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New generating functions for certain polynomial systems associated with the H-functions

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Abstract

This paper presents a number of new generating functions for several interesting classes of polynomials whose coefficients involve the H-function of C. Fox [*Trans. Amer. Math. Soc.* **98** (1961), 395-429]. The first of these main results stems from an attempt to provide a generalization of certain bilateral generating functions, due to H. M. Srivastava and R. Panda [*J. Reine Angew. Math.* **283/284** (1976), 265-274] and R. K. Raina [*Proc. Nat. Acad. Sci. India Sect. A* **46** (1976), 300-304], for a general class of hypergeometric polynomials or polynomials with essentially arbitrary coefficients; the other results are natural further generalizations of (or motivated by) the first one.

It is observed how readily these new generating functions can be extended to hold for the H-function of several complex variables, which was defined in the aforementioned paper by Srivastava and Panda.

1. Introduction

In an earlier paper H. M. Srivastava and R. Panda [6] gave a bilateral generating function, involving the H-function of C. Fox [1], for a general class of hypergeometric polynomials and also considered its multivariable analogue associated with the H-function of several complex variables. If, for convenience, we let (a_p), $\{(a_p, \alpha_p)\}$ and $\{(a_p; \alpha_p, \alpha'_p)\}$ abbreviate the p-parameter arrays

$$a_1, \cdots, a_p; \ (a_1, \alpha_1), \cdots, (a_p, \alpha_p);$$

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and

$$(a_1\,;\ \alpha_1, \alpha_1'), \cdots, (a_p\,;\ \alpha_p, \alpha_p')\,,$$

respectively, with similar interpretations for

$$(b_q),\ \{(b_q, \beta_q)\},\ \{(b_q\,;\ \beta_q, \beta_q')\}\,,$$

et cetera, we may recall the aforementioned result of Srivastava and Panda [6] in the following (essentially equivalent) form (*cf.* [6], p. 267, Eq. (2. 1)):

$$\text{(1. 1)} \qquad \sum_{n=0}^{\infty} {}_{m+p}F_q\left[\begin{matrix}\Delta(m, -n), (a_p)\,; \\ \\ (b_q)\,;\end{matrix}\ x\right] H_{u+1,v}^{r,s+1}\left[y\left|\begin{matrix}(1-\lambda-n, \varepsilon),\ \{(c_u, \gamma_u)\} \\ \{(d_v, \delta_v)\}\end{matrix}\right.\right]\frac{t^n}{n!}$$

$$= (1-t)^{-\lambda}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)} H_{1,0:\ [p,q+1];\ [u,v]}^{0,1:\ (1,p);\ (r,s)}$$

$$\left[\begin{matrix}-x\left(\dfrac{t}{m(t-1)}\right)^m \\ \\ \dfrac{y}{(1-t)^\varepsilon}\end{matrix}\left|\begin{matrix}(1-\lambda;\ m, \varepsilon):\qquad \{(1-a_p, 1)\};\ \{(c_u, \gamma_u)\} \\ \\ \text{———}:(0, 1), \{(1-b_q, 1)\};\ \{(d_v, \delta_v)\}\end{matrix}\right.\right],$$

where m, p, q, r, s, u, v are integers such that $m \geqq 1$, $p \geqq 0$, $q \geqq 0$, $0 \leqq r \leqq v$, and $0 \leqq s \leqq u$, $\varepsilon > 0$, $\Delta(m, -n)$ abbreviates the m-parameter array $(-n+j-1)/m$, $j=1, \cdots, m$,

$$\text{(1. 2)} \qquad H_{p,q}^{m,n}\left[z\left|\begin{matrix}\{(a_p, \alpha_p)\} \\ \{(b_q, \beta_q)\}\end{matrix}\right.\right] = \frac{1}{2\pi i}\int_{\mathscr{L}} \theta(\xi)\, z^\xi\, d\xi\,,$$

with, of course,

$$\text{(1. 3)} \qquad \theta(\xi) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\xi)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)}\,,$$

is the familiar H-function of Fox ([1], p. 408; see also [6], p. 265, Eq. (1. 1) *et seq.*), and the function on the right-hand side of (1. 1) is an H-function of two variables (*cf.*, *e.g.*, [6], p. 266, Eq. (1. 5)). For a natural further extension of (1. 1) to hold for a multivariable H-function defined by Srivastava and Panda ([6], p. 271, Eq. (4. 1) *et seq.*), see [6], p. 273, Eq. (4. 10).

