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Isometric imbedding of a compact orientable flat Riemannian manifold into Euclidean space

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1. Introduction. The well-known integral formula of Minkowski was generalized by C. C. Hsiung to integral formulas of a closed hypersurface in an $(n+1)$-dimensional Euclidean space [2]. Y. Katsurada and H. Kojyô obtained integral formulas of a more general case, namely of a closed submanifold in a Riemannian manifold [4]. These integral formulas are used to characterize some closed hypersurfaces or submanifolds. For example we have the following theorem [8]

THEOREM A. *Let M be a closed convex hypersurface of a Euclidean space E^{n+1} and M_l the l-th mean curvature. If M_l is constant for an integer l, $1\leqq l\leqq n-1$, then M is totally umbilical, hence a sphere.*

Theorems which take the place of the above theorem in the case of a closed hypersurface and a closed submanifold in a Riemannian manifold were obtained by M. Tani [7] and Y. Katsurada [3] respectively.

In [8] Newton's formulas and some related formulas are used. These formulas were first used by M. Konishi (her name was M. Tani at that time) in [7]. In the present paper we derive them by a simpler method and then use Yano's method in [8] after a slight modification to get some integral formulas for a closed submanifold and finally to get a theorem similar in some respect to Theorem A. In our theorem, however, the Riemannian connection induced on the submanifold is assumed to be flat and our aim is to get a necessary and sufficient condition of the submanifold to lie on some hypersphere of the ambient Euclidean space.

In § 2 we prove Newton's formulas and their derived ones by using a generating function and get some formulas which may be useful in various cases. In § 3 we prove the main theorem. In the present paper we always use the following technique. When H is an n-matrix valued function on M with only real eigenvalues, we define $H(c)=H+cE$ where E is the unit n-matrix and c is a constant such that all eigenvalues of $H(c)$ are positive on M. In § 4 this technique is applied to a closed hypersurface so that we can take off the convexity condition in Theorem A. This section does not fit the title as the Riemannian connection of the hypersurface considered is not flat.

2. *Preliminaries.* Let M be an n-dimensional compact orientable Riemannian manifold and H a (1, 1)-tensor field on M satisfying

$$\nabla_k h_j^i = \nabla_j h_k^i . \tag{2.1}$$

Here and in the sequel Latin indices $i, j, k, \cdots$ run the range $1, \cdots, n$, h_j^i are the local components of H and ∇ denotes Riemannian connection. It will be convenient to consider H as a matrix valued function on M where the (i, j)-element is h_j^i. We define $H(c)$ by

$$H(c) = H + cE \tag{2.2}$$

where E is the unit n-matrix and denote its eigenvalues by $\lambda_1(c), \cdots, \lambda_n(c)$. If we put

$$\det(\lambda E + H(c)) = \lambda^n + \cdots + s_l(c)\lambda^{n-l} + \cdots + s_n(c) , \tag{2.3}$$

then $s_l(c)$ is the l-th fundamental symmetric function of $\lambda_1(c), \cdots, \lambda_n(c)$. From $s_l(c)$ we define $M_l(c)$ by

$$s_l(c) = \binom{n}{l} M_l(c) . \tag{2.4}$$

As is well-known we have the identity

$$\log(1+\lambda_i u) = \lambda_i u - \frac{1}{2}(\lambda_i u)^2 + \frac{1}{3}(\lambda_i u)^3 - \cdots$$

from which we can easily deduce

$$\sum_{l=0}^{\infty} s_l(c) u^l = \exp\left[\sum_{m=1}^{\infty} (-1)^{m+1} \frac{1}{m} p_m(c) u^m\right] \tag{2.5}$$

where

$$p_m(c) = \sum_{i=1}^{n} (\lambda_i(c))^m ,$$

$s_0(c) = 1$ and $s_k(c) = 0$ if $k > n$.

