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Cut loci of Berger's spheres

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1. Introduction. In the present note we determine the structure of the cut loci of Berger's spheres explicitly. Let M be a compact riemannian manifold with a fixed point o. Then for every geodesic γ_X (parametrized by arc-length) from o with unit initial direction $X \in T_o M$, we define the cut point $\gamma_X(t)$ of o along γ_X as the last point on γ_X to which the geodesic arc of γ_X minimizes the distance from o. $tX \in T_o M$ will be called the tangent cut point of o along γ_X. The set of all (tangent) cut points of o is called the (tangent) cut loculs of o. The cut locus contains the essential information on the topology of M. Now the structure of cut locus is interesting in connection with the singularlity of the exponential mapping Exp: $T_o M \rightarrow M$. Recently in case of analytic riemannian structure or in generic case much progress has been made ([2], [3], [4], [14]). But since their works appeal to the powerful general theory (Hironaka's or Mather's theory), it is not clear how to apply these methods to concrete cases. On the other hand for compact symmetric spaces the structure of cut loci has been completely analyzed in terms of root system by the author and M. Takeuchi ([10], [11], [12]). Thus it is a natural problem to study the cut loci of more general homogeneous riemannian manifolds. But it seems difficult to establish a general theory which analyzes the detailed and concrete structure of cut loci in all homogeneous spaces. So we consider here some examples which seems to be the first step for the above problem. Namely we consider Berger's spheres M_α $(0<\alpha\leq\pi/2)$ —for the construction see §2— which are homogeneous spaces diffeomorphic to the three dimensional sphere. These riemannian structures may be obtained from the canonical riemannian structure $M_{\pi/2}$ of constant curvature by deforming the metric along the fibers of the Hopf fibering $S^3 \rightarrow S^2$ (see [9]) or may be realized as the distance spheres in CP^2 with Fubini-Study metric ([15]) and give nice examples in riemannian geometry ([5], [13], [16]).

Now in the present note, by determining the structure of the cut locus of a point in M_α explicitly we see that the cut locus is the 2-disc whose boundary consists of the conjugate points of order 1 and the cut locus contracts to a point as α converges to $\pi/2$.

In the case of the compact simply connected symmetric spaces the cut locus coincides with the first conjugate locus by Crittenden's theorem. But in our case they don't coincide but have the intersection. Moreover it turns out that the first conjugate points have strong influence upon the cut locus. We suspect that this holds also in the case of general simply connected homogeneous spaces.

Although we treat here three dimensional case for the sake of simplicity, the same method seems to be valid also for higher dimensional case.

In § 4 we give an explicit estimate for injectivity radii of Wallach's manifolds which are seven dimensional manifolds of positive curvature. This is a supplementary remark to a Huang's recent work ([7]).

The author wishes to express his sincere thanks to S. Tanno for the check of tedious computations.

2. Preliminaries.

2.1. In this section we review the construction of the Berger's spheres (see [1], [5]). Let $\mathfrak{g}$ be the Lie algebra of the Lie group $G:=SU(2)\times \boldsymbol{R}^1$. Then $\mathfrak{g}$ carries a bi-invariant inner product which is obtained from the Killing form $B(X, Y):=-1/2\ \mathrm{tr}\ XY$ on $\mathfrak{su}(2)$ and the canonical one on $\boldsymbol{R}$.

We set for $0<\alpha\leq\pi/2$,

$$S_1:=\begin{pmatrix}\sqrt{-1} & 0 & 0\\ 0 & -\sqrt{-1} & 0\\ 0 & 0 & 0\end{pmatrix},\quad D:=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\ (\text{generator of } \boldsymbol{R}),$$

$$\xi:=\sin\cdot S_1-\cos\alpha\, D,$$

$$e_1:=\begin{pmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},\quad e_2:=\begin{pmatrix}0 & \sqrt{-1} & 0\\ \sqrt{-1} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Note that $\{e_1, e_2, S_1, D\}$ forms an orthonormal basis of $\mathfrak{g}$. If we put $\mathfrak{h}_\alpha:=$ linear span of $\cos\alpha\, S_1+\sin\alpha\, D$, then

$$\exp\mathfrak{h}_\alpha=\left\{\begin{pmatrix}e^{l\cos\alpha\cdot\sqrt{-1}} & 0 & 0\\ 0 & e^{-l\cos\alpha\cdot\sqrt{-1}} & 0\\ 0 & 0 & l\sin\alpha\end{pmatrix};\ l\in\boldsymbol{R}\right\}$$

defines a closed subgroup H_α of G and we get the normal homogeneous riemannian manifolds $M_\alpha:=G/H_\alpha$ which are diffeomorphic to S^3 and are called Berger's spheres. Note that $M_{\pi/2}$ is the standard sphere of constant curvature 1. In the following we assume that $0<\alpha<\pi/2$. The tangent space T_oM ($o:=\pi(e)$; $\pi: G\to M_\alpha$ canonical projection) may be identified with the subspace $\mathfrak{m}_\alpha :=$ linear span of $\{\xi, e_1, e_2\}$ of $\mathfrak{g}$. We get easily

$$\mathfrak{g}=\mathfrak{h}_\alpha+\mathfrak{m}_\alpha,\ [\mathfrak{h}_\alpha, \mathfrak{m}_\alpha]\subset\mathfrak{m}_\alpha,\ [\mathfrak{h}_\alpha, \xi]=0,\ [\mathfrak{h}_\alpha, e_1]\neq 0\,.$$