Now, for Wright's generalized hypergeometric function defined by

$$(1.4)\qquad {}_pF_q^*\left[\begin{matrix}\{(a_p,\alpha_p)\}\,;\\ \\ \{(b_q,\beta_q)\}\,;\end{matrix}\;z\right]=\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}(a_j)_{n\alpha_j}}{\prod_{j=1}^{q}(b_j)_{n\beta_j}}\,\frac{z^n}{n!},$$

where, as usual,

$$(1.5)\qquad (\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1, & n=0,\\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \forall n\in\{1,2,3,\cdots\},\end{cases}$$

$\alpha_j>0$ $(j=1,\cdots,p)$, $\beta_j>0$ $(j=1,\cdots,q)$, and *either*

$$(1.6)\qquad \Lambda\equiv 1+\sum_{j=1}^{q}\beta_j-\sum_{j=1}^{p}\alpha_j>0\,,\qquad |z|<\infty\,,$$

or

$$(1.7)\qquad \Lambda=0\quad\text{and}\quad |z|<R\equiv\prod_{j=1}^{p}\alpha_j^{-\alpha_j}\prod_{j=1}^{q}\beta_j^{\beta_j}\,,$$

it is easily verified that

$$(1.8)\qquad \sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\,{}_{p+1}F_q^*\left[\begin{matrix}(-n,m),\ \{(a_p,\alpha_p)\}\,;\\ \\ \{(b_q,\beta_q)\}\,;\end{matrix}\;x\right]t^n$$
$$=(1-t)^{-\lambda}\,{}_{p+1}F_q^*\left[\begin{matrix}(\lambda,m),\ \{(a_p,\alpha_p)\}\,;\\ \\ \{(b_q,\beta_q)\}\,;\end{matrix}\;x\left(\frac{t}{t-1}\right)^m\right],\qquad |t|<1\,,$$

m being a positive integer.

Making use of (1.8), instead of its well-known special case (*cf.*, *e.g.*, [6], p. 268, Eq. (2.4)) when $\alpha_j=1$ $(j=1,\cdots,p)$ and $\beta_j=1$ $(j=1,\cdots,q)$, we can readily find an obvious generalization of (1.1) in the form:

$$(1.9)\qquad \sum_{n=0}^{\infty}{}_{p+1}F_q^*\left[\begin{matrix}(-n,m),\ \{(a_p,\alpha_p)\}\,;\\ \\ \{(b_q,\beta_q)\}\,;\end{matrix}\;x\right]H_{u+1,v}^{r,s+1}\left[y\,\middle|\,\begin{matrix}(1-\lambda-n,\varepsilon),\ \{(c_u,\gamma_u)\}\\ \{(d_v,\delta_v)\}\end{matrix}\right]\frac{t^n}{n!}$$
$$=(1-t)^{-\lambda}\frac{\prod_{j=1}^{q}\Gamma(b_j)}{\prod_{j=1}^{p}\Gamma(a_j)}H_{1,0:\ [p,q+1];\ [u,v]}^{0,1:\ (1,p);\ (r,s)}$$
$$\left[\begin{matrix}-x\left(\dfrac{t}{t-1}\right)^m\\ \dfrac{y}{(1-t)^{\varepsilon}}\end{matrix}\,\middle|\,\begin{matrix}(1-\lambda;\ m,\varepsilon): & \{(1-a_p,\alpha_p)\}\,;\ \{(c_u,\gamma_u)\}\\ \text{———}: & (0,1),\ \{(1-b_q,\beta_q)\}\,;\ \{(d_v,\delta_v)\}\end{matrix}\right].$$