Differentiating both members of (2.5) with respect to u and then multiplying by u we get

$$\sum_{l=1}^{\infty} l s_l(c) u^l = \sum_{k=0}^{\infty} s_k(c) u^k \sum_{m=1}^{\infty} (-1)^{m+1} p_m(c) u^m .$$

As u is an arbitrary variable, this gives Newton's formulas,

(2. 6) $$l s_l(c) - s_{l-1}(c) p_1(c) + \cdots + (-1)^{l-1} s_1(c) p_{l-1}(c) + (-1)^l p_l(c) = 0 .$$

Suppose H depends on a variable t differentiably. Then we get from (2. 5)

$$\sum_{l=0}^{\infty} s'_s(c) u^l + \sum_{k=0}^{\infty} s_k(c) u^k \sum_{m=1}^{\infty} (-1)^m \frac{1}{m} p'_m(c) u^m = 0$$

hence

(2. 7) $$s'_l(c) - s_{l-1}(c)\, p'_1(c) + \frac{1}{2} s_{l-2}(c)\, p'_2(c) + \cdots + (-1)^{l-1} \frac{1}{l-1} s_1(c)\, p'_{l-1}(c) + (-1)^l \frac{1}{l} p'_l(c) = 0$$

where f' stands for $\partial f / \partial t$. We can also consider the gradient of each function and get

(2. 8) $$\nabla s_l(c) - s_{l-1}(c) \nabla p_1(c) + \frac{1}{2} s_{l-2}(c) \nabla p_2(c) + \cdots + (-1)^{l-1} \frac{1}{l-1} s_1(c) \nabla p_{l-1}(c) + (-1)^l \frac{1}{l} \nabla p_l(c) = 0 .$$

Now we define the matrix valued function $B_l(c)$ by

(2. 9) $$B_l(c) = \sum_{m=0}^{l} (-1)^m s_m(c)\, H(c)^{l-m} .$$

If we take any vector field v on M and if a (1, 1)-tensor field V is defined as the covariant derivative of v, namely $\nabla_u v = Vu$ for any vector field u, then we have

(2. 10) $$\operatorname{div}(B_l(c)\, v) = \operatorname{trace}(B_l(c) V) ,$$

where, at each point of M, V is considered as a matrix whose (i, j)-element is the local component $\nabla_j v^i$ of V for any i and j.

In order to prove (2. 10) we just observe

$$\nabla_k h_j{}^i(c) = \nabla_j h_k{}^i(c)$$

where $h_j{}^i(c) = h_j{}^i + c\delta^i_j$. Hence we can thoroughly follow the method in [8] and get

$$\begin{aligned} &\operatorname{div}\left(H(c)^l v\right) \\ &= \nabla_i \left(h_{i_1}{}^i(c) h_{i_2}{}^{i_1}(c) \cdots\cdots h_{i_l}{}^{i_{l-1}}(c)\, v^{i_l}\right) \\ &= \operatorname{trace}\left(H(c)^l V\right) + \sum_{m=1}^{l} \frac{1}{m} \left(\nabla_i p_m(c)\right) \left(H(c)^{l-m} v\right)^i , \end{aligned}$$

$$\begin{aligned}
&\operatorname{div}\left(B_l(c)\,v\right)\\
&\quad=\sum_{k=0}^{l}(-1)^k \operatorname{div}\left(s_k(c)\,H(c)^{l-k}v\right)\\
&\quad=\operatorname{trace}\left(B_l(c)\,V\right)\\
&\qquad+\sum_{k=0}^{l}(-1)^k\left(\nabla_i s_k(c)\right)\left(H(c)^{l-k}v\right)^i\\
&\qquad+\sum_{k=0}^{l}(-1)^k s_k(c)\sum_{m=1}^{l-k}\frac{1}{m}\left(\nabla_i p_m(c)\right)\left(H(c)^{l-k-m}v\right)^i .
\end{aligned}$$

As the two sums in the last member cancel each other because of (2. 8), we get (2. 10). Thus we have proved the following lemma.

LEMMA 2. 1. *If an n-matrix valued function H on M satisfies* (2. 1), *then the n-matrix valued function $B_l(c)$ defined by* (2. 2), (2. 3) *and* (2. 9) *satisfies* (2. 10) *for any vector field v on M.*

Now let us assume that M admits a non-trivial vector field v satisfying

(2. 11) $$\nabla_k\nabla_j v^i=\nabla_j\nabla_k v^i$$

and H is given by

(2. 12) $$h_j{}^i=\nabla_j v^i .$$

Then we have $V=H$, hence $V=H(c)-cE$, and in view of (2. 10)

$$\operatorname{div}\left(B_l(c)\,v\right)=\operatorname{trace}\left(B_l(c)H(c)\right)-c\operatorname{trace}\left(B_l(c)\right).$$