2.2. From the above we see that $\mathrm{Ad}\,H_\alpha$ acts as an isometry group transitively on the unit circle in the orthogonal complement $\xi^\perp$ of ξ in $\mathfrak{m}_\alpha$. This means that the tangent cut locus and conjugate locus of o are surfaces of revolution around the axis ξ. Thus to study the tangent cut locus it suffices to determine the tangent cut point of o along the geodesic γ_θ emanating from o with the initial direction $\cos\theta\cdot\xi+\sin\theta\cdot e_2$ $(0\leq\theta\leq\pi)$. Recall that if $\gamma_\theta(t)$ is the cut point of o along γ_θ then either $\gamma_\theta(t)$ is the first conjugate point to o along γ_θ or there exists another minimizing geodesic γ' $(\neq\gamma_\theta)$ emanating from o with $\gamma_\theta(t)=\gamma'(t)$.

Now the first conjugate point of o along γ_θ has been determined by I. Chavel ([5]).

2.3. LEMMA ([5]). *Let $t_0(\theta)$ denote the first conjugate point to o along γ_θ. Then $t_0(\theta)$ is the unique solution of*

(1) $(\tan xt)/(tx)=-\cot^2\alpha\cdot\sin^2\theta,\ x=\sqrt{\cos^2\theta\sin^2\alpha+\sin^2\theta}$,
which satisfies $\pi/2<xt_0\leq\pi$.

2.4. REMARK. It is easy to see that $t_0(\theta)$ is strictly monotone decreasing on $[0, \pi/2]$ and $\left.\dfrac{dt_0}{d\theta}\right|_{\theta=0}=\left.\dfrac{dt_0}{d\theta}\right|_{\theta=\pi/2}=0,\ t_0(0)=\pi/\sin\alpha,\ t_0(\pi-\theta)=t_0(\theta)$. In fact, $\dfrac{dt_0}{d\theta}\{1/\cos^2 t_0x+\sin^2\theta\cot^2\alpha\}=-\sin 2\theta/(2x^2)\{t_0\cos^2\alpha(1/\cos^2 t_0x+\cot^2\alpha\sin^2\theta)+2t_0x^2\cot^2\alpha\}$.

Thus for our purpose it suffices to determine $t(\theta):=\mathrm{Inf}\,\{0<t\leq t_0(\theta);$ there exists another minimizing geodesic γ' $(\neq\gamma_\theta)$ with $\gamma_\theta(t)=\gamma'(t)\}$. Namely, for a given θ we have to search for the minimum value of positive $t\leq t_0(\theta)$ such that

(2) $\mathrm{Exp}\,t(\cos\theta\cdot\xi+\sin\theta\cdot e_2)=\mathrm{Exp}\,t\,\mathrm{Ad}\,h(\cos\theta_1\cdot\xi+\sin\theta_1\cdot e_2)$
holds for some $\theta_1\in[0, \pi]$, $h\in H_\alpha$ with $\cos\theta\cdot\xi+\sin\theta\cdot e_2\neq\mathrm{ad}\,h(\cos\theta_1\cdot\xi+\sin\theta_1\cdot e_2)$ and $t\leq t_0(\theta)$, $t_0(\theta_1)$. Since in our case $\mathrm{Exp}\,tX=\pi\cdot\exp tX$, where exp denotes the exponential map of Lie group, (2) is clearly equivalent to the following:

(3) $\exp\left(-t(\cos\theta\cdot\xi+\sin\theta\cdot e_2)\right)h\exp t(\cos\theta_1\cdot\xi+\sin\theta_1\cdot e_2)\in H_\alpha$.

3. Tangent cut locus and cut locus.

3.1. Since we have $\exp t(\cos\theta\cdot\xi+\sin\theta\cdot e_2)=M(t, \theta, x):=$

$$=\begin{pmatrix} \cos tx+\sqrt{-1}\,\dfrac{\sin tx}{x}\cos\theta\sin\alpha & \sqrt{-1}\,\dfrac{\sin tx}{x}\sin\theta & 0 \\ \sqrt{-1}\,\dfrac{\sin tx}{x}\sin\theta & \cos tx-\sqrt{-1}\,\dfrac{\sin tx}{x}\cos\theta\sin\alpha & 0 \\ 0 & 0 & -t\cos\theta\cdot\cos\alpha \end{pmatrix},$$

(3) is equivalent to

$$M(-t,\theta,x)\begin{pmatrix} e^{l\cos\alpha\cdot\sqrt{-1}} & 0 & 0 \\ 0 & e^{-l'\cos\alpha\cdot\sqrt{-1}} & 0 \\ 0 & 0 & l\sin\alpha \end{pmatrix}M(t,\theta_1,x_1)$$

$$=\begin{pmatrix} e^{l'\cos\alpha\cdot\sqrt{-1}} & 0 & 0 \\ 0 & e^{-l\cos\alpha\cdot\sqrt{-1}} & 0 \\ 0 & 0 & l'\sin\alpha \end{pmatrix} \quad \text{with } x_1:=\sqrt{\cos^2\theta_1\sin^2\alpha+\sin^2\theta_1} \quad (0\le\theta_1\le\pi)$$

where α, θ, x are known and t, θ_1, l, l' are unknown. In the above note that for (3, 3)-components matrix multiplication is given by addition. Again this is equivalent to the following transcendental equations:

