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On the construction of p-adic L-functions

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Let $\boldsymbol{Q}$ be the rational number field, $\bar{\boldsymbol{Q}}$ the algebraic closure of $\boldsymbol{Q}$, $\boldsymbol{C}$ the complex number field, p a prime number, $\boldsymbol{Q}_p$ the p-adic rational number field, $\boldsymbol{Z}_p$ the integer ring of $\boldsymbol{Q}_p$, $\boldsymbol{C}_p$ the completion of the algebraic closure of $\boldsymbol{Q}_p$ and let $\mathfrak{m}$ be the maximal ideal of the integer ring of $\boldsymbol{C}_p$. We fix an imbedding of $\bar{\boldsymbol{Q}}$ into $\boldsymbol{C}$ and also fix an imbedding of $\bar{\boldsymbol{Q}}$ into $\boldsymbol{C}_p$. Let $L_i(z)=L_i(z_1, \cdots, z_r)=\sum_{1\leqq j\leqq r} a_{ij}z_j$ be linear forms of r variables, where i ranges from 1 to n, r and n are natural numbers. We suppose that the coefficients a_{ij} are algebraic numbers and satisfy the following conditions: a_{ij} are real positive when considered as complex numbers, and $a_{ij}\in\mathfrak{m}$ when considered as p-adic numbers. Let $L_j^*(t)=L_j^*(t_1, \cdots, t_n)=\sum_{1\leqq i\leqq n} a_{ij}t_i$ be linear forms with above coefficients a_{ij}, where j ranges from 1 to r.

In the following, let us agree that the suffix i ranges from 1 to n and the suffix j ranges from 1 to r. We also agree that an algebraic number may be considered both as a complex number and as a p-adic number by the above fixed imbeddings.

Let $\chi_j:(\boldsymbol{Z}/d_j\boldsymbol{Z})^\times\to\bar{\boldsymbol{Q}}^\times$ be Dirichlet characters defined modulo d_j, which may be not necessarily primitive (here $R^\times$ denotes the multiplicative group of invertible elements of a ring R and $\boldsymbol{Z}$ denotes the ring of rational integers). Let $\xi_j\in\bar{\boldsymbol{Q}}^\times$ be such that $\xi_j^{d_j}\not\equiv 1 \pmod{\mathfrak{m}}$ and $|\xi_j|\leqq 1$ where $|\xi_j|$ is the absolute value of ξ_j considered as a complex number. Let x_j be real algebraic number such that $0\leqq x_j<1$ and $L_i(x)\equiv 1 \pmod{\mathfrak{m}}$ for $i=1, \cdots, n$, where we have put $x=(x_1, \cdots, x_r)$.

Now we define a function $Z(s)=Z(s_1, \cdots, s_n)$ of n complex variables $s=(s_1, \cdots, s_n)$ by

$$Z(s)=\sum_{m_1,\cdots,m_r=0}^{\infty}\frac{\chi_1(m_1)\cdots\chi_r(m_r)\,\xi_1^{m_1}\cdots\xi_r^{m_r}}{L_1(x+m)^{s_1}\cdots L_n(x+m)^{s_n}}$$

where $x+m=(x_1+m_1, \cdots, x_r+m_r)$.
It is easy to see that this series is absolutely convergent when the real parts of $s_1, \cdots, s_n$ are sufficiently large to give there a complex analytic function.

Next we define a mermorphic (i.e., meromorphic in each variable) function $G(t)=G(t_1, \cdots, t_n)$ of n complex variables $t=(t_1, \cdots, t_n)$ by

$$G(t)=\prod_{1\leq j\leq r}\frac{\sum_{0\leq m<d_j}\exp((x_j+m)L_j^*(t))\chi_j(m)\xi_j^m}{1-\exp(d_jL_j^*(t))\xi_j^{d_j}}$$

In this note we shall prove the following two theorems.