As we have

$$\begin{aligned}
\operatorname{trace}\left(B_l(c)\right)&=\sum_{m=0}^{l}(-1)^m s_m(c)\operatorname{trace}\left(H(c)^{l-m}\right)\\
&=\sum_{m=0}^{l}(-1)^m s_m(c)\,p_{l-m}(c),\\
\operatorname{trace}\left(B_l(c)\,H(c)\right)&=\sum_{m=0}^{l}(-1)^m s_m(c)\,p_{l+1-m}(c),
\end{aligned}$$

we get in view of (2. 6)

$$\operatorname{trace}\left(B_l(c)\right)=(-1)^l\,(n-l)s_l(c),$$

$$\operatorname{trace}\left(B_l(c)\,H(c)\right)=(-1)^l\,(l+1)s_{l+1}(c),$$

hence

(2. 13) $$\operatorname{div}\left(B_l(c)\,v\right)=(-1)^l\left\{(l+1)s_{l+1}(c)-c(n-l)s_l(c)\right\}.$$

Substituting (2. 4) into the second member and integrating over M, we get the following integral formulas

$$\int_M \left(M_{l+1}(c) - cM_l(c) \right) *1 = 0, \quad l = 0, 1, \cdots, n-1 . \tag{2.14}$$

If M admits a non-trivial vector field u satisfying

$$\nabla_k \nabla_j u^i = \nabla_j \nabla_k u^i , \tag{2.15}$$

we can take H where the local components are given by

$$h_j^i = \delta_j^i - \nabla_j u^i . \tag{2.16}$$

Then we have div $(B_l(c)u)$=trace$(B_l(c)U)$ where $U=(1+c)E-H(c)$, hence

$$\operatorname{div}\left(B_l(c)u\right) = (1+c)\operatorname{trace}\left(B_l(c)\right) - \operatorname{trace}\left(B_l(c)H(c)\right).$$

This gives the integral formulas

$$\int_M \left(M_{l+1}(c) - (1+c) M_l(c) \right) *1 = 0 , \quad l = 0, 1, \cdots, n-1 . \tag{2.17}$$

Thus we have proved the following lemma.

LEMMA 2. 2. *Let us assume that a non-trivial vector field v on a compact orientable Riemannian manifold M of dimension n satisfies* (2. 11) *and that a* (1, 1)*-tensor field* (*namely an n-matrix valued function*) $H(c)$ *is defined by*

$$h_j^i(c) = \nabla_j v^i + c\, \delta_j^i .$$

If $M_0(c)$, $M_1(c)$, $\cdots$, $M_n(c)$ are defined by (2. 3) *and* (2. 4), *then we have the integral formulas* (2. 14). *If u is a non-trivial vector field on M satisfying* (2. 15), $H(c)$ *is defined by*

$$h_j^i(c) = (1+c)\, \delta_j^i - \nabla_j u^i ,$$

then we have the integral formulas (2. 17).

The integral formulas (2. 14) and (2. 17) are essentially the same.

3. Main theorem. Let M be an n-dimensional submanifold in a Euclidean $(n+p)$-space E^{n+p} isometric to a compact orientable flat Riemannian manifold. We consider only the case $n \geqq 2$, $p \geqq 2$.

A fixed point O is taken in E^{n+p} which serves as the initial point of any position vector X. The scalar product of vectors A and B of E^{n+p} is denoted by $A \cdot B$. Any point of M is denoted by a position vector X and at the same time by local coordinates $x^1, \cdots, x^n$ in a relevant coordinate neighborhood. Thus X is a function of $x^1, \cdots, x^n$ there. The Riemannian metric on M is locally given by the components $g_{ji}=(\partial_j X)\cdot(\partial_i X)$ where $\partial_i X=\partial X/\partial x^i$ and the Riemannian connection is denoted by ∇.

Let us define a vector field u on M by the local components

$$u^i = g^{ik}(X\cdot\partial_k X)\,.$$

H_{ji} defined by

$$H_{ji} = \nabla_j \partial_i X \qquad i, j = 1, \cdots, n$$

compose the second fundamental tensor of M and we get

$$\nabla_j u^i = \delta_j^i + (X\cdot H_{jk})\, g^{ki}\,.$$

As the Riemannian connection of M is flat, u satisfies (2. 15) and the matrix valued function $H=[h_j{}^i]$ where the (i, j)-component is given by

$$h_j{}^i = -(X\cdot H_{jk})\, g^{ki}$$

satisfies (2. 1). Hence, if $H(c)=[h_j{}^i(c)]$ is defined by

$$h_j{}^i(c) = -(X\cdot H_{jk})\, g^{ki} + c\, \delta_j^i \tag{3. 1}$$

and $M_0(c)$, $M_1(c), \cdots, M_n(c)$ by (2. 3) and (2. 4), the latter satisfy the integral formulas (2. 17).