$$(4)\quad \frac{\sin tx\sin tx_1}{xx_1}\sin(\theta_1-\theta)\ \sin\alpha\cos u=$$

$$=\left(\frac{\sin tx}{x}\cos tx_1\sin\theta+\frac{\sin tx_1}{x_1}\cos tx\sin\theta_1\right)\sin u$$

$$\frac{\sin tx\sin tx_1}{xx_1}\sin(\theta_1+\theta)\sin\alpha\sin u=$$

$$=\left(\frac{\sin tx}{x}\cos tx_1\sin\theta-\frac{\sin tx_1}{x_1}\cos tx\sin\theta_1\right)\cos u\,.$$

$$(5)\quad \cos u\cos A+\sin u\sin A=$$

$$=\left(\cos tx\cos tx_1+\frac{\sin tx\sin tx_1}{xx_1}(\cos\theta\cos\theta_1\sin^2\alpha+\sin\theta\sin\theta_1)\right)\cos u$$

$$+\left(\frac{\sin tx}{x}\cos tx_1\cos\theta-\frac{\sin tx_1}{x_1}\cos tx\cos\theta_1\right)\sin\alpha\sin u\,.$$

$$\sin u\cos A-\cos u\sin A=$$

$$=\left(\cos tx\cos tx_1+\frac{\sin tx\sin tx_1}{xx_1}(\cos\theta\cos\theta_1\sin^2\alpha-\sin\theta\sin\theta_1)\right)\sin u$$

$$-\left(\frac{\sin tx}{x}\cos tx_1\cos\theta-\frac{\sin tx_1}{x_1}\cos tx\cos\theta_1\right)\sin\alpha\cos u,$$

where we have put $u=l\cos\alpha$, $A=t(\cos\theta_1-\cos\theta)\cot\alpha\cos\alpha$ and l' is determined by $l' \cos\alpha = u - A$.

3.2. Firstly we consider the equation (4). Considering (4) as homogeneous linear equation with respect to unknowns $\sin u$ and $\cos u$, we get the *n.s.c.* of having solutions:

$$(6) \quad \frac{\sin^2\theta_1}{\sin^2\theta} = \frac{\sin^2\alpha + x_1^2\cot^2 tx_1}{\sin^2\alpha + x^2\cot^2 tx} \qquad \left(0 < t \leq t_0(\theta_1),\ t_0(\theta)\right)$$

3.3. LEMMA. (6) *implies that* $\sin\theta = \sin\theta_1$.

PROOF. Setting $v := tx$, $v_1 := tx_1$ and noting that $\sin^2\theta = (x^2 - \sin^2\alpha)/\cos^2\alpha$, we get from (6)

$$(7) \quad \frac{v_1^2 - \sin^2\alpha\cdot t^2}{\sin^2\alpha\cdot t^2 + v_1^2\cot^2 v_1} = \frac{v^2 - \sin^2\alpha\cdot t^2}{\sin^2\alpha\cdot t^2 + v^2\cot^2 v}$$

Now suppose that $\sin\theta_1 \neq \sin\theta$. Then we may assume $0 \leq \theta < \theta_1 \leq \pi/2$. For fixed $t(\leq t_0(\theta_1), t_0(\theta))$, we consider the function f of $\bar\theta \in [\theta, \theta_1]$ defined by

$$f(\bar\theta) := \frac{v^2(\bar\theta) - \sin^2\alpha\cdot t^2}{\sin^2\alpha\cdot t^2 + v^2(\bar\theta)\cot^2 v(\bar\theta)}, \quad v(\bar\theta) = tx(\bar\theta).$$

Then since $dx/d\theta(\bar\theta) = \sin 2\bar\theta\cdot\cos^2\alpha/2x(\bar\theta)$, we have

$$df/d\theta(\bar\theta) =$$

$$= \frac{t^4\sin 2\bar\theta\cdot\cos^2\alpha}{(\sin^2\alpha\cdot t^2 + v^2\cot^2 v)^2}\left\{\frac{\sin^2\alpha}{\sin^2 v} + \frac{\cot v}{\sin^2 v}\, v\sin^2\bar\theta\cos^2\alpha\right\}.$$

Now put $v_0(\bar\theta) := t_0(\bar\theta)x(\bar\theta)$. Then for $\bar\theta \in [\theta, \theta_1]$ we get $v(\bar\theta) \leq v_0(\bar\theta)$, since $t_0(\bar\theta)$ is monotone decreasing and $t \leq t_0(\theta_1)\,(< t_0(\theta))$. On the other hand the function $v \to v\cot v$ is strictly monotone decreasing for positive v and we see that $v\cot v \geq -\tan^2\alpha/\sin^2\bar\theta$. Neamely we have

$$df/d\theta(\bar\theta) \geq 0$$

and $f(\bar\theta)$ is strictly monotone increasing on $[\theta, \theta_1]$. Thus we get $f(\theta) < f(\theta_1)$ which contradicts (7). q.e.d.

3.4. LEMMA. $\theta_1 = \pi - \theta\ (\theta \neq \theta_1)$.

PROOF. Suppose that $\theta = \theta_1$. Then from (4) we get $\sin tx\cos tx\cdot\sin\theta\cdot\sin u = 0$, $\sin^2 tx\cdot\sin 2\theta\cdot\sin u = 0$, namely, $\sin u = 0$, $tx = \pi$, $\sin\theta = 0$, or $\theta = t = \pi/2$.