A natural unification of the bilateral generating functions (1. 1) and (1. 9) is due to R. K. Raina [3] who gave the result [*op. cit.*, p. 301, Eq. (2. 1)]

$$(1.10)\qquad \sum_{n=0}^{\infty} \sigma_n^m(x)\, H_{u+1,v}^{r,s+1}\left[y\,\middle|\,\begin{matrix}(1-\lambda-n,\varepsilon),\ \{(c_u,\gamma_u)\}\\ \{(d_v,\delta_v)\}\end{matrix}\right]\frac{t^n}{n!}$$

$$=(1-t)^{-\lambda}\sum_{k=0}^{\infty}\frac{\Omega_k}{(mk)!}\left[x\left(\frac{t}{1-t}\right)^m\right]^k$$

$$\cdot H_{u+1,v}^{r,s+1}\left[\frac{y}{(1-t)^{\varepsilon}}\,\middle|\,\begin{matrix}(1-\lambda-mk,\varepsilon),\ \{(c_u,\gamma_u)\}\\ \{(d_v,\delta_v)\}\end{matrix}\right],$$

where, for convenience,

$$(1.11)\qquad \sigma_n^m(x)=\sum_{k=0}^{[n/m]}\binom{n}{mk}\Omega_k x^k,$$

the coefficients Ω_k being arbitrary. Indeed, Raina [3] also stated an *erroneous* version of (1. 1) [3, p. 302, Eq. (3. 2)] as a special case of the result (1. 10), and further derived an extension of (1. 10) involving the (Srivastava-Panda) H-function of several complex variables [*op. cit.*, p. 303, Eq. (4. 1)].

In an attempt to provide a generalization of the bilateral generating functions (1. 1), (1. 9) and (1. 10), we first introduce the following class of polynomials whose coefficients obviously involve Fox's H-function:

$$(1.12)\qquad P_{n,m}^{(\alpha,\beta)}(x,y)=\sum_{k=0}^{[n/m]}(-n)_{mk}\Omega_k x^k$$

$$\cdot H_{u+1,v+1}^{r,s+1}\left[y\,\middle|\,\begin{matrix}(-\alpha-(\beta+1)n,\varepsilon),\ \{(c_u,\gamma_u)\}\\ \{(d_v,\delta_v)\},\ (-\alpha-\beta n-mk,\varepsilon)\end{matrix}\right],$$

and let $\Phi[z]$ possess a power series expansion given by

$$(1.13)\qquad \Phi[z]=\sum_{n=0}^{\infty}\Omega_n z^n,\qquad \Omega_0\neq 0,$$

where α, β, c_j $(j=1,\cdots,u)$, and d_j $(j=1,\cdots,v)$ are arbitrary complex parameters, $\{\Omega_n\}_{n=0}^{\infty}$ is an arbitrary complex sequence, $\varepsilon>0$, $\gamma_j>0$ $(j=1,\cdots,u)$, $\delta_j>0$ $(j=1,\cdots,v)$, and (as before) m, r, s, u, v are integers such that $m\geqq 1$, $0\leqq r\leqq v$, and $0\leqq s\leqq u$. The proposed generalization of (1. 1), (1. 9) and (1. 10) may then be stated as

$$(1.14)\qquad \sum_{n=0}^{\infty}P_{n,m}^{(\alpha,\beta)}(x,y)\frac{t^n}{n!}=\frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\Phi\left[x(-\zeta)^m\right]$$

$$\cdot H_{u,v}^{r,s}\left[y(1+\zeta)^{\varepsilon}\,\middle|\,\begin{matrix}\{(c_u,\gamma_u)\}\\ \{(d_v,\delta_v)\}\end{matrix}\right],$$

where ζ is a function of t defined by

(1. 15) $\zeta=t(1+\zeta)^{\beta+1}, \qquad \zeta(0)=0.$

In order to recover the bilateral generating relation (1. 10) as a special case of (1. 14), we set $\alpha=\lambda-1$ and $\beta=0$, so that $\zeta=t/(1-t)$, and appeal to a certain elementary operational technique involving Laplace transforms, which augments a desirable numerator parameter in the H-functions occurring in (1. 12) and (1. 14). And, of course, the reducibility of (1. 10) to (1. 1) and (1. 9) demands appropriate specializations of the coefficients Ω_k, $k\geqq 0$.

