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Mean curvatures for certain ν-planes in quaternion Kählerian manifolds

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Introduction.

Let (M, g) be an n-dimensional Riemannian manifold with metric tensor g. We denote by $K(X, Y)$ the sectional curvature for a 2-plane spanned by tangent vectors X and Y at $P\in M$, and by π a ν-plane at $P\in M$. Let $\{e_1, \cdots, e_n\}$ be an orthonormal base of the tangent space at P such that $\{e_1, \cdots, e_\nu\}$ spans π. S. Tachibana [3] defined the mean curvature $\rho(\pi)$ for π by

$$\rho(\pi)=\frac{1}{\nu(n-\nu)}\sum_{b=\nu+1}^{n}\sum_{a=1}^{\nu}K(e_a, e_b),$$

which is independent of the choice of an adapted base for π. He obtained the following

THEOREM A (S. Tachibana [3]). *In an $n(>2)$-dimensional Riemannian manifold (M, g), if the mean curvature for ν-plane is independent of the choice of ν-planes at each point, then*

(i) *for $\nu=1$ or $n-1$, (M, g) is an Einstein space,*

(ii) *for $1<\nu<n-1$ and $2\nu\neq n$, (M, g) is of constant curvature,*

(iii) *for $2\nu=n$, (M, g) is conformally flat.*

The converse is true.

Taking holomorphic 2λ-planes or antiholomorphic ν-planes, instead of ν-planes, analogous results in Kählerian manifolds are also obtained.

THEOREM B (S. Tachibana [4] and S. Tanno [5]). *In a Kählerian manifold (M, g, J) of dimension $n=2l\geqq 4$, if the mean curvature for holomorphic 2λ-plane is independent of the choice of holomorphic 2λ-planes at each point, then*

(i) *for $1\leqq\lambda\leqq l-1$ and $2\lambda\neq l$, (M, g, J) is of constant holomorphic sectional curvature,*

(ii) *for $2\lambda=l$, the Bochner curvature tensor vanishes.*

The converse is true.

THEOREM C (K. Iwasaki and N. Ogitsu [2]). *In a Kählerian manifold (M, g, J) of dimension $n=2l\geqq 4$, if the mean curvature for antiholomorphic*

ν-plane is independent of the choice of antiholomorphic ν-planes at each point, then

(i) *$\nu=1$, (M, g, J) is an Einstein space,*

(ii) *$2\leqq\nu\leqq l-1$, (M, g, J) is of constant holomorphic sectional curvature,*

(iii) *$\nu=l$, the Bochner curvature tensor vanishes.*

The converse is true.

L. Vanhecke ([6], [7]) generalized Theorems B and C.

The main purpose of this paper is to prove analogous results in quaternion Kählerian manifolds.

§ 1. Quaternion Kählerian manifolds (cf. [1]).

Let (M, V) be an almost quaternion manifold of dimension $n=4m$, that is, a manifold M which admits a 3-dimensional vector bundle V consisting of tensors of type $(1, 1)$ over M satisfying the following condition: In any coordinate neighborhood U of M, there is a local base $\{J_1, J_2, J_3\}$ of V such that

$$J_p J_q = -\delta_{pq} J_0 + \sum_{r=1}^{3} \delta_{pqr} J_r$$

for p and q in a set $\{1, 2, 3\}$, where J_0 is the identity tensor of type $(1, 1)$ on M, δ_{pq} is the Kronecker's delta and δ_{pqr} is 1 or -1 according as (p, q, r) is even or odd permutation of $(1, 2, 3)$ and 0 otherwise. And it is well known that $\Lambda=\sum_{p=1}^{3} J_p \otimes J_p$ is a tensor of type $(2, 2)$ defined globally on M.