Case 1. $\sin u = 0$: Then $u = n\pi$, $n \in \boldsymbol{Z}$ and $\mathrm{Ad}\, h = \mathrm{Ad}\begin{pmatrix} (-1)^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & n\pi\tan\alpha \end{pmatrix}$ fixes the initial direction $\cos\theta\cdot\xi + \sin\theta\cdot e_2$ which contradicts the assumption in (2). *Case 2.* $\sin\theta=0$: In this case Ad H_α again leaves $\cos\theta\cdot\xi+\sin\theta\cdot e_2$ fixed. *Case 3.* $tx=\pi$ implies that $t_0x=\pi$ and $\sin\theta=0$. *Case 4.* $\theta=t=\pi/2$ implies $\sin u=0$ by (5). q. e. d.

Now by solving $(4)_1$ under this condition we get

(8) $\sin\alpha\tan u=(\tan tx/x)\sin\alpha\cos\theta\sin\theta$.

This means that $u=l\cos\alpha$ is determined uniquely modulo π from the value of t unless $\sin\theta\neq0$, namely Ad h is determined uniquely from the value of t unless $\sin\theta\neq0$. In the case of $\sin\theta=0$, *i.e.*, $\theta_1=\pi$ or 0, we get for any $h\in H_\alpha$ ad $h(\cos\theta_1\cdot\xi+\sin\theta_1\cdot e_2)=\cos\theta_1\cdot\xi+\sin\theta_1\cdot e_2$ $(=-\xi$ or $\xi)$ which is also uniquely determined.

3.5. Remark. For $\theta=\pi/2$, there exist no minimizing geodesics different from $\gamma_{\pi/2}$ which intersect $\gamma_{\pi/2}$ at the value of $t(=xt)\leq t_0(\pi/2)$.

3.6. Remark. From the above we have

(9) $A=-2t\cos\theta\cot\alpha\cos\alpha$.

3.7. Next we consider the equation (5). By lemma 3.4, (5) takes the following form:

$$\cos A\cos u+\sin A\sin u=$$
$$=\left(\cos^2 tx-\frac{\sin^2 tx}{x^2}(\cos^2\theta\sin^2\alpha-\sin^2\theta)\right)\cos u+$$
$$+2\sin\alpha\frac{\sin tx}{x}\cos tx\cdot\cos\theta\cdot\sin u,$$
$$-\sin A\cos u+\cos A\sin u=$$
$$=(\cos^2 tx-\sin^2 tx)\sin u-2\sin\alpha\cdot\frac{\sin tx}{x}\cos tx\cdot\cos\theta\cdot\cos u.$$

Considering again this as the homogeneous linear equation with respect to $\cos u$ and $\sin u$ we get the *n. a. s.* condition of having solutions:

$$\cos 2tx\left[2\cos A-\frac{\sin^2\theta}{x^2}(1+\cos A)\right]+\sin 2tx\left[\frac{2\sin\alpha}{x}\cos\theta\sin A\right]$$
$$=2-\frac{\sin^2\theta}{x^2}(1+\cos A).$$

Noting that

$$\left[2\cos A-\frac{\sin^2\theta}{x^2}(1+\cos A)\right]^2+\left[\frac{2\sin\alpha}{x}\cos\theta\cdot\sin A\right]^2=$$
$$=\left[2-\frac{\sin^2\theta}{x^2}(1+\cos A)\right]^2,$$

the above condition is equivalent to the following:

(10) $\cos(2tx-B)=1$,

where B is defined by $\sin B=2\sin\alpha\cos\theta\sin A/xC$, $\cos B=(2\cos A-\sin^2\theta(1+\cos A)/x^2)/C$, with $C=2-\sin^2\theta(1+\cos A)/x^2$ and $A=-2t\cos\theta\cot\alpha\cos\alpha$.

Then this determines the value of t in principle. We study this in a more detail. Put $\sin\gamma=\sin\theta/x$, $\cos\gamma=\sin\alpha\cdot\cos\theta/x$. By the above we have $\tan(A/2)=\cos\gamma\cdot\tan(B/2)$. On the other hand we get $\tan(B/2)=\tan tx$ by virtue of (10). Thus we get

(11) $\tan(A/2)=\cos\gamma\cdot\tan tx$ $(\cos\gamma=\sin\alpha\cdot\cos\theta/x,$
$A=-2t\cos\theta\cot\alpha\cos\alpha)$.

Now (11) determines $t=t(\theta)$ uniquely. In fact firstly we determine $t(0)$. In this case (11) turns out to be $-\tan(t\sin\alpha\cdot\cot^2\alpha)=\tan(t\sin\alpha)$, *i.e.*, $t=n\pi\sin\alpha$, $n\in\boldsymbol{Z}$. Since $t(0)$ is the positive minimum value which satisfies the above, we have $t(0)=\pi\sin\alpha<\pi/\sin\alpha=t_0(0)$. Next we put

$$f(t,\theta):=\cos\gamma\cdot\tan tx+\tan(t\cos\theta\sin\alpha\cot^2\alpha)).$$

Then we get for $\theta\in[0,\pi/2]$

$$\partial f/\partial t=\cos\gamma\cdot x(1/\cos^2 tx+\cot^2\alpha/\cos^2(t\cos\theta\sin\alpha\cdot\cot^2\alpha))\geq 0,$$

where the equality holds iff $\cos\gamma=0$, *i.e.*, $\theta=\pi/2$. By the implicit function theorem $t=t(\theta)$ is determined uniquely for $\theta\in[0,\pi/2)$. On the other hand note that $t(\pi-\theta)=t(\theta)$ by (11).