2. Derivation of the generating function (1. 14)

In view of the definition (1. 2), the H-function occurring in (1. 12) may be replaced by a Mellin-Barnes contour integral, and we have

$$(2.1)\qquad P_{n,m}^{(\alpha,\beta)}(x,y)=\frac{1}{2\pi i}\int_{\mathscr{L}}\Gamma_\xi y^\xi \cdot\left\{\sum_{k=0}^{[n/m]}\frac{\Gamma(1+\alpha+(\beta+1)n+\varepsilon\xi)}{\Gamma(1+\alpha+\beta n+mk+\varepsilon\xi)}(-n)_{mk}\Omega_k x^k\right\}d\xi,$$

where, for convenience,

$$(2.2)\qquad \Gamma_\xi=\frac{\prod_{j=1}^{r}\Gamma(d_j-\delta_j\xi)\prod_{j=1}^{s}\Gamma(1-c_j+\gamma_j\xi)}{\prod_{j=r+1}^{v}\Gamma(1-d_j+\delta_j\xi)\prod_{j=s+1}^{u}\Gamma(c_j-\gamma_j\xi)},\qquad 0\leqq r\leqq v,\ 0\leqq s\leqq u,$$

an empty product being interpreted to be 1.

Substituting from (2. 1) into the left-hand side of the generating relation (1. 14), and formally inverting the order of integration and summation, we find that

$$(2.3)\qquad \Omega\equiv\sum_{n=0}^{\infty}P_{n,m}^{(\alpha,\beta)}(x,y)\frac{t^n}{n!}=\frac{1}{2\pi i}\int_{\mathscr{L}}\Gamma_\xi y^\xi \cdot\left\{\sum_{n=0}^{\infty}\binom{\alpha+\varepsilon\xi+(\beta+1)n}{n}t^n\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}\Omega_k x^k}{(1+\alpha+\varepsilon\xi+\beta n)_{mk}}\right\}d\xi,$$

where Γ_ξ is given by (2. 2).

By appealing to a result due to Srivastava (*cf.* [4], p. 232, Eq. (11))

$$(2.4)\qquad \sum_{n=0}^{\infty}\binom{\alpha+(\beta+1)n}{n}t^n\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}C_k}{(1+\alpha+\beta n)_{mk}}\frac{x^k}{k!}=\frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\sum_{k=0}^{\infty}C_k\frac{[x(-\zeta)^m]^k}{k!},$$

with α replaced by $\alpha+\varepsilon\xi$ and $C_k=k!\,\Omega_k$, $k\geqq 0$, we obtain from (2. 3)

$$(2.5)\qquad \Omega=\frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\sum_{k=0}^{\infty}\Omega_k x^k(-\zeta)^{mk} \cdot\frac{1}{2\pi i}\int_{\mathscr{L}}\Gamma_\xi y^\xi(1+\zeta)^{\varepsilon\xi}d\xi,$$

where ζ is a function of t defined by (1. 15), and Γ_ξ is given, as before, by (2. 2).

Now we interpret the infinite series and contour integral in (2. 5) by means of the definitions (1. 13) and (1. 2), respectively, and the second member of the generating relation (1. 14) follows at once.

This evidently completes our derivation of (1. 14) under the hypothesis that the series and integral involved in (2. 3) are absolutely convergent. Thus, in general, the generating relation (1. 14) holds true for such values of the various parameters and variables involved for which each of its members has a meaning. {See also the various conditions surrounding our general result (4. 6) below.}

3. A mild generalization of (1. 14)

An interesting generalization of Srivastava's formula (2. 4) was given subsequently by Srivastava and Buschman (*cf.* [5], p. 361, Theorem 2) in the form :

$$(3.1)\qquad \sum_{n=0}^{\infty}\binom{\alpha+(\beta+1)n}{n}t^n\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}(1+\alpha+(\beta+1)n)_{\omega k}}{(1+\alpha+\beta n)_{(\omega+m)k}}\Omega_k x^k =\frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\Phi\left[x(-\zeta)^m(1+\zeta)^\omega\right],$$

which evidently reduces to (2. 4) when the complex parameter $\omega=0$, and $\Omega_k=C_k/k!$, $k\geqq 0$.

