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Blocks with a normal defect group*

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To Professor Goro Azumaya to commemorate his sixtieth birthday

1. Introduction.

Let G be a finite group, p a fixed rational prime and P be a Sylow p-subgroup of G. In the following, we will consider the group algrbeas over a complete discrete valuation ring R with the unique maximal ideal $(\pi)\ni p$, where its residue class field $F=R/(\pi)$ of characterist c p is a splitting field for G.

In this paper we shall introduce two invariants $n(B)$, $m(B)$ (both positive integers) which can be associated with a given p-block B of G. Namely, $n(B)$ is the number of indecomposable direct summands of $B_{P\times P}$ (the restriction of a $G\times G$-modules B to $P\times P$), and $m(B)$ is the number of indecomposable direct summands of $B_{\Delta(P)}$, where Δ is the diagonal map from G to $G\times G$. These ideas are derived from module-theoretic concepts of a block ideal B which is due to works of J. A. Green ([10], [11], [13]).

On the other hand, Brauer investigated the relation between the invariants $k(B)$, $l(B)$ (the number of ordinary and modular irreducible characters in B, respectively) and the integer $v(B)$ defined by

$$\dim B=p^{2a-d}v(B)\text{ ,}$$

where $p^{a}=|P|$ and d is the defect of B (see section 2 in this paper and Brauer [5]). Following R. Brauer, we shall obtain an elementary inequality

$$\text{(2E, 1)}\qquad p^{a-d}v(B)\leqq m(B)\leqq p^{a}n(B)\leqq p^{a}v(B)\text{ .}$$

Our main interest is in the "extreme" cases of (2E, 1), namely,

$$n(B)=v(B)\quad\text{and}\quad m(B)=p^{a}v(B)\text{ .}$$

Then, in section 3, we will give the structure of G in the above cases. Consequently, for example, it is proved that if $B=B_{0}$, the principal block, then $n(B_{0})=v(B_{0})$ if and only if $G=O_{p'pp'}(G)$, and $m(B_{0})=p^{a}v(B_{0})$ if and only if $G=O_{p'p}(G)$ and P is abelian. In section 4, we will consider another

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"extreme" cases for the invariants $k(B)$, $l(B)$, namely,

$$l(B)=v(B) \quad \text{and} \quad k(B)=p^a v(B).$$

Then, it turns out that there exist some relations among these four "extreme" cases.

NOTATION. For the gereral notation and terminology in the theory of modular representations of finite groups, we shall refer to the books of Curtis-Reiner [6], Dornhoff [7], Feit [8] and Gorenstein [9].

We denote by $Irr(B)$, $IBr(B)$ *and* $Pin(B)$ the set of all irreducible ordinary, Brauer characters and principal indecomposable characters in B, respectively. Let $Ker\, B=\bigcap_{\chi\in Irr(B)} Ker\,\chi$ and $Ker\, B^*=\bigcap_{\varphi\in IBr(B)} Ker\,\varphi$. For a group H, R_H means the trivial RH-module with R-rank 1. We identify a block with a block ideal of RG, and we write B in both cases. Concluding this section we shall summerize the results of Green.

LEMMA (1 A) (Green [11], Theorem 1). *Let B be a block of G with defect group D. Then B is an indecomposable $R(G\times G)$-submodule of RG with vertex $\Delta(D)$.*

Let X be a p-group acting on Ω. Then the module $[\Omega]$ whose basis consists of all elements of Ω is an indecomposable RX-module with vertex the stabilizer $X(a)$ of $a\in\Omega$. Moreover, now, $G\times G$ acts transitively on G, by the rule $a(x,y)=x^{-1}ay$.

LEMMA (1 B) (Green, [11]). *Let B be a block of G with defect group D, and X be a p-subgroup of $G\times G$ which contains $\Delta(D)$. Then*

1) $B_X\cong\bigoplus_{\text{some } a^X}[a^X]$, *where a^X is an X-orbit of G,*
2) $X(a)\underset{G\times G}{\subseteqq}\Delta(D)$ *for any a^X in the right of* 1),
3) *there exists $a_0{}^X$ in the right of* 1) *such that* $X(a_0)\underset{X}{=}\Delta(D)$.

In Lemma (1 B), if $X=P\times Q$ for p-subgroups P, Q of G, then $X(a)=(P,Q,a)=\{(x,x^a)\mid x\in P\cap Q^{a^{-1}}\}\subseteqq P\times Q$, and if $X=\Delta(P)$, then $X(a)=\Delta(C_P(a))\subseteqq\Delta(P)$ for $a\in G$.

2. The invariants $n(B)$, $m(B)$.

Let X be a p-subgroup of $G\times G$, and B be a block of G. Then, as in section 1, every indecomposable direct summand of B_X is isomorphic to the module $[a^X]$ where a^X is an X-orbit of $a\in G$. If $X=P\times Q$, for p-subgroups P, Q of G, then $a^X=PaQ$, and if $X=\Delta(P)$, then $a^X=a^P$ (P-conjugate class contains a).