Now we can write (2. 17) in the form

$$\int_M M_l(c)*1 = (1+c)^l \int_M *1\,.$$

If $M_l(c)$ is constant for some l, we have

$$\begin{aligned} M_l(c)\int_M M_1(c)*1 &= (1+c)\, M_l(c)\int_M *1 = (1+c)\int_M M_l(c)*1 \\ &= (1+c)^{l+1}\int_M *1 = \int_M M_{l+1}(c)*1\,, \end{aligned}$$

hence

$$\int_M \left\{M_l(c)\, M_1(c) - M_{l+1}(c)\right\}*1 = 0\,. \tag{3. 2}$$

On the other hand, if c is such that the eigenvalues of $H(c)$ are all positive at every point of M, we have

$$M_l(c) M_1(c) - M_{l+1}(c) \geqq 0\,.$$

Hence we have in this case

$$M_l(c) M_1(c) - M_{l+1}(c) = 0\,.$$

This proves that the eigenvalues $\lambda_i(c)$ of $H(c)$ satisfy $\lambda_1(c)=\lambda_2(c)=\cdots=\lambda_n(c)$ as it is already known [8] and consequently H has also the same property. Thus we have proved the following lemma.

LEMMA 3.1. *If, for some real number c such that $H(c)$ has only positive eigenvalues at every point of M, one of $M_1(c), \cdots, M_{l-1}(c)$ is constant on M, then the eigenvalues λ_i of H satisfy $\lambda_1=\lambda_2=\cdots=\lambda_n$ at each point of M.*

If the condition of Lemma 3.1 is satisfied, we have

$$X\cdot H_{ji}=\varphi g_{ji} \tag{3.3}$$

where φ is a constant because of (2.1). In view of the identity

$$\nabla_j(X\cdot\partial_i X)=g_{ji}+X\cdot H_{ji}$$

we get from (3.3)

$$\nabla^j(X\cdot\partial_i X)=(1+\varphi)g_{ji}$$

and

$$\frac{1}{2}\nabla_i(||X||^2)=n(1+\varphi)\,.$$

As M is compact and orientable we get $1+\varphi=0$, hence $||X||$ is a constant, namely, M lies on some hypersphere of E^{n+p} whose center is O.

In order to express $M_l(c)$ by $M_1(0), \cdots, M_l(0)$, namely, the scalars belonging to H itself, we expand both members of the identity

$$\det(\lambda E+H)=\det\Big((\lambda-c)E+H(c)\Big)$$

into polynomials of λ and $(\lambda-c)$ respectively. Then we get

$$M_l(c)=\sum_{m=0}^{i}\binom{l}{m}M_m(0)c^{l-m}\,. \tag{3.4}$$

Thus we can state the following main theorem.

THEOREM 3.2. *Let M be an n-dimensional closed submanifold of E^{n+p}, $n\geqq 2$, $p\geqq 2$, isometric to a compact orientable flat Riemannian manifold. Let X be the position vector of the point of M, g_{ji} the Riemannian metric and ∇ the Riemannian connection. From the matrix valued function H on M defined by $H=[h_j{}^i]$ where*

$$h_j{}^i=-X\cdot\nabla_j\nabla^i X$$

we define M_l by

$$\det(\lambda E+H)=\sum_{l=0}^{n}\binom{n}{l}M_l\lambda^{n-l}\,.$$

Then a necessary and sufficient condition for the submanifold M to lie on a hypersphere of E^{n+p} centered at the origin O is that there exist a constant c and an integer l such that $H(c)=H+cE$ has only positive eigenvalues at every point of M, $1\leqq l\leqq n-1$, and

$$\sum_{m=0}^{l}\binom{l}{m}M_m\, c^{l-m}$$

is constant on M.

That the condition is necessary is almost obvious. The essential part of this theorem is that the condition is sufficient. This theorem reminds us of a paper [3] of Y. Katsurada, but in our theorem the submanifold has flat Riemannian connection. The field X of position vector in E^{n+p} is a vector field of homothetic transformation of E^{n+p}. When restricted on the submanifold M we can put $X=\alpha N_{(X)}+B_{(X)}$ where $N_{(X)}$ is a field of unit normal vector to M and $B_{(X)}$ is the tangential component of X. In [3] $N_{(X)}\cdot H_{ji}$ plays the central role but in our study $X\cdot H_{ji}$, namely, $\alpha N_{(X)}\cdot H_{ji}$, $\alpha=N_{(X)}\cdot X$, plays the central role.