3.8. LEMMA. $dt/d\theta(\theta)=\dfrac{\tan\theta((\tan tx)/x+\sin^2\theta\cot^2\alpha\cdot t)}{(x^2/\sin^2\alpha+\tan^2 tx)}$

PROOF. Note that $dx/d\theta=\cos\theta\cdot\sin\theta\cdot\cos^2\alpha/x$, $d\cos\gamma/d\theta=-\sin\alpha\cdot\sin\theta/x^3$ and $dA/d\theta=-2\sin\alpha\cdot\cot^2\alpha(dt/d\theta\cdot\cos\theta-t\sin\theta)$. The by differentiating the both sides of $\tan A/2=\cos\gamma\tan tx$ and noting that $\dfrac{1}{\cos^2 A/2}=1+\cos^2\gamma\tan^2 tx$, $1+\cot^2\alpha\cos^2\gamma=\dfrac{1}{x^2}$, we get the lemma. q.e.d.

Now we compute $\lim\limits_{\theta\to\pi/2}\tan tx$ for completeness. By l' Hospital's rule we get

$$\lim_{\theta\to\pi/2}\tan tx=\lim_{\theta\to\pi/2}\tan(A/2)/\cos\gamma=\lim_{\theta\to\pi/2}(d/d\theta\tan(A/2))/(d/d\theta\cos\gamma)$$
$$=\lim_{\theta\to\pi/2}\cot^2\alpha(\sin^2\alpha\tan t-\sin^2\alpha\cdot t-t\sin^2\alpha\tan^2 t)/(1+\sin^2\alpha\tan^2 t);$$

From this we have $\lim\limits_{\theta\to\pi/2}\sin^2\alpha/\cos^2 t\cdot(t\cot^2\alpha+\tan t)=0$, *i.e.*,

$$\tan(\lim_{\theta\to\pi/2} t(\theta))/\lim_{\theta\to\pi/2} t(\theta) = -\cot^2\alpha \quad \text{(compare with (1))}.$$

Finally we study the sign of $dt/d\theta$ $(0\leq\theta<\pi/2)$. Clearly $dt/d\theta(0)=0$.

(i) If $0<tx<\pi/2$, then $dt/d\theta>0$,

(ii) If $tx=\pi/2$, then $dt/d\theta=0$,

(iii) In the case of $tx>\pi/2$. We may assume $tx\leq v_0$ (for the definition of v_0 see the proof of lemma 3.3) and therefore

$(\tan tx)/tx\leq(\tan v_0)/v_0=-\sin^2\theta\cdot\cot^2\alpha$. Thus we get

$\tan tx/x+\sin^2\theta\cdot\cot^2\alpha\cdot t\leq 0$,

where equality holds iff $tx=v_0$.

3.9. To sum up we can see how the function $t=t(\theta)$ behaves:

Case 1. $t(0)=\pi\sin\alpha\geq\pi/(2\sin\alpha)$: Then $t=t(\theta)$ can not intersect $tx=\pi/2$ by virtue of (ii). On the other hand $t=t(\theta)$ can not intersect $t=t_0(\theta)$ for $0\leq\theta<\pi/2$. Otherwise at the intersection we can estimate that $dt/d\theta=0$, $dt_0/d\theta<0$ $(0<\theta<\pi/2)$. (See 2.4. Remark). From this we see that $t=t_0(\theta)$ and $t=t(\theta)$ can not intersect at more one than point for $\theta\in(0,\pi/2)$. So assume that $t=t_0(\theta)$ intersects $t=t(\theta)$ at exactly one point given by $0<\theta_0<\pi/2$. Now we consider the tangent cut locus: firstly take the curve C_1 in (ξ, e_2)-plane defined by

$$C_1(\theta):=\begin{cases} t(\theta)(\cos\theta\cdot\xi+\sin\theta\cdot e_2) & 0\leq\theta\leq\theta_0 \\ t_0(\theta)(\cos\theta\cdot\xi+\sin\theta\cdot e_2) & \theta_0\leq\theta\leq\pi/2. \end{cases}$$

Secondly consider the curve C_2 in (ξ, e_2)-plane obtained by reflecting C_1 about the e_2-axis. Then the tangent cut locus is the surface of revolution obtained by rotating the curve $C_1\cup C_2$ around the ξ-axis. The cut locus is then given by projecting the tangent cut locus under the exponential mapping and must contain homotopically non-trivial closed curves. This contradicts the fact that the cut locus is simply connected iff the given manifold is simply connected. Thus $t=t(\theta)$ is monotone decreasing on $[0,\pi/2]$ and $\lim_{\theta\to\pi/2} t(\theta)=t_0(\pi/2)$. Also we get $\lim_{\theta\to\pi/2} dt/d\theta=0$ and the cut point of $\gamma_{\pi/2}$ is the first conjugate point. The last fact follows also from Remark 3.5. (see diagram 1).