DEFINITION (2 A). *Let B be a block of G, and P, Q be p-subgroups* *of G. We denote by $n_{P,Q}(B)$ the number of indecomposable direct summands of $B_{P\times Q}$, and by $m_P(B)$ the number of indecomposable direct summands of $B_{\Delta(P)}$.*

Now we shall show that $n_{P,Q}(B)$ and $m_P(B)$ can be given in terms of characters in B, and those are a block refinement of the number of (P, Q)-double cosets of G and that of P-conjugate classes of G. The following formulas are frequently used in section 3 and are helpfull to investigate examples.

PROPOSITION (2 B). *Let B be a block of G, and P, Q be p-subgroups of G. Then*

1) $n_{P,Q}(B)=\sum_{\chi\in Irr(B)}(\chi_P, 1_P)(\chi_Q, 1_Q)$.

2) $m_P(B)=\sum_{\chi\in Irr(B)}(\chi_P, \chi_P)$, *in particular* $m_P(B)\geqq k(B)$.

PROOF. Let X be a p-subgroup of $G\times G$, and K be the quotient field of R of characteristic 0 which is a splitting field for G. We consider the K-character afforded by B_X. Then it is $\sum_{\chi\in Irr(B)}(\bar{\chi}\chi)_X$. On the other hand, since $B_X\cong\bigoplus_{\text{some } a^X}[a^X]$ and X acts on a^X transitively, the multiplicity of 1_X in the K-character afforded by every module $[a^X]$ is 1. Therefore the number of indecomposable direct summands of B_X is equal to

$$\sum_{\chi\in Irr(B)}\left((\bar{\chi}\chi)_X, 1_X\right).$$

Set $X=P\times Q$, $\Delta(P)$, then it is easily verified that 1), 2) holds respectively from the above equation. The last comment is clear from $k(B)=\sum_{\chi\in Irr(B)}(\chi, \chi)$.

From now we fix a Sylow p-subgroup P of order p^a of G, and let us set $n(B)=n_{P\times P}(B)$ and $m(B)=m_P(B)$. In the following we will discuss relation among the numbers $l(B)$, $k(B)$, $n(B)$ and $m(B)$ by the medium of R-rank of a block ideal B. As in section 1, rank $B=p^{2a-d}v(B)$. Brauer has shown that $v(B)$ decomposes into two parts, namely,

$$v(B)=u(B)^2w(B),$$

where $u(B)$ is the G. C. D. of $\varphi(1)/p^{a-d}$ for all $\varphi\in IBr(B)$, and it is also the G. C. D. of $\chi(1)/p^{a-d}$ for all $\chi\in Irr(B)$, furthermore it is also the G. C. D. of $\Phi(1)/p^a$ for all $\Phi\in\mathrm{Pin}(B)$. Then Brauer obtained the following.

PROPOSITION (2 C) (Brauer [5], Theorem 4). *Let B be a block of G of defect d. Then*

1) $l(B)\leqq w(B)$.

2) $l(B)=1$ *iff* $w(b)=1$.

3) $l(B)=w(B)$ *iff* $\varphi(1)=p^{a-d}u(B)$, $\Phi(1)=p^a u(B)$ *for every* $\varphi\in IBr(B)$, $\Phi\in Pin(B)$.

4) $l(B)=v(B)$ *iff* $l(B)=w(B)$ *and* $u(B)=1$.

Similar argument immediately yields the following.

PROPOSITION (2 D).

1) $k(B)\leqq p^d w(B)$.

2) $k(B)=p^d w(B)$ *iff* $k(B)=p^d$ *and* $w(B)=1$ *iff* $\chi(1)=p^{a-d}u(B)$ *for all* $\chi\in Irr(B)$.

3) $k(B)=p^d v(B)$ *iff* $k(B)=p^d$ *and* $v(B)=1$ *iff* $\chi(1)=p^{a-d}$ *for all* $\chi\in Irr(B)$.

PROPOSITION (2 E).

1) $p^{a-d}v(B)\leqq m(B)\leqq p^a n(B)\leqq p^a v(B)$.

2) $n(B)=1$ *iff* $v(B)=1$, *in particular if* $n(B)=1$, *then* $l(B)=1$.

3) $p^{a-d}v(B)=m(B)$ *iff* $n(B)=p^{-d}v(B)$ *iff* $d=0$.