If an almost quaternion manifold (M, V) admits the metric tensor g such that

$$(1.1)\qquad g(X, \phi Y)+g(\phi X, Y)=0\,, \qquad \nabla\Lambda=0$$

for any cross-section ϕ of V and any vectors X and Y, (M, g, V) is called a quaternion Kählerian manifold, where ∇ is the Riemanian connection induced from g. We have known that if $m\geqq 2$, (M, g, V) is an Einstein space and satisfies

$$(1.2)\qquad R(X, Y, Z, W)=R(X, Y, J_p Z, J_p W) + \frac{S}{4m(m+2)}\left\{g(X, J_q Y)\, g(J_q Z, W)+g(X, J_r Y)\, g(J_r Z, W)\right\},$$

where (p, q, r) is a permutation of $(1, 2, 3)$, R and S are the curvature tensor and the scalar curvature of (M, g, V), respectively, and we put

$$R(X, Y, Z, W)=g\big(R(X, Y)\, Z, W\big)\,.$$

Throughout this paper, we assume that $m \geqq 2$, and indices p, q, r run over the range $\{1, 2, 3\}$ unless stated otherwise.

§ 2. Lemmas.

Let $T_p(M)$ be a tangent space at a point P of (M, g, V). The sectional curvature $K(X, Y)$ for a 2-plane spanned by X, Y in $T_P(M)$ is defined by

$$K(X, Y) = -\frac{R(X, Y, X, Y)}{g(X, X)\,g(Y, Y)-(g(X, Y))^2} \tag{2.1}$$

From (1.1), (1.2) and (2.1), we have

$$K(X, J_p X) = \frac{S}{4m(m+2)} + \frac{R(X, J_p X, J_q X, J_r X)}{(g(X, X))^2} \tag{2.2}$$

for an even permutation (p, q, r) of $(1, 2, 3)$ (cf. [1]). From (2.2) and the first Bianchi identity, we have

LEMMA 1. $\sum_{p=1}^{3} K(X, J_p X) = \dfrac{3S}{4m(m+2)}$.

Similarly, we get

LEMMA 2. *For a permutation* (p, q, r) *of* $(1, 2, 3)$,

$$K(J_p X, Y) = K(X, J_p Y),\quad K(J_p X, J_p Y) = K(X, Y),$$
$$K(J_p X, J_q Y) = K(X, J_r Y).$$

Next, by $Q(X)$ we denote the 4-plane spanned by $\{X, J_1 X, J_2 X, J_3 X\}$ for $X \in T_P(M)$, and such a 4-plane is called the Q-section determined by X. Now assume that two Q-sections $Q(X)$ and $Q(Y)$ are orthogonal to each other and $g(X, X) = g(Y, Y) = 1$. Then we have

$$R(X, J_p X, J_q Y, J_r Y) = R(X, J_p X, J_p Y, Y) - \frac{S}{4m(m+2)},$$
$$R(X, J_p Y, J_q Y, J_r X) = -R(X, J_p Y, X, J_p Y),$$
$$R(X, J_p Y, J_p X, Y) = -R(X, J_p Y, X, J_p Y),$$
$$R(X, Y, J_p Y, J_p X) = -R(X, Y, X, Y)$$

for an even permutation (p, q, r) of $(1, 2, 3)$. Using these identities, we have

$$\begin{aligned} &K\big(X+Y, J_p(X+Y)\big) + K\big(X-Y, J_p(X-Y)\big) \\ &= \frac{1}{2}\Big\{K(X, J_p X) + K(Y, J_p Y) + 4K(X, J_p Y) \\ &\qquad + 2R(X, J_p X, J_p Y, Y)\Big\}, \end{aligned}$$

$$K(X+J_pY, J_pX-Y)+K(X-J_pY, J_pX+Y)$$
$$=\frac{1}{2}\Big\{K(X, J_pX)+K(Y, J_pY)+4K(X, Y)$$
$$+2R(X, J_pX, J_pY, Y)\Big\},$$

$$K\big(X+J_pY, J_q(X+J_pY)\big)+K\big(X-J_pY, J_q(X-J_pY)\big)$$
$$=\frac{1}{2}\Big\{K(X, J_qX)+K(Y, J_qY)+4K(X, J_rY)$$
$$-2R(X, J_qX, J_qY, Y)+\frac{S}{2m(m+2)}\Big\}$$

for a permutation (p, q, r) of $(1, 2, 3)$, from which, we get

LEMMA 3. *For unit vectors X and Y in $T_P(M)$ whose Q-sections are orthogonal to each other,*

$$6\sum_{p=0}^{3}K(X, J_pY)=\sum_{p=0}^{3}\sum_{q=1}^{3}\Big\{K\big(X+J_pY, J_q(X+J_pY)\big)$$
$$+K\big(X-J_pY, J_q(X-J_pY)\big)\Big\}$$
$$-2\sum_{p=1}^{3}\Big\{K(X, J_pX)+K(Y, J_pY)\Big\}-\frac{3S}{2m(m+2)}.$$