THEOREM 1. *Under the above assumptions, the function $Z(s)$ has an analytic continuation to a meromorphic (i.e., meromorphic in each variable) function to the whole space $\boldsymbol{C}^n$. Moreover, its value at non-positive integers, i.e., the value at $s_1=-a_1, \cdots, s_n=-a_n$ with non-negative integers $a_1, \cdots, a_n$, is evaluated as the coefficient of $\frac{t_1^{a_1}}{a_1!}\cdots\frac{t_n^{a_n}}{a_n!}$ in the Laurent expansion at the origin of the function $G(t)$.*

THEOREM 2. *Under the same assumptions as in Theorem 1, there exists a p-adic analytic function $Z_p(s)=Z_p(s_1, \cdots, s_n)$ of n variables such that $Z_p(-a)=Z(-a)$ for $a=(a_1, \cdots, a_n)$ with non-negative integers $a_1, \cdots, a_n$ (this function $Z_p(s)$ is also an analogue of Iwasawa function of n variables).*

The method of proof is essentially due to N. Koblitz [6] which gives a simple proof of the existence of p-adic Dirichlet L-functions.

We remark that a variant of an abelian L-function of a totally real algebraic number field may be expressed as a finite linear combination of certain special types of functions we are considering (c.f., T. Shintani [10] and P. Cassou-Noguès [1], especially [1] Théorème 4). Hence we obtain another (somewhat simplified) proof of the following theorem (c.f., Théorème 26 of P. Cassou-Noguès [1]) which states the existence of the p-adic L-function for a totally real algebriac number field.

THEOREM. *Let K be a totally real algebraic number field of finite degree, M a totally real finite abelian extension of K with Galois group $G(M/K)$. Let $\chi: G(M/K)\to\bar{\boldsymbol{Q}}^\times$ be a character with trivial kernel. Let $\omega: \boldsymbol{Z}_p^\times\to\boldsymbol{Z}_p^\times$ be the homomorphism defined by $\omega(x)=\lim_{n\to\infty}x^{p^n}$. Let θ be the character of the ideal group of K defined by $\theta(\mathfrak{a})=\omega(N(\mathfrak{a}))$ for an ideal $\mathfrak{a}$ of K, where $N(\mathfrak{a})$ is the absolute norm of $\mathfrak{a}$. Then there exists a function $L_p(\chi, s)$ defined over $s\in\boldsymbol{Z}_p$ such that $L_p(\chi, 1-m)=L(\chi\theta^{-m}, 1-m)$ for any positive integer m.*

PROOF OF THEOREM 1. When the real parts of $s_1, \cdots, s_n$ are sufficiently large, we have

$$\prod_i L_i(x+m)^{-s_i}=$$

$$\prod_i\Gamma(s_i)^{-1}\int_0^\infty\cdots\int_0^\infty\exp\Big(-t_1L_1(x+m)\Big)t_1^{s_1-1}\cdots\exp\Big(-t_nL_n(x+m)\Big)t_n^{s_n-1}dt_1\cdots dt_n$$

$$= \prod_i \Gamma(s_i)^{-1} \int_0^\infty \cdots \int_0^\infty \exp\Big(-\sum_j (x_j + m_j)\, L_j^*(t)\Big)\, t_1^{s_1-1} \cdots t_n^{s_n-1} dt_1 \cdots dt_n\,,$$

where $x+m=(x_1+m_1, \cdots, x_r+m_r)$ and $t=(t_1, \cdots, t_n)$.
After multiplying $\prod_j (\chi_j(m_j)\, \xi_j^{m_j})$ both sides, we sum up over $m_1, \cdots, m_r$. We remark that

$$\sum_{m_1,\cdots,m_r=0}^{\infty} \Big(\exp\Big(-\sum_j (x_j+m_j)\, L_j^*(t)\Big) \prod_j \Big(\chi_j(m_j)\, \xi_j^{m_j}\Big)\Big)$$
$$= \exp\Big(-\sum_j x_j L_j^*(t)\Big) \prod_j \frac{\sum_{0 \le m < d_j} \exp(-mL_j^*(t))\, \chi_j(m)\, \xi_j^m}{1-\exp(-d_j L_j^*(t))\, \xi_j^{d_j}}$$