PROOF. Now we have $B_{\Delta(P)}\cong\bigoplus_{\text{some } y^P}[y^P]$ and $B_{P\times P}\cong\bigoplus_{\text{some } PxP}[PxP]$. We shall compare R-rank of the above equations. It follows from Lemma (1 B), 2) that the rank of the modules $[y^P]$, $[PxP]$ in the right of the above equations is divisible by p^{a-d}, p^{2a-d} respectively. Then there exists an integer $0\leqq e(y^P)\leqq d$, $0\leqq e(PxP)\leqq d$ such that rank $[y^P]=p^{a-d+e(y^P)}$, rank $[PxP]=p^{2a-d+e(PxP)}$. Then we have that

$$p^{2a-d}v(B)=p^{a-d}\sum p^{e(y^P)}=p^{2a-d}\sum p^{e(PxP)}\,.$$

Hence

$$m(B)\leqq p^a v(B)=\sum p^{e(y^P)}\leqq p^d m(B)\,,$$

and also

$$n(B)\leqq v(B)=\sum p^{e(PxP)}\leqq p^d n(B)\,.$$

It follows from Lemma (1 B), 3) that 3) holds. The remainder of 1) is proved as follows; let $B_{P\times P}\cong\bigoplus[Px_iP]$, where $i=1,\cdots,n(B)$. If Px_iP decomposes into r_i P-conjugate classes of G, namely, $y_{i1}^P,\cdots,y_{ir_i}^P$, then

$$B_{\Delta(P)}\cong\bigoplus_{i=1}^{n(B)}\bigoplus_{j=1}^{r_i}[y_{ij}^P]\,.$$

Therefore $m(B)=\sum_{i=1}^{n(B)}r_i$ and $p^{2a-d+e(Px_iP)}=p^{a-d}\sum_j p^{e(y_{ij}^P)}$. Now, since y_{ij}^P is a $\Delta(P)$-orbit of Px_iP, the stabilizer of y_{ij}^P is contained in a stabilizer of Px_iP. This means that $e(y_{ij}^P)\geqq e(Px_iP)$ for all $j=1,\cdots,r_i$. Hence

$$p^{2a-d+e(Px_iP)}\geqq r_i p^{a-d+e(Px_iP)}\,.$$

Therefore $r_i \leqq p^a$, and we have that $m(B) \leqq p^a n(B)$.

2) Suppose $n(B)=1$, then $B_{P \times P} \cong [PxP]$ for some $x \in G$. It follows from Lemma (1 B), 3) that $e(PxP)=0$ and hence $v(B)=1$. By Proposition (2 C), 2), if $n(B)=1$, then $l(B)=1$.

REMARK (2 F). It is immediate from the proof of (2 E) and (1 B, 2) that the "extreme" cases occur if and only if R-rank of every indecomposable direct summand of $B_{P \times P}$, $B_{\Delta(P)}$ is just p^{2a-d}, p^{a-d}, respectively.

REMARK (2 G). *Concerning* (2 E, 2), *in fact, it holds that* $l(B) \leqq n(B)$. *This result will be presented in another paper.*

REMARK (2 H). Relating to (2 D, 1), Brauer has proved a stronger result that $k(B) \leqq p^d l(B)$ in [2], (5 D). Therefore we have $l(B) \leqq k(B) \leqq p^d l(B) \leqq p^d w(B)$. And this relation is similar to that between $n(B)$ and $m(B)$ in (3 C, 1). If Brauer's conjecture "$k(B) \leqq p^d$" is valid, then it should hold that $k(B)=p^d l(B)$ if and only if $k(B)=p^d w(B)$, and hence it also seems to hold that $m(B)=p^a n(B)$ if and only if $m(B)=p^a v(B)$. In the next section, it will be clarified that the above conjecture is related to Glauberman's Z^*-theorem (see Theorem 67. 1 in [7]).

3. The structure of G whose block B satisfies $m(B)=p^a v(B)$ or $n(B)=v(B)$.

Let $K \subseteq H$ be subgroups of G. We call K strongly closed in H with respect to G if $K^x \cap H \subseteq K$ for $x \in G$. Let K be a subset of $H \subseteq G$. We call K weakly closed in H with respect to G if $K=K^x$ when $K^x \subseteq H$ for $x \in G$. The main tool of this section is the character theory of G. The purpose in this section is to prove the following theorems.

THEOREM (3 A). *Let B be a block of G with defect group D which is contained in $P \in Syl_p(G)$. Suppose D is strongly closed in P with respect to G, then the following are equivalent.*

a) $n(B)=v(B)$.

b) *If* $\chi_P \supset 1_P$, *then* $D \subseteq Ker\, \chi$ *for all* $\chi \in Irr(B)$.

c) $D \cdot Ker\, B \lhd G$.

THEOREM (3 B). *Let B be a block of G with defect group D which is contained in $P \in Syl_p(G)$. Then*

1) *the following are equivalent.*

a) $m(B)=p^a n(B)$.

b) *Every element of D is weakly closed in P with respect to G, in particular D is abelian.*

2) *the following are equivalent.*

a) $m(B)=p^a v(B)$.

b) $[G, D]\subseteq Ker\, B$.