By virtue of Lemmas 1 and 3, we obtain

LEMMA 4. *For the same X and Y as in Lemma 3,*

$$\sum_{p=0}^{3}K(X, J_pY)=\frac{S}{4m(m+2)}.$$

For the same X and Y as above, we have

$$R(X, J_pX, X, J_pY)=R(X, J_pX, Y, J_pX),$$
$$R(X, J_pX, Y, J_pY)=R(X, Y, X, Y)+R(X, J_pY, X, J_pY),$$

from which, we get

LEMMA 5. *For the same X and Y as in Lemma 3,*

$$K\big(X+Y, J_p(X-Y)\big)$$
$$=\frac{1}{4}\Big\{K(X, J_pX)+K(Y, J_pY)-2K(X, Y)-2K(X, J_pY)\Big\}.$$

§ 3. Mean curvature for quaternionic 4μ-plane.

The 4μ-plane π in $T_P(M)$ is called a quaternionic 4μ-plane if $J_p\pi\subset\pi$ $(p=1, 2, 3)$. Hence we can take the orthonormal base $\{\tilde{e}_\alpha|\alpha=1, \cdots, 4m\}$ of $T_P(M)$ such that

$$\tilde{e}_{4i+p-3}=J_p e_i\,, \qquad i=1, \cdots, m\,;\; p=0, \cdots, 3$$

and $\{\tilde{e}_\alpha|\alpha=1, \cdots, 4\mu\}$ spans π. Then, the mean curvature $\rho(\pi)$ for π is following:

$$\begin{aligned}\rho(\pi)&=\frac{1}{16\mu(m-\mu)}\sum_{\beta=4\mu+1}^{4m}\sum_{\alpha=1}^{4\mu}K(\tilde{e}_\alpha, \tilde{e}_\beta)\\ &=\frac{1}{16\mu(m-\mu)}\sum_{j=\mu+1}^{m}\sum_{i=1}^{\mu}\sum_{p,q=0}^{3}K(J_p e_i, J_q e_j)\,.\end{aligned}$$

Using Lemmas 2 and 4, we have

$$\begin{aligned}\rho(\pi)&=\frac{1}{4\mu(m-\mu)}\sum_{j=\mu+1}^{m}\sum_{i=1}^{\mu}\sum_{p=0}^{3}K(e_i, J_p e_j)\\ &=\frac{S}{16m(m+2)}\,.\end{aligned}$$

Therefore we can obtain

THEOREM 1. *In a quaternion Kählerian manifold of dimension* $4m\geqq 8$, *the mean curvature for quaternionic* 4μ*-plane is always constant for* $1\leqq\mu\leqq m-1$, *and its value is equal to* $\dfrac{S}{16m(m+2)}$.

§ 4. Mean curvature for antiquaternionic ν-plane.

We now assume that the sectional curvature $K(X, Y)$ is independent of the choice of X and Y whose Q-sections are orthogonal to each other. Then, from Lemma 5, we get

$$K(X, J_pX)+K(Y, J_pY)=8k \tag{4.1}$$

where we put $k=K(X, Y)$ and $g(X, X)=g(Y, Y)=1$. Similarly, for a unit $Z\in T_P(M)$ orthogonal to $Q(X)$, we have

$$K(X, J_pX)+K(Z, J_pZ)=8k\,. \tag{4.2}$$

On the other hand, from (1.2), we have

$$\begin{aligned}R(J_qY, J_pY, J_qY, J_rY)&=-R(Y, J_pY, J_qY, J_pY)\,,\\ R(Y, J_rY, J_qY, J_rY)&=-R(Y, J_rY, Y, J_pY)\end{aligned}$$

for an even permutation (p, q, r) of $(1, 2, 3)$. Putting $Z=(Y+J_q Y)/\sqrt{2}$, from these identities and (2. 2), we have

$$K(Z, J_p Z) = K(Y, J_r Y). \tag{4. 3}$$

From (4. 1), (4. 2) and (4. 3), it follows that

$$K(Y, J_p Y) = K(Y, J_r Y).$$

Therefore we can obtain

THEOREM 2. *In a quaternion Kählerian manifold* (M, g, V) *of dimension* $4m \geqq 8$, *if the sectional curvature* $K(X, Y)$ *is independent of the choice of* X *and* Y *at each point whose Q-sections are orthogonal to each other,* (M, g, V) *is of constant Q-sectional curvature. The converse is true.*