In view of (3. 1), we may define the system of polynomials

$$(3.2)\qquad Q_{n,m}^{(\alpha,\beta)}(\omega;x,y)=\sum_{k=0}^{[n/m]}(-n)_{mk}\Omega_k x^k \cdot H_{u+1,v+1}^{r,s+1}\left[y\left|\begin{matrix}(-\alpha-(\beta+1)n-\omega k,\varepsilon),\ \{(c_u,\gamma_u)\}\\ \{(d_v,\delta_v)\},\ (-\alpha-\beta n-(\omega+m)k,\varepsilon)\end{matrix}\right.\right],$$

and the method of derivation of (1. 14), detailed in the preceding section, can be applied *mutatis mutandis* to obtain the following (slightly more general) result :

$$(3.3)\qquad \sum_{n=0}^{\infty} Q_{n,m}^{(\alpha,\beta)}(\omega\,;x,y)\frac{t^n}{n!}=\frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta}\Phi\left[x(-\zeta)^m(1+\zeta)^{\omega}\right]$$
$$\cdot H_{u,v}^{r,s}\left[y(1+\zeta)^{\varepsilon}\left|\begin{matrix}\{(c_u,\gamma_u)\}\\ \{(d_v,\delta_v)\}\end{matrix}\right.\right],$$

where ζ and $\Phi[z]$ are given by (1.15) and (1.13), respectively.

Since

$$(3.4)\qquad Q_{n,m}^{(\alpha,\beta)}(0\,;x,y)=P_{n,m}^{(\alpha,\beta)}(x,y),$$

as we indicated above, (3.3) would reduce to the generating relation (1.14) in its special case when $\omega=0$.

4. Further generalizations

In this section we give a general class of generating relations for the polynomials $Q_{n,m}^{(\alpha,\beta)}(\omega;\ x,y)$, which can be derived by applying Theorem 3 (p. 366) of Srivastava and Buschman [5] or as a *direct* consequence of Gould's identity [2, p. 196]

$$(4.1)\qquad \sum_{n=0}^{\infty}\frac{\gamma}{\gamma+(\beta+1)n}\binom{\alpha+(\beta+1)n}{n}t^n$$
$$=(1+\zeta)^{\alpha}\sum_{n=0}^{\infty}(-1)^n\binom{\alpha-\gamma}{n}\binom{n+\gamma/(\beta+1)}{n}^{-1}\left(\frac{\zeta}{1+\zeta}\right)^n,$$

where α, β and γ are arbitrary complex numbers, and ζ is defined by (1.15).

Our generalizations are contained in the following

THEOREM. *Corresponding to the power series given by* (1.13), *let*

$$(4.2)\qquad \Theta(n,m\,;\ \alpha,\beta,\gamma,\omega\,;\ x,y)$$
$$=\sum_{k=0}^{\infty}\frac{\gamma}{\gamma+(\beta+1)mk}\binom{n+mk+\gamma/(\beta+1)}{n}^{-1}\Omega_k x^k$$
$$\cdot H_{u+1,v+1}^{r,s+1}\left[y\left|\begin{matrix}(\gamma-\alpha-\omega k,\varepsilon),\ \{(c_u,\gamma_u)\}\\ \{(d_v,\delta_v)\},\ (\gamma-\alpha-\omega k+n,\varepsilon)\end{matrix}\right.\right],$$

where α, β, γ, ω, c_j $(j=1,\cdots,u)$ *and* d_j $(j=1,\cdots,v)$ *are arbitrary complex numbers,* $\varepsilon>0$, $\gamma_j>0$ $(j=1,\cdots,u)$ *and* $\delta_j>0$ $(j=1,\cdots,v)$, *and the integers* m, n, r, s, u, v *are constrained by* $m\geqq 1$, $n\geqq 0$, $1\leqq r\leqq v$, *and* $0\leqq s\leqq u$, *such that either*

$$(4.3)\qquad \chi\equiv\sum_{j=1}^{v}\delta_j-\sum_{j=1}^{u}\gamma_j>0\quad and\quad 0<|y|<\infty,$$

or

(4. 4) $\qquad \chi=0 \quad and \quad 0<|y|<S\equiv \prod_{j=1}^{u} \gamma_j^{-\gamma_j} \prod_{j=1}^{v} \delta_j^{\delta_j},$

it being understood, as before, that the various exceptional values of the parameters and variables involved are tacitly excluded. Also let the polynomials $Q_{n,m}^{(\alpha,\beta)}(\omega;\ x, y)$ *be defined by* (3. 2), *and suppose that*

(4. 5) $$\Psi[x, y;\ z]=\sum_{n=0}^{\infty} \Theta(n, m;\ \alpha, \beta, \gamma, \omega;\ x, y) \frac{z^n}{n!}.$$

Then

(4. 6) $$\sum_{n=0}^{\infty} \frac{\gamma}{\gamma+(\beta+1) n} Q_{n,m}^{(\alpha,\beta)}(\omega;\ x, y) \frac{t^n}{n!}$$
$$=(1+\zeta)^{\alpha} \Psi\left[x(-\zeta)^m(1+\zeta)^{\omega},\ y(1+\zeta)^{\varepsilon};\ -\zeta/(1+\zeta)\right],$$

where ζ *is given by* (1. 15).