PROOF of Theorem (3 A). At first we will prove

$$v(B)=\sum_{\chi}\left(\chi(1)/p^{a-d}\right)\cdot(\chi_P, 1_P)\,.$$

Let $\zeta=\sum_{\chi\in Irr(B)}\chi(1)\cdot\chi$, then $\zeta_{P^\#}=0$ by the orthogonality relation. Therefore $\zeta_P=p^{a-d}v(B)\cdot\rho_P$, where ρ_P is the regular character of P. Hence

$$p^{a-d}v(B)=(\zeta_P, 1_P)=\sum\chi(1)\,(\chi_P, 1_P)\,.$$

a)$\Leftrightarrow$b). Since D is strongly closed in P with respect to G, $\chi(x)=0$ for any p-element x which is not contained in D by the theorem of Brauer or Green (see Feit [8] IV. 2. 4). Hence

$$(\chi_P, 1_P)=(1/p^{a-d})\,(\chi_D, 1_D)\leqq(1/p^{a-d})\,\chi(1)\,,$$

and the equality holds if and only if $D\subseteq Ker\,\chi$. Since $n(B)=\sum(\chi_P, 1_P)^2$ from Proposition (2 B) and $v(B)=\sum(\chi(1)/p^{a-d})\,(\chi_P, 1_P)$, we have $n(B)=v(B)$ if and only if $D\subseteq Ker\,\chi$ for $\chi\in Irr(B)$ satisfying $(\chi_P, 1_P)\neq 0$.

b)$\Rightarrow$c). Case 1. $\chi=1_G$ if $\chi_P\supset 1_P$.

In this case we have $n(B)=1$, and from Proposition (2 E), 2) it follows $l(B)=1$. Since in our case B is the principal block, $G=O_{p'p}(G)$ by the theorem of Brauer. Thus our assertion holds in this case.

Case 2. There exists $\chi\neq 1_G\in Irr(B)$ such that $\chi_P\supset 1_P$. Let $K=Ker\,\chi$, then $D\subseteq K\trianglelefteq G$ by by our assumption. Since $\chi\in Irr(G/K)$ and $\chi\in Irr(B)$, D must contain a Sylow p-subgroup of K. This means that D is a Sylow p-subgroup of K. On the other hand, B covers the principal block $B_0(K)$ of K, and hence if $\zeta_D\supset 1_D$ for $\zeta\in Irr(B_0(K))$, then there exists $\chi'\in Irr(B)$ such that $\chi'_K\supset\zeta$ and $\chi'_P\supset 1_P$. (Since we may take $D=P\cap K\lhd P$, it holds in general that $\chi'_P\supset 1_P$.) From our assumption b), $D\subseteq Ker\,\chi'\cap K\subseteq Ker\,\zeta$. Thus K with $B_0(K)$ has the property b), and so we have from induction on $|G|$ that $K=O_{p'pp'}(K)$, since $Ker\,B_0(K)=O_{p'}(K)$ by the theorem of Brauer (Theorem 1 in [1]) and D is a Sylow p-subgroup of K. Then $D\cdot O_{p'}(K)=O_{p'p}(K)$ *char* $K\lhd G$, and hence $D\cdot O_{p'}(K)\lhd G$. Therefore we obtain $D\cdot Ker\,B\lhd G$. (So this direction holds in general, but really it is easy to see that if $D\subseteq Ker\,\chi$ for $\chi\in Irr(B)$, then D is strongly closed in P with respect to G.)

c)$\Rightarrow$a). This direction holds also in general. We may assume $Ker\,B=1$ by induction on $|G|$. Then our assumption $D\lhd G$ implies that $\chi(x)=0$ for any p-element x which is not contained in D. Therefore, if $x_P\supset 1_P$, then

$(\chi_P, 1_P)=(1/p^{a-d})\ (\chi_D, 1_D)=\chi(1)$ by the theorem of Clifford. Hence we have $n(B)=v(B)$. This completes the proof of Theorem (3 A).

PROOF of Theorem (3 B). We have already proved in Proposition (2 E), 1) that $m(B)\leqq p^a n(B)$. However we will, here, again prove it in terms of characters, in order to clarify the structure of D in G.

From Proposition (2 F), we have

$$\begin{aligned} n(B) &= \sum_{\chi\in Irr(B)} (\chi_P, 1_P)^2 \\ &= \sum_{\chi}\Big((1/p^a)\sum_{x\in P}\chi(x)\Big)\Big((1/p^a)\sum_{y\in P}\overline{\chi(y)}\Big) \\ &= (1/p^{2a})\sum_{x,y}\Big(\sum_{\chi}\chi(x)\,\overline{\chi(y)}\Big)\,. \end{aligned}$$