The ν-plane π in $T_P(M)$ is called an antiquaternionic ν-plane if $J_p\pi$ $(p=1, 2, 3)$ are orthogonal to π. Hence we can take the orthonormal base $\{\tilde{e}_\alpha | \alpha=1, \cdots, 4m\}$ of $T_P(M)$ such that

$$\tilde{e}_{4i+p-3} = J_p e_i, \qquad i=1, \cdots, m; \ p=0, \cdots, 3$$

and $\{e_1, \cdots, e_\nu\}$ spans π. Then, the mean curvature $\rho(\pi)$ for π is following:

$$\begin{aligned} \rho(\pi) &= \frac{1}{\nu(4m-\nu)}\left\{\sum_{i,j=1}^{\nu}\sum_{p=1}^{3} K(e_i, J_p e_j) + \sum_{j=\nu+1}^{m}\sum_{i=1}^{\nu}\sum_{p=0}^{3} K(e_i, J_p e_j)\right\} \\ &= \frac{1}{\nu(4m-\nu)}\left\{\sum_{i=1}^{\nu}\sum_{p=1}^{3} K(e_i, J_p e_i) - \sum_{\substack{i,j=1\\ i\neq j}}^{\nu} K(e_i, e_j)\right. \\ &\quad \left. + \sum_{\substack{i,j=1\\ i\neq j}}^{\nu}\sum_{p=1}^{3} K(e_i, J_p e_j) + \sum_{j=\nu+1}^{m}\sum_{i=1}^{\nu}\sum_{p=0}^{3} K(e_i, J_p e_j)\right\}. \end{aligned} \tag{4. 4}$$

From (4. 4) and Lemmas 1, 2 and 4, we have

$$\rho(\pi) = \frac{1}{\nu(4m-\nu)}\left\{\frac{\nu}{4m} S - 2\sum_{1\leqq i<j\leqq \nu} K(e_i, e_j)\right\}. \tag{4. 5}$$

We now assume that the mean curvature for antiquaternionic ν-plane is independent of the choice of antiquaternionic ν-planes. Since a ν-plane π_1 spanned by $\{e_1, J_p e_2, e_3, \cdots, e_\nu\}$ is also antiquaternionic, we have $\rho(\pi)=\rho(\pi_1)$, from which we have

$$K(e_1, e_2) + \sum_{i=3}^{\nu} K(e_2, e_i) = K(e_1, J_p e_2) + \sum_{i=3}^{\nu} K(J_p e_2, e_i). \tag{4. 6}$$

Similarly, using antiquaternionic ν-planes spanned by $\{J_p e_1, e_2, \cdots, e_\nu\}$ and $\{J_p e_1, J_p e_2, e_3, \cdots, e_\nu\}$, we have

$$K(J_p e_1, e_2) + \sum_{i=3}^{\nu} K(e_2, e_i) = K(J_p e_1, J_p e_2) + \sum_{i=3}^{\nu} K(J_p e_2, e_i). \tag{4. 7}$$

From (4.6), (4.7) and Lemmas 2 and 4, we know that $K(e_1, e_2)$ is constant.

Let X and Y be arbitrary unit tangent vectors at $P \in M$ whose Q-sections are orthogonal to each other. Then we can take an orthonormal base $\{J_p e_i | i=1, \cdots, m ; p=0, \cdots, 3\}$ of $T_P(M)$ such that $e_1 = X$ and $e_2 = Y$.

Summing up the arguments developed above, by virtue of Theorem 2, we can obtain

THEOREM 3. *In a quaternion Kählerian manifold* (M, g, V) *of dimension* $4m \geqq 8$, *if the mean curvature for antiquaternionic ν-plane is independent of the choice of antiquaternionic ν-planes at each point for* $2 \leqq \nu \leqq m$, (M, g, V) *is of constant Q-sectional curvature. The converse is true.*

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