Case 2. $t(0)=\pi\sin\alpha<\pi/2\sin\alpha$. In this case for small positive values of θ, $t=t(\theta)$ is monotone increasing. Since $\lim_{\theta\to\pi/2} t(\theta)=t_0(\pi/2)>\pi/2$, $t=t(\theta)$ intersects a point of $tx=\theta/2$. Then we see by the same arguement as above that $t=t(\theta)$ converges to $t_0(\pi/2)$ monotone decreasingly with $\lim_{\theta\to\pi/2} dt/d\theta=0$.

(see diagram 2).

3.10. REMARK. From the above construction unknowns θ_1, t, l, and l' are uniquely determined. Then the equation (5) is automatically satisfied (In fact we have $A=2u$). It is also clear that $t=t(\theta)$ defines the tangent cut point of o along the geodesic γ_θ.

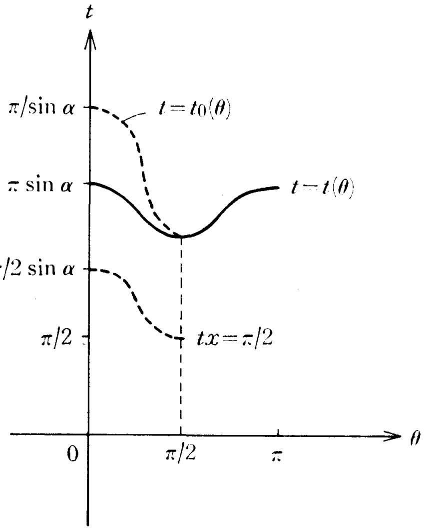
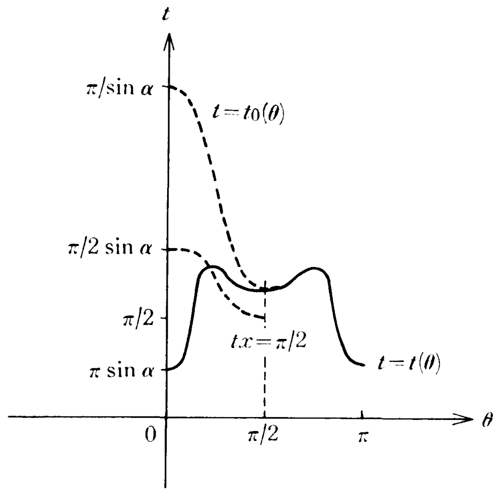


Diagram 1 $(\alpha\geq\pi/4)$



Diagam 2 $(\alpha<\pi/4)$

Thus the whole tangent cut locus of o is obtained by rotating the diagrams 3, 4 around the axis ξ. Under the exponential mapping every point of the northern hemisphere is identified with the unique point in the southern hemisphere, which is determined by the value of l and $\theta_1=\pi-\theta$ (see §§ 3.4.). On the equator the exponential mapping is injective. Since M is homogeneous we have

3.11. THEOREM. *The cut locus of any point of Berger's sphere M_α $(0<\alpha<\pi/2)$ is the disc whose boundary consists of conjugate points of order 1. This disc contracts to a point when α converges to $\pi/2$.*

3.12. REMARK. For $0<\alpha<\pi/4$, the injectivity radius of M (: $=\operatorname{Inf} d$ (o, the cut locus of o)) is given by $\pi\sin\alpha$. On the other hand for $\alpha\geq\pi/4$ this value gives the diameter of M and the injectivity radius is given by $t_0(\pi/2)$ which defines the first conjugate point. The first conjugate points are given by non-isotropic Jacobi fields in the sense of Chavel ([5]). Thus the first conjugate locus has strong influence upon the cut locus and we suspect that this holds also for general homogeneous spaces.

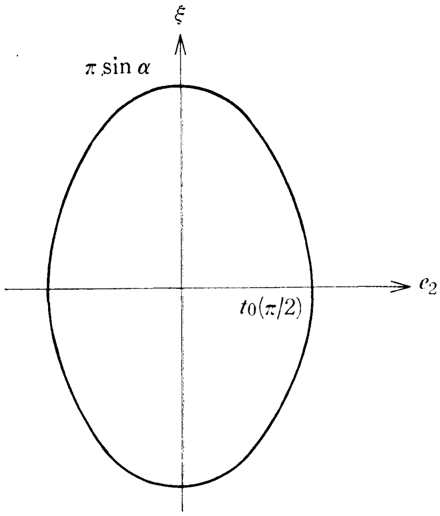


Diagram 3 $(\alpha \geq \pi/4)$

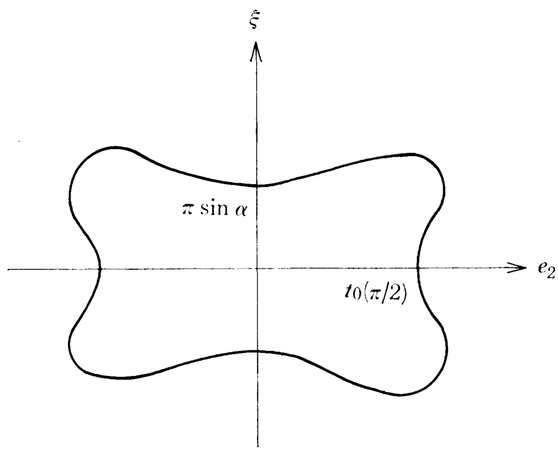


Diagram 4 $(\alpha < \pi/4)$

4. Examples of Wallach.

4.1. N. Wallach constructed the following 7-dimensional manifolds of positive curvature. We shall follow the notation of Huang ([7]). Let $T(p, q)$ (p, q are relatively prime integers) be the circle in $SU(3)$ defined by