because χ_j is a character defined modulo d_j. Let $g(t)$ denote the right hand side of the above equality. Then we have

$$Z(s) = \prod_i \Gamma(s_i)^{-1} \int_0^\infty \cdots \int_0^\infty g(t)\, t_1^{s_1-1} \cdots t_n^{s_n-1} dt_1 \cdots dt_n\,.$$

For a positive number $\varepsilon<1$, C_ε denotes the integral path in $\boldsymbol{C}$ consisting of the interval $(+\infty, \varepsilon]$, counterclockwise circle of radius ε around the origin and the interval $[\varepsilon, +\infty)$.

Since $L_1^*, \cdots, L_r^*$ are linear forms with positive coefficients and $\xi_j^{d_j} \neq 1$, for sufficiently small $\varepsilon<1$, we have

$$Z(s) = \prod_i \Big(\Gamma(s_i) \Big(\exp(2\pi\sqrt{-1}\, s_i) - 1\Big)\Big)^{-1} \int \cdots \int_{(C_\varepsilon)^n} g(t)\, t_1^{s_1-1} \cdots t_n^{s_n-1} dt_1 \cdots dt_n\,.$$

It is easy to see that, as a function of $s=(s_1, \cdots, s_n)$, the above integral is meromorphic (i.e., meromorphic in each variable) in the whole space $\boldsymbol{C}^n$. Moreover, since

$$\prod_{1 \le i \le n} \Big(\Gamma(s_i) \Big(\exp(2\pi\sqrt{-1}\, s_i) - 1\Big)\Big)^{-1}$$
$$= (2\pi\sqrt{-1})^{-n} \prod_{1 \le i \le n} \Big(\Gamma(1-s_i) \exp(-\pi\sqrt{-1}\, s_i)\Big)\,,$$

the value of the integral at $s_1=-a_1, \cdots, s_n=-a_n$ is equal to $(-1)^{\sum_i a_i} \prod_i (a_i\,!)$ times the coefficient of $t_1^{a_1} \cdots t_n^{a_n}$ in the Laurent expansion at the origin of the function $g(t)$. As $G(t)=g(-t)$, theorem 1 is now proved.

Proof of Theorem 2. First, we review the results of Koblitz [6]. For a positive rational integer d, let $X_0 = \varprojlim_N (\boldsymbol{Z}/dp^N \boldsymbol{Z})$. Let $m+dp^N \boldsymbol{Z}_p$, $0 \le m < dp^N$, denote the set of $x \in X_0$ which map to m under the natural map $X_0 \to \boldsymbol{Z}/dp^N \boldsymbol{Z}$. A character defined modulo d can be pulled back to X_0 via the map $X_0 \to \boldsymbol{Z}/d\boldsymbol{Z}$. We also have a projection $\pi: X_0 \to \boldsymbol{Z}_p$ which "forgets the mod d information". If f is a function on $\boldsymbol{Z}_p$, we also use f to denote the function $f\circ\pi$ on X_0. For example, for fixed small $t\in\boldsymbol{C}_p$ (namely, for ord $_pt>1/(p-1)$), the sum $\sum_{n=0}^{\infty}(tx)^n/n!$ $x\in\boldsymbol{Z}_p$, converges to give a function $\exp(tx)$ on $\boldsymbol{Z}_p$, which we also consider as a function $\exp(tx)$ on X_0.