Then the term in the above bracket is 0 if $x \underset{G}{\neq} y$ by the orthogonality relation (see Feit, p. 245, (6. 3) in [8]), therefore

$$\begin{aligned} n(B) &= (1/p^{2a})\sum_{x}|x^G\cap P|\sum_{\chi}\left|\chi(x)\right|^2 \\ &\geqq (1/p^{2a})\sum_{x}\sum_{\chi}\left|\chi(x)\right|^2 \\ &= (1/p^a)\sum_{\chi}(\chi_P, \chi_P) \\ &= (1/p^a)\ m(B)\,. \end{aligned}$$

Then the equality holds in the above equation if and only if $|x^G\cap P|=1$ for $x\in P$ such that $\sum_{\chi\in Irr(B)}|\chi(x)|^2\neq 0$. Since $\sum_{\chi}|\chi(x)|^2\neq 0$ for $x\in P$ if and only if $x\in D$, the above condition holds if and only if $|x^G\cap P|=1$ for $x\in P$ and $x \underset{G}{\in} D$.

Suppose $|x^G\cap P|=1$ for $x\in P$ and $x\underset{G}{\in} D$, and if $z\in D$ and $z^G\cap P\ni y$, then $z^G\cap P\ni z, y$ and from our assumption $z=y$, hence if $z\in D$, then it holds $z^G\cap P=\{z\}$. Conversely, suppose $z^G\cap P=\{z\}$ for $z\in D$, and if $x\in D$, and if $x\in P$ and $x^t\in D$ for some $t\in G$, then $x=(x^t)^{t^{-1}}\in(x^t)^G\cap P$ and from our assumption $(x^t)^G\cap P=\{x^t\}$, hence $x=x^t$, and therefore $|x^G\cap P|=1$. Then, consequently, $m(B)=p^a n(B)$ if and only if every element of D is weakly closed in P with respect to G.

In particular, $D\subseteq Z(N_G(D))$, and hence D is abelian. Thus we have proved 1).

2). a)$\Rightarrow$b). In general $n(B)\leqq v(B)$. Therefore $m(B)=p^a v(B)$ implies $m(B)=p^a n(B)$. Then from 1), we have $\chi(x)=0$ for any p-element x which is not contained in D, when $\chi\in Irr(B)$. Now we have

$$\begin{aligned} m(B) &= \sum_{\chi \in Irr(B)} (\chi_P, \chi_P) \\ &= (1/p^a) \sum_{\chi} \sum_{x \in P} |\chi(x)|^2 \\ &= (1/p^a) \sum_{\chi} \sum_{x \in D} |\chi(x)|^2 \\ &\leqq (1/p^{a-d}) \sum_{\chi} \chi(1)^2 \\ &= p^a v(B). \end{aligned}$$

Since now it holds $m(B)=p^a v(B)$, we have $\chi(1)=|\chi(x)|$ for all $x \in D$. This implies that $[G, D] \subseteq Ker\ B$.
b)$\Rightarrow$a). We may assume $Ker\ B=1$ by induction on $|G|$. Then $D \subseteq Z(G)$ in our assumption, and it holds that $\chi(x)=0$ for any p-element x which is not contained in D, when $\chi \in Irr(B)$. Hence, in the above equation, the equality just holds and we have $m(B)=p^a v(B)$. This completes the proof of Theorem (3 B).

COROLLARY (3 C). *Let B_0 be the principal block of G. Then*
1) *the following are equivalent,*
  a) $n(B_0)=v(B_0)$
  b) *if* $\chi_P \supset 1_P$, *then* $P \subseteq Ker\ \chi$ *for* $\chi \in Irr(B_0)$
  c) $G=O_{p'pp'}(G)$,
2) *the following are equivalent,*
  a) $m(B_0)=p^a n(B_0)$
  b) $m(B_0)=p^a v(B_0)$
  c) $G=O_{p'p}(G)$ *and* P *is abelian.*

PROOF. 1) Clear by $Ker\ B_0=O_{p'}(G)$.
2) a)$\Rightarrow$c). It follows from Theorem (3 B), 1) that $P \subseteq Z(N_G(P))$. Hence G has a normal p-complement by the theorem of Burnside (see Gorenstein, 7. 4. 3 in [9]). Other direction is clear.

REMARK. 1) of Corollary (3 C) is a generalization of Corollary 2 in [5], in the sense that $v(B_0)=1$ means $n(B_0)=v(B_0)=1$.

EXAMPLE (3 D). In Theorem (3 A) the assumption that D is strongly closed in P with respect to G cannot be dropped.

1). Solvable case.

Let G be the split extension of $GL(2, 3)$ by an elementary abelian group V of order 9. Let $p=2$, and H be the subgroup $SL(2, 3) \cdot V$ of G. Then H has the unique irreducible character ζ of degree 8, and then G has the 2-block B of defect 1 (see Feit [8], X, p. 546). $Irr(B)=\{\chi_1, \chi_2\}$, and $\chi_1+\chi_2=\zeta^G$ and $\chi_i(1)=8$. A defect group D of B is not contained in H, and then D is not normal in any Sylow 2-subgroup of G. And $n(B)=v(B)=1$, but $D\cdot Ker\, B$ is not normal in G.