$$T(p,q):=\left\{\begin{pmatrix} e^{2\pi ip\theta} & 0 & 0 \\ 0 & e^{2\pi iq\theta} & 0 \\ 0 & 0 & e^{-2\pi i(p+q)\theta}\end{pmatrix};\ \begin{array}{l}\theta\in\boldsymbol{R},\ p^2+q^2>0 \\ i=\sqrt{-1}\end{array}\right\}$$

and define $M(p, q):=SU(3)/T(p, q)$, which is simply connected and $H^4(M(p, q);\boldsymbol{Z})\cong\boldsymbol{Z}_r$ with $r=|p^2+q^2+pq|$. Then the Lie algebra $\mathfrak{g}$ of $SU(3)$ is decomposed into the vector space direct sum: $\mathfrak{g}=H+V_1+V_2$ where

$$H:=\text{ Lie algebra of } T(p,q)$$

$$V_1:=\left\{\begin{pmatrix} Z & 0 \\ 0 & -trZ\end{pmatrix};\ Z=\begin{pmatrix} a & v \\ -\bar{v} & b\end{pmatrix},\ \begin{array}{l} a(2p+q)+b(p+2q)=0 \\ a, b\in\boldsymbol{R},\ v\in\boldsymbol{C}\end{array}\right\}$$

$$V_2:=\left\{\begin{pmatrix} 0 & 0 & z_1 \\ 0 & 0 & z_2 \\ -\bar{z}_1 & -\bar{z}_2 & 0\end{pmatrix};\ z_i\in\boldsymbol{C},\ i=1,2\right\}.$$

Let $\langle\ ,\ \rangle$ be the $\mathrm{Ad}_{SU(3)}$-invariant inner product on $\mathfrak{g}$ defined by $\langle X, Y\rangle=-1/2\ \mathrm{tr}\ XY$. We define the inner product $\langle\ ,\ \rangle_t$ on $\mathfrak{p}=V_1+V_2$ by

$$\langle X_1+X_2, Y_1+Y_2\rangle_t:=(1+t)\langle X_1, Y_1\rangle+\langle X_2, Y_2\rangle\quad(-1<t<0),$$

and extend $\langle\ ,\ \rangle_t$ to an inner product on $\mathfrak{g}$ making H and $\mathfrak{p}$ orthogonal and choosing $\langle\ ,\ \rangle_t=\langle\ ,\ \rangle$ on H. Then $\langle\ ,\ \rangle_t$ is $\mathrm{Ad}_{T(p,q)}$-invariant and induces a homogeneous riemannain structure on $M(p, q)$ which turns out to be of positive curvature. Note that this metric can not be normal.

4.2. Now Huang showed that the pinching number of $M(i, i+1)$ converges to that of $M(1, 1)$ as $i \to \infty$ and computed explicitly the pinching number of $M(1, 1)$ for $t=-1/2$ $(2/37 \leq K_\sigma \leq 29/8)$. Then using the finiteness theorem due to Cheeger and Weinstein he concluded that the infinimum of the injectivity radii of $M(i, i+1)$ is equal to zero.

Now we remark that it is possible to give an explicit estimate from the above of the injectivity radius of $(M(p, q), \langle\ ,\ \rangle_t)$. Our idea is the following: Let T^2 be the maximal torus containing $T(p, q)$. Then $N:=SU(3)/T^2$ carries a structure of hermitian manifold of positive curvature and we have the circle bundle $M(p, q) \to N$. Then the fibers of this bundle will give short closed geodesics. This happens in the case of Berger's spheres ([6]).

For that purpose we consider the element

$$(12) \quad \overline{M}_7 := \left(1/\sqrt{3(1+t)}\,\Delta\right) \begin{pmatrix} (2q+p)i & 0 & 0 \\ 0 & -(2p+q)i & 0 \\ 0 & 0 & (p-q)i \end{pmatrix} \in \mathfrak{p}$$

with $\Delta = \sqrt{p^2+pq+q^2}$ which defines a unit vector in $T_oM(p, q) \cong V_1 + V_2$, $o=\pi(e)$. Then using the multiplication table for Lie bracket in [7], we easily see that

$$\nabla_{\overline{M}_7} \overline{M}_7 = 1/2\,[\overline{M}_7, \overline{M}_7] - ad^*_{\overline{M}_7} \overline{M}_7 = 0\,.$$

Namely $s \to \pi \circ \exp s\overline{M}_7$ defines a geodesic in $(M(p, q), \langle\ ,\ \rangle_t)$.

Now we look for the minimum positive number s_0 of s which satisfies $\exp s\overline{M}_7 \in T(p, q)$. This holds iff

$$\begin{cases} \exp\left((2q+p)si/\sqrt{3+(1+t)}\,\Delta\right) = \exp ip\theta \text{ for some } \theta \in \boldsymbol{R}, \\ \exp\left(-(2p+q)si/\sqrt{3(1+t)}\,\Delta\right) = \exp iq\theta \end{cases} \quad \textit{i. e.,}$$

$$(13) \quad \begin{cases} (2q+p)s/\sqrt{3(1+t)}\,\Delta = p\theta + 2\pi m, \quad m, n \in \boldsymbol{Z}\,. \\ -(2p+q)s/\sqrt{3(1+t)}\,\Delta = q\theta + 2\pi n \end{cases}$$