For each p-adic number $\xi\in\boldsymbol{C}_p$ such that $\xi^{dp^N}\neq 1$ for all N, we define a $\boldsymbol{C}_p$-valued finitely additive set function μ_ξ (i. e., μ_ξ is a map from the set of open-compact subsets of X_0 to $\boldsymbol{C}_p$, which is finitely additive) by the following formula:

$$\mu_\xi(m+dp^N\boldsymbol{Z}_p)=\frac{\xi^m}{1-\xi^{dp^N}}\,,\qquad 0\leqq m<dp^N\,. \tag{2.1}$$

The results of Koblitz [6] state that μ_ξ is always finitely additive (i. e., μ_ξ can be extended to all open-compact subsets of X_0, which is finitely additive), and μ_ξ is bounded (i. e., the p-adic absolute values of $\mu_\xi(U)$, U open-compact subsets of X_0, are bounded) if and only if $\xi^d\not\equiv 1$ mod $\mathfrak{m}$. If μ_ξ is bounded, we can integrate a $\boldsymbol{C}_p$-valued continuous function f on X_0 by the "measure" μ_ξ:

$$\int_{X_0}fd\mu_\xi=\lim_{N\to\infty}\sum_{0\leqq m<dp^N}f(m)\,\mu_\xi(m+dp^N\boldsymbol{Z}_p)\,. \tag{2.2}$$

Now we return to our previous notations. With the same notations at the beginning of this note, let $X_j=\varprojlim_N(\boldsymbol{Z}/d_jp^N\boldsymbol{Z})$ and let $\mu_{\xi_j}(m+d_jp^N\boldsymbol{Z}_p)=\frac{\xi_j^m}{1-\xi_j^{d_jp^N}}$ be the measure on X_j. Let $X=\prod_{1\leqq j\leqq r}X_j$ be the product space and let $\mu_\xi=\prod_{1\leqq j\leqq r}\mu_{\xi_j}$ be the product measure on X. Fix p-adic variables $t=(t_1,\cdots,t_n)$ such that $\exp(\sum_j(x_j+y_j)\,L_j^*(t))$ is convergent for any $y=(y_1,\cdots,y_r)\in X$. A simple calculation using (2.1) and (2.2) shows that

$$\int_X\exp\Big(\sum_j(x_j+y_j)\,L_j^*(t)\Big)\prod_j\chi_j(y_j)\,d\mu_\xi(y)$$
$$=\prod_j\frac{\sum_{0\leqq m<d_j}\exp((x_j+m)\,L_j^*(t))\,\chi_j(m)\,\xi_j^m}{1-\exp(d_jL_j^*(t))\,\xi_j^{d_j}}=G(t)\,.$$

Expanding $\exp(\sum_{1\leqq j\leqq r}(x_j+y_j)\,L_j^*(t))=\exp(\sum_{1\leqq i\leqq n}t_iL_i(x+y))$, equating the coefficient of $\frac{t_1^{a_1}}{a_1!}\cdots\frac{t_n^{a_n}}{a_n!}$, we have

$$Z(-a)=Z(-a_1,\cdots,-a_n)=\int_X\prod_{1\leqq i\leqq n}L_i(x+y)^{a_i}\prod_{1\leqq j\leqq r}\chi_j(y_j)\,d\mu_\xi(y)\,.$$

From the hypotheses that the coefficients of L_i are contained in $\mathfrak{m}$ and $L_i(x)\equiv 1 \bmod \mathfrak{m}$, we have $L_i(x+y)\equiv 1 \bmod \mathfrak{m}$ for any $y\in X$. Hence the value $L_i(x+y)^{s_i}$ is well-defined for any $s_i\in \boldsymbol{Z}_p$. Now define

$$Z_p(s)=Z_p(s_1, \cdots, s_n)=\int_X \prod_{1\leq i\leq n}\left(L_i(x+y)\right)^{-s_i}\prod_{1\leq j\leq r}\chi_j(y_j)\, d\mu_\xi(y)\,.$$

This is the function what we want; i.e., $Z_p(s)$ is a p-adic analytic function (also an analogue of Iwasawa function) such that $Z_p(-a)=Z(-a)$ for $a=(a_1, \cdots, a_n)$ with non-negative integers $a_1, \cdots, a_n$.

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