2). Non-solvable case.

Let G be the Symmetric group of degree 5, and $p=2$. Let B be a block of defect 1 which consists of irreducible characters χ_1, χ_2 of degree 4. Then we have

1) D is not normal in any 2-Sylow subgroup of G, and

2) $n(B)=v(B)=1$,

but $D\cdot Ker\, B$ is not normal in G.

4. The structure of G whose block B satisfies that $k(B)=p^d v(B)$ or $l(B)=v(B)$.

In section 3 we investigated the structure of G whose block B satisfies the condition that $m(B)=p^a v(B)$ or $n(B)=v(B)$. Theorems (3 A), (3 B) and Propositions (2 C), (2 D) and (2 E) suggest that there exists relation between above conditions and those in terms of characters in B. Then, in this section, we shall investigate more closely such relation. The proof is rather module-theoretical in which Green correspondence f plays an important role (see Green, [12], [14]).

Now our purpose in this section is to prove the following theorems.

THEOREM (4 A). *Let B be a block of G with defect group D. We assume that $D\lhd P$ for a Sylow p-subgroup P of G. If $l(B)=v(B)$, then $D\cdot Ker\, B\lhd G$.*

THEOREM (4 B). *Let B be a block of G with defect group D and defect d. We assume that $D\lhd P$ for a Sylow p-subgroup P of G. If $k(B)=p^d v(B)$, then $[G, D]\subseteq Ker\, B$.*

PROOF of Theorem (4 A). Let $N=N_G(D)$, and b be the block of N with defect group D satisfying $b^G=B$. We know from Proposition (2 C), 4) that $l(B)=v(B)$ if and only if $\varphi(1)=p^{a-d}$ for all $\varphi\in IBr(B)$ and $\Phi(1)=p^a$ for all $\Phi\in Pin(B)$. But, in fact, it is not necessary to use the condition on Φ to prove Theorem (4 A).

Step 1. φ_N is irreducible and belongs to b for all $\varphi\in IBr(B)$.

Let V be an irreducible FG-module which affords φ, then V has a vertex D, since φ is of height 0 in B (see Green [10], Theorem 9 and Theorem 12). Then V_N is indecomposable. Otherwise, $V_N\cong W\oplus W'$, where $W=f(V)$ is an indecomposable FN-module with vertex D. Then $W\in b$, and $\dim_F W$

is divisible by p^{a-d}, since our assumption implies $P\subseteq N$. Since $\dim_F V=p^{a-d}$, we have $V_N=W$. As $W\in b$, every irreducible constitutent of W belongs to b, and its dimension is divisible by p^{a-d}. Therefore $V_N=W$ must be irreducible and belongs to b.

Step 2. $D\cdot Ker\, B\lhd G$.

Now, since $D=O_p(N)$, D is contained in the kernel of every irreducible Brauer character of N, and hence $D\subseteq Ker\,\varphi_N\subseteq Ker\,\varphi$ for any $\varphi\in IBr(B)$ by Step 1. We may assume $Ker\, B=1$ by induction on $|G|$. Since $Ker\, B^*/Ker\, B=O_p(G/Ker\, B)$ (see Brauer [4], Propositon (3 D)), now $D\subseteq Ker\, B^*=O_p(G)$. As D is a defect group of B, we have $D\supseteq O_p(G)$. Thus $D\lhd G$, and we complete the proof of Theorem (4 A).

Proof of Theorem (4 B). From Proposition (2 D), 3), $k(B)=p^d v(B)$ if and only if $\chi(1)=p^{a-d}$ for all $\chi\in Irr(B)$, in particular $v(B)=1$. Hence $l(B)=v(B)=1$. Therefore under the assumption $D\lhd P$, we have $D\cdot Ker\, B\lhd G$ by Theorem (4 A). We may assume $Ker\, B=1$ by induction on $|G|$, and so now we have $D\lhd G$.

Step 1. We may assume $PC_G(D)=G$.

Let us set $H=PC_G(D)$. Since $D\lhd G$ and χ has height 0 for every $\chi\in Irr(B)$, D is abelian by the theorem of Reynolds (see Reynolds [18], Theorem 9). Suppose $H\lneqq G$. As $C_G(D)\subseteq H$, there exists a block B' of H satisfying $B'^G=B$ (for instance see Solomon [19], Lemma 3). Then B' has a defect group D.

At first we will show $k(B')=p^d v(B')$. Since $C=C_G(D)\lhd G$ and $C\subseteq H\subseteq G$, we can find a block b of C which satisfies the following conditions;

a) $b^H=B'$ and b is covered by B', and

b) $b^G=B$ and b is covered by B.