4. A sufficient condition of a closed hypersurface of E^{n+1} to be totally umbilical. We treat in this section a closed hypersurface where naturally the Riemannian connection is not flat. The reason that such a section is added in this paper is that we use the technique used in § 2 and § 3. Besides using this technique the method in [8] especially pages 82∼86 is imitated. In the last part, however, some other means are also taken. After all our aim is to find a more lenient condition as a necessary and sufficient condition.

Let h_{ji} be the local components of the second fundamental tensor of a closed hypersurface M in E^{n+1} where the inner normal is taken and H be a matrix valued function on M, $H=[h_j{}^i]$, $h_j{}^i=h_{jk}g^{ki}$. If k is the least value of the principal curvature of M, $H(c)=H+cE$ has only positive eigenvalues on M for c satisfying $c+k>0$.

Let $M_1(c), \cdots, M_n(c)$ be the scalars obtained by (2. 3), (2. 4) from $H(c)$ and $M_1, \cdots, M_n$ be the corresponding ones from H. Then we have (3. 4) again. $M_1, \cdots, M_n$ satisfy the integral formulas [8]

$$\text{(4. 1)} \qquad \int_M (M_l+\alpha M_{l+1})*1=0 \qquad l=0, 1, \cdots, n-1$$

from which we get in view of (3. 4)

$$\text{(4. 2)} \qquad \int_M \left\{(1-c\alpha)\, M_l(c)+\alpha\, M_{l+1}(c)\right\}*1=0 \qquad l=0, 1, \cdots, n-1\,.$$

If we have $M_l(c)=$constant for some integer, $1\leqq l\leqq n-1$, then we can deduce from (4. 2)

$$\int_M \alpha\left\{M_l(c)\,M_1(c)-M_{l+1}(c)\right\}*1=0\,. \tag{4. 3}$$

If moreover $c+k>0$, then we have $M_l(c)M_1(c)\geqq M_{l+1}(c)$. Let us assume $\alpha>0$. Then we get $M_l(c)M_1(c)=M_{l+1}(c)$ from (4. 3) and the eigenvalues $\lambda_i(c)$ of $H(c)$ satisfy $\lambda_1(c)=\lambda_2(c)=\cdots=\lambda_n(c)$. In such a way we can conclude that M is totally umbilical.

Thus we can state the following theorem.

THEOREM 4. 1. *Let M be a closed hypersurface of E^{n+1} such that at no point P of M the position vector lies in the tangent hyperplane at P, and let k be the least value of the principal curvature of M. If there exist a number c and an integer l, $1\leqq l\leqq n-1$, such that $c+k>0$ and*

$$\sum_{m=0}^{l}\binom{l}{m}M_m\,c^{l-m}=\text{constant} \tag{4. 4}$$

on M, then M is totally umbilical, hence a sphere.

If $l=1$ we have the well-known sufficient condition $M_1=$constant [1] [5] [6].

If M is convex we can replace k by 0. In this case we can prove the following theorem.

THEOREM 4. 2. *If a closed convex hypersurface M of E^{n+1} satisfies*

$$a_1M_1+a_2M_2+\cdots+a_{n-1}M_{n-1}=\text{constant}$$

for some set of non negative constants $a_1,\cdots,a_{n-1}$ where $a_1+\cdots+a_{n-1}$ is positive, then M is totally umbilical.

PROOF. Using the integral formulas in [8], page 86, we get from

$$\int_M\sum_{l=1}^{n-1}a_lM_l(1+\alpha M_1)*1=\sum_{l=1}^{n-1}a_lM_l\int_M(1+\alpha M_1)*1=0$$

and

$$\int_M\sum_{l=1}^{n-1}a_l(M_l+\alpha M_{l+1})*1=0\,,$$

the following formula,

$$\int_M\alpha\sum_{l=1}^{n-1}a_l(M_1M_l-M_{l+1})*1=0\,.$$

Since M is convex we can choose the origin of the position vector in such a way that α has definite sign and moreover $M_1, \cdots, M_n$ satisfy $M_1M_l-M_{l+1}\geqq 0$. As $a_1, \cdots, a_{n-1}$ are non negative we get $M_1M_l-M_{l+1}=0$ if $a_l>0$. As such an integer l exists, M is totally umbilical.

Similarly, we can loosen also the sufficient condition of Theorem 4.1. A similar modification is also possible in Lemma 3.1 and Theorem 3.2.

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