Proof. The proof of the theorem is fairly straightforward. We start from the first member of the generating function (4. 6), replace $Q_{n,m}^{(\alpha,\beta)}(\omega;\ x, y)$ by its definition (3. 2) in terms of the Mellin-Barnes contour integral obtainable from (1. 2), change the order of summations and integration, and then make use of Gould's identity (4. 1). Interpreting the resulting expressions by means of (1. 2), (4. 2) and (4. 5), successively, we arrive at the righthand side of (4. 6).

Alternatively, as we suggested above, the theorem can be deduced as an application of a result due to Srivastava and Buschman ([5], p. 366, Eq. (27)), and we skip the details involved.

Some special or confluent cases of our theorem are worthy of mention here. For $\gamma=\alpha$, the generating relation (4. 6) would simplify considerably. On the other hand, its confluent case when $\gamma \rightarrow \infty$ corresponds formally to our generating function (3. 3). Obviously, therefore, for bounded γ, $\gamma \neq \alpha$. our theorem may be looked upon as being independent of the generating function (3. 3).

5. An associated system of polynomials

We define the system of polynomials

(5. 1) $$R_{n,m}^{(\alpha,\beta)}(\omega;\ x, y)=\sum_{k=0}^{[n/m]}(-n)_{mk} \Omega_k x^k$$
$$\cdot H_{u+1, v+1}^{r, s+1}\left[y \left| \begin{matrix} (1-\alpha-(\beta+1) n-\omega k, \varepsilon),\ \{(c_u, \gamma_u)\} \\ \{(d_v, \delta_v)\},\ (-\alpha-\beta n-(\omega+m) k, \varepsilon) \end{matrix} \right.\right],$$

where α, β and ω are arbitrary complex numbers, m is a positive integer, and the coefficients Ω_k, $k \geqq 0$, are given by (1.13). For these associated polynomials we give here a *new* class of generating functions of type (4.6) with $\gamma=\alpha$. Indeed, if we let

$$\Xi[x, y] = \sum_{k=0}^{\infty} \Omega_k x^k \cdot H_{u+1,v+1}^{r,s+1}\left[y \middle| \begin{matrix} (1-\alpha-(\beta+1)\,mk-\omega k, \varepsilon),\ \{(c_u, \gamma_u)\} \\ \{(d_v, \delta_v)\},\ (-\alpha-(\beta+1)\,mk-\omega k, \varepsilon) \end{matrix}\right], \tag{5.2}$$

and apply the identity (4.1) with $\gamma=\alpha$, we shall readily obtain the generating relation

$$\sum_{n=0}^{\infty} R_{n,m}^{(\alpha,\beta)}(\omega;\ x, y)\frac{t^n}{n!} = (1+\zeta)^{\alpha}\,\Xi\Big[x(-\zeta)^m(1+\zeta)^{\omega},\ y(1+\zeta)^{\varepsilon}\Big], \tag{5.3}$$

which incidentally is not contained in the substantially more general result (4.6) given by our theorem; ζ is defined by (1.15) and the various parameters and variables involved are constrained as in the theorem.

6. Extensions involving H-functions of several variables

Each of the generating functions given in the preceding sections admits itself of a straightforward generalization involving the multivariable H-function defined by Srivastava and Panda ([6], p. 271, Eq. (4.1) *et seq.*). Following the various conventions, notations and abbreviations explained fairly fully in Section 4 (pp. 271–272) of the aforementioned paper by Srivastava and Panda [6], we content ourselves by merely stating the multivariable extensions of the generating functions (4.6) and (5.3).