Eliminating θ from the last equation we get

$$s = \pi\sqrt{3(1+t)}\,/\Delta \cdot (mq-np)\,.$$

Then θ is determined automatically from the above (13). Now since p, q are relatively prime we can choose m, n so that $mq-np=1$. Thus we see that the smallest value s_0 of such s is given by

$$s_0=\pi\sqrt{3(1+t)}/\sqrt{p^2+pq+q^2}\ .$$

This means that $s\rightarrow\pi\circ\exp s\overline{M}_7$; $s\in[0, s_0]$ defines a geodesic loop in $M(p, q)$ which turns out to be a closed geodesic of length $\pi\sqrt{3(1+t)}/\sqrt{p^2+pq+q^2}$ (see [8]). Since the injectivity radius is not greater than half the length of closed geodesics we get

4.3. PROPOSITION. *For the injectivity radius* $i(M(p, q))$ *of* $M(p, q)$ *we get the following inequality*:

$$i(M(p, q))\leq\pi\sqrt{3(1+t)}/2\sqrt{p^2+pq+q^2}\ .$$

In this occasion I would like to correct some errors in my previous paper "ON THE INDEX THEOREM OF AMBROSE" (Vol. IV (1975), 227-233) Hokkaido Math. J.

P 228 ↑ 12 H'-vector fields$\rightarrow H^1$-vector fields
P 229 ↑ 16 $2\alpha(X_h, X_v), (Y_h, Y_v))\rightarrow\alpha((X_h, X_v), (Y_h, Y_v))$
↑ 7 $\perp\dot{c}(b):=S_1\oplus T_1\oplus N\oplus A\rightarrow\perp\dot{c}(b):=(S_1+T_1)\oplus N\oplus A$, where S_1+T_1 denotes the linear span of S_1 and T_1.
↑ 2 $pr_NA_Tpr_Nx\rightarrow pr_NA_Tpr_Tx$
P 230 ↓ 1 $pr_{T_1}A_Tpr_NY(b)\rightarrow pr_{T_1}A_Tpr_TY(b)$
↓ 2 $pr_NA_Tpr_NY(b)\rightarrow pr_NA_Tpr_TY(b)$
↓ 4 $pr_{S^*(b)}(\nabla Y(b)-\nabla x(b))\rightarrow pr_{S^*(b)}(\nabla Y(b)-\nabla X(b))$
↓ 17 $\Psi=A_{S^*(b)}-pr_NA_T\rightarrow\Psi=A_{S^*(b)}-pr_{S^*(b)}A_T$
P 232 ↓ 3 Insert the following sentense before "So by lemma 4, $\cdots$". "Similary for any $\tilde{U}(a)\in W(a)$ we have $\langle\xi(a), \nabla U(a)\rangle=0$ from $D^2E(c)(\xi, \zeta_0\oplus\zeta_1)=0$."
↑ 3 H'-vector fields$\rightarrow H^1$-vector fields.

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References

[1] BERGER, M.: Les variétés riemanniennes homogénes normales simplement connexes a curbure strictment positive, Ann. Scuola Norm. Sup. Pisa, **15** (1961), 179-146.

[2] BUCHNER, M.: Simplicial structure of the real analytic cut locus, Proc. A. M. S., **64** (1977), 118-121.

[3] BUCHNER, M.: Stability of the cut locus in dim$\leq$6, Inventiones Math., **43** (1977), 199–231.

[4] BUCHNER, M.: The structure of the cut locus in dim$\leq$6, to appear.

[5] CHAVEL, I.: A class of Riemannian homogeneous spaces, J. Diff. Geo., **4** (1970), 13–20.

[6] CHEEGER, J. and EBIN, G.: Comparison theorems in Riemannian geometry, North Holland-American Elsever., New York, 1975.

[7] HUANG, H. M.: Some results in the pinching problem, Doctorial Thesis, State Univ. of New York at Stony Brook, 1976.

[8] KOSTANT, B.: On differential geometry and homogeneous spaces, Proc. Nat. Acad. Sci. **42** (1958), 258–261, 354–357.

[9] SAKAI, T: The contact structures on $\{SU(n+1)\times R/SU(n)\times R\}_\alpha$ and $\{Sp(n)\times SU(2)/Sp(n-1)\times S\,U(2)\}_\alpha$ of Berger, Tôhoku Math. J., **25** (1973), 145–155.

[10] SAKAI, T.: On the structure of cut loci in compact riemannian symmetric spaces, Math. Ann., **235** (1978), 129–148.

[11] SAKAI, T.: Cut loci of compact symmetric spaces, Minimal submanifolds and Geodesics, Kaigai Pub. Ltd., Tokyo, 1978.

[12] TAKEUCHI, M.: On conjugate loci and cut loci of compact symmetric space I, II, Tsukuba J. Math., **2** (1978), 35–68, **3** (1979), 1–29.

[13] TANNO, S.: The first eigenvalue of the Laplacian on spheres, Tôhoku Math. J., **31** (1979), 179–185.

[14] WALL, C. T. C.: Geometric properties of generic differentiable manifolds, Geometry and Topology, Lecture notes in Math. **597** (1977), Springer, Berlin-Heidelberg-New York., 707–774.

[15] WEINSTEIN, A.: Distance spheres in complex projective spaces, Proc. A. M. S., **39** (1973).

[16] ZILLER, W.: Closed geodesics on homogeneous spaces, Math. Z., **152** (1976), 67–88.

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