For, as $C\subseteq H$, B' is regular with respect to C, therefore there exists a block b of C satisfies a) (see Feit [8], V. 3). Then $b^G=(b^H)^G=B'^G=B$, and B covers b, since B is regular with respect to C. Now b has a defect group D. Therefore by the theorem of Reynolds, every $\varepsilon\in Irr(b)$ has height 0, and since b is covered by B, every $\varepsilon\in Irr(b)$ has degree of a power of p by the theorem of Clifford. This implies $k(b)=p^d v(b)$. Also now it holds that every $\zeta\in Irr(B')$ has height 0, and since B' covers b, it follows from the theorem of Clifford

$$\zeta(1)=e\sum_{i=1}^{r}\varepsilon_i(1)\,,\qquad \text{where } \zeta\in Irr(B'),\ \varepsilon_1\in Irr(b)$$

and $r=|H:I_H(\varepsilon_1)|$. Since $H=PC$ and $I_H(\varepsilon_1)\supseteq C$ and e is an integer which divides $|I_H(\varepsilon_1):C|$, we have that e, r and $\varepsilon_i(1)$ are a power of p. Therefore $\zeta(1)$ is a power of p, and hence $k(B')=p^d v(B')$.

Secondly, since $H \lneqq G$, it follows $[H, D] \subseteq Ker\, B'$ by induction on $|G|$, in particular $D \subseteq Z(P)$ since $Ker\, B'$ is a p'-group. Then it holds that

$$\chi_D = e \sum_{i=1}^{s} \lambda_i\,, \qquad \text{where } \chi \in Irr\,(B),\ \lambda_i \in Irr\,(D)$$

and $s = |G : I_G(\lambda_1)|$. Since $D \subseteq Z(P)$, we have $I_G(\lambda_1) \supseteq P$ and so s is an integer prime to p. Therefore $\chi(1) = p^{a-d}$ means $s=1$. Hence

$$\left|\chi(x)\right| = \chi(1) \qquad \text{for all } \chi \in Irr\,(B) \text{ and all } x \in D^{\bullet}$$

This implies $[G, D] \subseteq Ker\, B = 1$, and we may assume $H = G$.

Step 2. Suppose $D \not\subseteq Z(G)$, then we may assume $|G : C_G(D)| = p$. From Step 1 we have that $|G : C_G(D)|$ is a power of p. Let H be a normal subgroup of G of index p which contains $C_G(D)$. Then there exists a block b of H such that $b^G = B$ and b is covered by B as in the proof of Step 1. Since D is abelian normal in G, a similar argument in Step 1 yields that $k(b) = p^d v(b)$. Then by induction on $|G|$ we have $[H, D] \subseteq Ker\, b$. If P_0 is a Sylow p-subgroup of H, then $D \subseteq Z(P_0)$ and hence $P_0 \subseteq C_G(D)$. Since $H = P_0 C_G(D)$, this implies $H = C_G(D)$. Thus we may assume $|G : C_G(D)| = p$.

Step 3. Final assertion.

Let $C = C_G(D)$, then by Step 2 we have $C \times C \lhd G \times G$ with index p^2. Suppose $D \not\subseteq Z(G)$, then there exists $x \in G - C$, and hence $(x^{-1}, x^{p-1}) \in G \times G - C \times C$ and normalizes $\Delta(D)$. Therefore $M = N_{G \times G}(\Delta(D))$ must be a normal subgroup of $G \times G$ of index p which contains $C \times C$. Now B is an indecomposable $R(G \times G)$-module with vertex $\Delta(D)$. If B_M is not indecomposable, then $B_M \cong b_1 \oplus \cdots \oplus b_p$, where b_i is an indecomposable RM-module, and they are $G \times G$-conjugate each other (see Isaacs and Scott [15], Lemma 2). Therefore $b_i^{G \times G} \cong B$ for all i, but Green correspondence means that B must determine uniquely such b_i that $b_i^{G \times G} \cong B$. This is a contradiction. Theorefore B_M is indecomposable, and again we have from Green correspondence that

$$(B_M)^{G \times G} \cong B \oplus B'\,,$$

where every indecomposable direct summand of B' has a vertex properly contained in $\Delta(D)$ under a $G \times G$-conjugation. Comparing the rank in the above isomorphism,

$$p^{2a-d+1} v(B) = p^{2a-d} v(B) + \mathrm{rank}_R\, B'\,,$$

and now $\mathrm{rank}_R\, B'$ is divisible by p^{2a-d+1} by the theorem of Green. This is impossible, since now $v(B) = 1$. Hence our assumption yields a contradiction, and we complete the proof of Theorem (4 B).

REMARK. Example (3 D) also shows that the condition $D \lhd P$ in Theorem (4 A) and (4 B) cannot be dropped.