If we let

$$S_{n,m}^{(\alpha,\beta)}(\omega;\ x, y_1, \cdots, y_r) = \sum_{k=0}^{[n/m]} (-n)_{mk}\Omega_k x^k \cdot H_{A+1,C+1:\ [B',D'];\cdots;[B^{(r)},D^{(r)}]}^{0,\lambda+1:\ (\mu',\nu');\cdots;(\mu^{(r)},\nu^{(r)})}\left(\begin{matrix}[-\alpha-(\beta+1)\,n-\omega k : \varepsilon_1, \cdots, \varepsilon_r], \\ [-\alpha-\beta n-(\omega+m)\,k : \varepsilon_1, \cdots, \varepsilon_r], \end{matrix}\right. \left.\begin{matrix}[(a) : \theta', \cdots, \theta^{(r)}] :\ [(b') : \phi'];\cdots;[(b^{(r)}) : \phi^{(r)}]; \\ [(c) : \psi', \cdots, \psi^{(r)}] :\ [(d') : \delta'];\cdots;[(d^{(r)}) : \delta^{(r)}]; \end{matrix}\ y_1, \cdots, y_r\right), \tag{6.1}$$

and define

$$\Theta^*(n, m;\ \alpha, \beta, \gamma, \omega;\ x, y_1, \cdots, y_r) = \sum_{k=0}^{\infty}\frac{\gamma}{\gamma+(\beta+1)\,mk}\binom{n+mk+\gamma/(\beta+1)}{n}^{-1}\Omega_k x^k \tag{6.2}$$

$$\cdot H_{A+1,C+1:\ [B',D'];\cdots;[B^{(r)},D^{(r)}]}^{0,\lambda+1:\ (\mu',\nu');\cdots;(\mu^{(r)},\nu^{(r)})}\left(\begin{matrix}[\gamma-\alpha-\omega k:\varepsilon_1,\cdots,\varepsilon_r],\\ [\gamma-\alpha-\omega k+n:\varepsilon_1,\cdots,\varepsilon_r],\end{matrix}\right.$$
$$\left.\begin{matrix}[(a):\theta',\cdots,\theta^{(r)}]:[(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\cdots,\psi^{(r)}]:[(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ y_1,\cdots,y_r\right),$$

then the generating function (4.6) can easily be generalized to the form:

$$\sum_{n=0}^{\infty}\frac{\gamma}{\gamma+(\beta+1)n}S_{n,m}^{(\alpha,\beta)}(\omega;\ x,y_1,\cdots,y_r)\frac{t^n}{n!}\tag{6.3}$$
$$=(1+\zeta)^{\alpha}\Psi^*\Big[x(-\zeta)^m(1+\zeta)^{\omega},\ y_1(1+\zeta)^{\varepsilon_1},\cdots,y_r(1+\zeta)^{\varepsilon_r};$$
$$-\zeta/(1+\zeta)\Big],$$

where ζ is given by (1.15) and, for convenience,

$$\Psi^*[x,y_1,\cdots,y_r;\ z]\tag{6.4}$$
$$=\sum_{n=0}^{\infty}\Theta^*(n,m;\ \alpha,\beta,\gamma,\omega;\ x,y_1,\cdots,y_r)\frac{z^n}{n!}.$$

Similarly, for the associated system of polymomials defined by

$$T_{n,m}^{(\alpha,\beta)}(\omega;\ x,y_1,\cdots,y_r)=\sum_{k=0}^{[n/m]}(-n)_{mk}\Omega_k x^k\tag{6.5}$$
$$\cdot H_{A+1,C+1:\ [B',D'];\cdots;[B^{(r)},D^{(r)}]}^{0,\lambda+1:\ (\mu',\nu');\cdots;(\mu^{(r)},\nu^{(r)})}\left(\begin{matrix}[1-\alpha-(\beta+1)n-\omega k:\varepsilon_1,\cdots,\varepsilon_r],\\ [-\alpha-\beta n-(\omega+m)k:\varepsilon_1,\cdots,\varepsilon_r],\end{matrix}\right.$$
$$\left.\begin{matrix}[(a):\theta',\cdots,\theta^{(r)}]:[(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\cdots,\psi^{(r)}]:[(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ y_1,\cdots,y_r\right),$$

