\mathfrak{K}}\mid xa=a\bar{\sigma}(x)$ for all $x\in\Lambda^{\mathfrak{K}}\}$ for each $\bar{\sigma}\in\bar{\mathfrak{G}}$ ($\bar{\mathfrak{G}}=\mathfrak{G}/\mathfrak{K}$), and let $\mathfrak{G}=\sigma_1\mathfrak{K}\cup\sigma_2\mathfrak{K}\cup\cdots\cup\sigma_r\mathfrak{K}$ be a decomposition of $\mathfrak{G}$ by cosets of $\mathfrak{K}$. Then, obviously $(\Sigma^{\oplus}_{\tau\in\mathfrak{K}}J_{\sigma_i\tau})\subseteq\bar{J}_{\bar{\sigma}_i}$, and we see $1\in J_{\sigma_i\tau}J_{\sigma_i^{-1}\tau^{-1}}\subseteq J_{\sigma_i^{-1}}J_{\bar{\sigma}_i^{-1}}$ for all $\bar{\sigma}_i\in\mathfrak{G}/\mathfrak{K}$. Thus $\bar{J}_{\bar{\sigma}}\bar{J}_{\bar{\sigma}^{-1}}=$ the center of $\Lambda^{\mathfrak{G}}$ for all $\bar{\sigma}\in\bar{\mathfrak{G}}$. Hence $\Lambda^{\mathfrak{G}}$ is H-separable over Γ. Thus we have shown (1) and (2). (3). Put $D=V_{\Lambda}(\Lambda^{\mathfrak{G}})$, which is equall to $\Sigma^{\oplus}_{\sigma\in\mathfrak{G}}J_{\sigma}$. (i)$\Rightarrow$(ii). By Prop. 4, $\Lambda^{\mathfrak{K}}=V_{\Lambda}(V_{\Lambda}(\Lambda^{\mathfrak{K}}))=V_{\Lambda}(D)=V_{\Lambda}(C')$. But since Δ is an Azumaya C'-algebra, we have that $V_{\Lambda}(C')=V_{\Lambda}(\Delta)\,\Delta=\Gamma\Delta(\cong\Gamma\otimes_{C'}\Delta)$. Hence $\Lambda^{\mathfrak{K}}=\Gamma\Delta$. (ii)$\Rightarrow$(i). Clear. (ii)$\Leftrightarrow$(iii). Put $\Lambda^{\mathfrak{K}}=\bar{\Lambda}$. Note that $V_{\bar{\Lambda}}(\Gamma)=\Lambda^{\mathfrak{K}}\cap V_{\Lambda}(\Gamma)=\Delta\cap\Lambda^{\mathfrak{K}}=\Delta$, $V_{\bar{\Lambda}}(\bar{\Lambda})=\bar{\Lambda}\cap V_{\Lambda}(\bar{\Lambda})=\bar{\Lambda}\cap D=D$. Hence Δ and C' are D-finitely generated projective by Definition 1 (b). By (2), we have $\Sigma^{\oplus}_{\bar{\sigma}\in\bar{\mathfrak{G}}}\bar{\Lambda}U_{\bar{\sigma}}\cong\mathrm{Hom}\,(\bar{\Lambda}_{\Gamma},\bar{\Lambda}_{\Gamma})$, and $\Sigma^{\oplus}_{\bar{\sigma}\in\bar{\mathfrak{G}}}\Delta U_{\bar{\sigma}}=(\Sigma^{\oplus}_{\bar{\sigma}\in\bar{\mathfrak{G}}}\bar{\Lambda}U_{\bar{\sigma}})_{\Gamma}\cong\mathrm{Hom}\,({}_{\Gamma}\bar{\Lambda}_{\Gamma},{}_{\Gamma}\bar{\Lambda}_{\Gamma})=\Delta\otimes_{D}\Delta^{0}$ (see (1. 5) [7]). Hence we have the following commutative diagram

$$\begin{array}{ccc}
\sum\limits^{\oplus}_{\bar{\sigma}\in\bar{\mathfrak{G}}}\Delta U_{\bar{\sigma}} & \longrightarrow & \mathrm{Hom}\,({}_{\Gamma}\bar{\Lambda}_{\Gamma},{}_{\Gamma}\bar{\Lambda}_{\Gamma})\cong\Delta\otimes_{D}\Delta^{0} \\
\| & & \downarrow \\
\sum\limits^{\oplus}_{\bar{\sigma}\in\bar{\mathfrak{G}}}\bar{\Delta}U_{\bar{\sigma}} & \xrightarrow{\;j\;} & \mathrm{Hom}\,(\Delta_{C'},\Delta_{C'})\cong\Delta\otimes_{C'}\Delta^{0}
\end{array}$$

Therefore, j is an isomorpgism if and only if $\Delta\otimes_{D}\Delta\cong\Delta\otimes_{C'}\Delta$, consequently, if and only if $D=C'$, because C' is D-finitely generated projective, and $C'<\oplus\Delta$.

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Hokkaido University