COROLLARY (4 C). *Let B_0 be the principal block of G. Then*
1) *the following are equivalent,*
   a) $l(B_0)=v(B_0)$.
   b) $G=O_{p'pp'}(G)$ *and* $G/O_{p'p}(G)$ *is abelian.*
2) *the following are equivalent,*
   a) $m(B_0)=p^a n(B_0)$.
   b) $m(B_0)=p^a v(B_0)$.
   c) $k(B_0)=p^a v(B_0)$.
   d) $G=O_{p'p}(G)$ *and P is abelian.*

PROOF. 1). a)$\Rightarrow$b). From Theorem (4 A) we have $G=O_{p'pp'}(G)$ since $Ker\, B_0=O_{p'}(G)$. Then $G/O_{p'}(G)$ has only the principal block. Now since $\varphi(1)=1$ for all $\varphi \in IBr(B_0)$ and $Ker\, B_0^*=O_{p'p}(G)$, we have $G/O_{p'p}(G)$ is abelian.
b)$\Rightarrow$a). It is easy to see that $\varphi(1)=1$ for all $\varphi \in IBr(B_0)$. And this fact easily yields that $\Phi(1)=p^a$ for all $\Phi \in Pin(B_0)$. 2). The direction c)$\Rightarrow$d) is easy to see from Theorem (4 B) and $Ker\, B_0=O_{p'}(G)$. The converse is trivial, and other direction is clear from Corollary (3 C), 2).

REMARK. Corollary (4 C), 1) is a generalization of Corollary 2 in [5], in the sense that $v(B_0)=1$ means $l(B_0)=v(B_0)=1$. And Corollaries (3 C), (4 C) provide some conditions for G to be of p-length 1 which are related to the results of Isaacs and Smith [16] and Okuyama [17].

From the results of sections 3, 4, there exists the following relation between the conditions that $m(B)=p^a v(B)$, $n(B)=v(B)$ and the conditions in terms of characters in B.

COROLLARY (4 D). *Let B be a block of G with defect group D and defect d. Then the following hold.*
1) *Suppose $D \lhd P$ for a Sylow p-subgroup P of G. If $k(B)=p^d v(B)$, then $m(B)=p^a v(B)$.*
2) *If $m(B)=p^a v(B)$, then $k(B)=p^d w(B)$. If $m(B)=p^a v(B)$ and $v(B)=1$, then $k(B)=p^d v(B)$.*
3) *Suppose $D \lhd P$ for Sylow p-subgroup P of G. If $l(B)=v(B)$, then $n(B)=v(B)$.*

PROOF. 1). It follows from Theorems (3 B), (4 B).
2). It follows from Theorem (3 B) and Theorem 3 of Reynolds in [18].
3). It follows from Theorems (3 A), (4 A). But, as we mentioned in Remark (2 G), in fact, it turns out that the assumption "$D \lhd P$" can be dropped.

As the final remark, we will state some examples which show that some of our theorems do not work in general.

EXAMPLE (4 E).
1). It is possible to construct Example (3 D) for any prime p. Let H be a finite group having the unique block b of defect 0, and assume that $\zeta(1)=|H|_p$ for $\zeta\in Irr(b)$. Let G be a semi direct product of H by an element x of order p which is contained in $Aut(H)$. Then $\zeta^G=\chi_1+\cdots+\chi_p$, and the set $\{\chi_1,\cdots,\chi_p\}$ forms $Irr(B)$, where B is a block of G of defect 1 with defect group $\langle x\rangle$ which is not normal in and Sylow p-subgroup of G. Therefore $k(B)=pv(B)$, and hence $v(B)=n(B)=l(B)=1$, however it fails that $D\cdot Ker\,B\lhd G$. Hence it also fails that $m(B)=p^a v(B)$.

G, in the above, is also an example that the condition "$D\lhd P$" cannot be dropped for the theorems of Okuyama [17] which is a generalization of Corollary 3 in [16] and that of Corollary 2 in [5].

For example, G is the following group; $H=PSL(2,p^p)$ and $G=\langle x\rangle H$, where x is the field automorphism of order p.
2). In the second statement of Corollary (4 D), 2) the condition $v(B)=1$ is necessary. Let B be a block of defect 0, and $\chi(1)\gtrless p^a$ for $\chi\in Irr(B)$, then it holds $m(B)=p^a v(B)$, but fails $k(B)=p^d v(B)$.
3). Theorem (4 B) does not hold when $k(B)=p^d w(B)$, even if $D\lhd P$. Let $G=M_{23}$ (Mathieu group of degree 23), $p=3$ and B be the block of defect 1, where $Irr(B)=\{\chi_1,\chi_2,\chi_3\}$ and $\chi_i(1)=231=3\times7\times11$. Then a defect group D is normal in P, since P is abelian of order 9. It holds $k(B)=p^d w(B)$ (and hence $l(B)=w(B)=1$), but fails $[G,D]\subseteq Ker\,B=1$.

Therefore G is also an example that Theorem (4 A) does not hold when $l(B)=w(B)$, even if $D\rhd P$.

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