we have the multivariable generating function

$$\sum_{n=0}^{\infty}T_{n,m}^{(\alpha,\beta)}(\omega;\ x,y_1,\cdots,y_r)\frac{t^n}{n!}\tag{6.6}$$
$$=(1+\zeta)^{\alpha}\Xi^*\Big[x(-\zeta)^m(1+\zeta)^{\omega},\ y_1(1+\zeta)^{\varepsilon_1},\cdots,y_r(1+\zeta)^{\varepsilon_r}\Big],$$

where, for convenience,

$$\Xi^*[x,y_1,\cdots,y_r]=\sum_{k=0}^{\infty}\Omega_k x^k\tag{6.7}$$
$$\cdot H_{A+1,C+1:\ [B',D'];\cdots;[B^{(r)},D^{(r)}]}^{0,\lambda+1:\ (\mu',\nu');\cdots;(\mu^{(r)},\nu^{(r)})}\left(\begin{matrix}[1-\alpha-(\beta+1)mk-\omega k:\varepsilon_1,\cdots,\varepsilon_r],\\ [-\alpha-(\beta+1)mk-\omega k:\varepsilon_1,\cdots,\varepsilon_r],\end{matrix}\right.$$
$$\left.\begin{matrix}[(a):\theta',\cdots,\theta^{(r)}]:[(b'):\phi'];\cdots;[(b^{(r)}):\phi^{(r)}];\\ [(c):\psi',\cdots,\psi^{(r)}]:[(d'):\delta'];\cdots;[(d^{(r)}):\delta^{(r)}];\end{matrix}\ y_1,\cdots,y_r\right),$$

and ζ is given, as before, by (1. 15).

In the generating functions (6. 3) and (6. 6) we assume, among other conditions analogous to those given in the preceding sections, that the multivariable H-functions are well defined. Thus we require, for example, that

$$\text{(6. 8)} \qquad \sum_{j=1}^{A} \theta_j^{(l)} + \sum_{j=1}^{B^{(l)}} \phi_j^{(l)} - \sum_{j=1}^{C} \psi_j^{(l)} - \sum_{j=1}^{D^{(l)}} \delta_j^{(l)} \leqq 0\,, \qquad \forall l \in \{1, \cdots, r\}\,,$$

where the equality holds for suitably restricted values of the complex variables $y_1, \cdots, y_r$, the θ's, ϕ's, ψ's, δ's and ε's are positive real numbers, and the integers λ, $\mu^{(l)}$, $\nu^{(l)}$, A, $B^{(l)}$, C and $D^{(l)}$ are constrained by the inequalities $0 \leqq \lambda \leqq A$, $1 \leqq \mu^{(l)} \leqq D^{(l)}$, $C \geqq 0$, and $0 \leqq \nu^{(l)} \leqq B^{(l)}$, $\forall l \in \{1, \cdots, r\}$.

Finally, we record a confluent case of the multivariable generating function (6. 3) when $\gamma \to \infty$. In terms of the power series $\Phi\,[z]$ given by (1. 13), and with ζ defined by (1. 15), this multivariable extension of the generating relation (3. 3) may be stated in the elegant form :

$$\text{(6. 9)} \qquad \sum_{n=0}^{\infty} S_{n,m}^{(\alpha,\beta)}(\omega\,;\; x, y_1, \cdots, y_r) \frac{t^n}{n!} = \frac{(1+\zeta)^{\alpha+1}}{1-\beta\zeta} \Phi\Big[x(-\zeta)^m (1+\zeta)^{\omega}\Big]$$

$$\cdot H_{A,C:\,[B',D'];\cdots;[B^{(r)},D^{(r)}]}^{0,\lambda:\,(\mu',\nu');\cdots;(\mu^{(r)},\nu^{(r)})} \begin{pmatrix} y_1(1+\zeta)^{\varepsilon_1} \\ \vdots \\ y_r(1+\zeta)^{\varepsilon_r} \end{pmatrix},$$

which is valid under essentially the same conditions as those given above for (6. 3) and (6. 6), and where the parameters of the H-function of several complex variables are those that are displayed in the defining equation ([6], p. 271, Eq. (4. 1)) with, of course, n replaced by r.

It may be of interest to conclude by remarking that the confluent cases of the various results of this section [*viz* (6. 3), (6. 6) and (6. 9)], when $\varepsilon_j \to 0$, $\forall j \neq l$, will obviously yield generating functions in which the parameters corresponding *only* to the H-function variable y_l, $1 \leqq l \leqq r$, are affected.

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