Hokkaido Mathematical Journal Vol. 11 (1982) p. 29-34

Linear parabolic equations in regions with re-entrant edges

By Ali AZZAM and Erwin KREYSZIG

(Received October 6, 1980; Revised January 12, 1981)

In a recent paper [1] we studied solutions of the parabolic equation

$$(1)\qquad Lu(x, t) = a_{ik}(x)u_{x_i x_k} + a_i(x, t)\, u_{x_i} + a(x, t)\, u - u_t = f(x, t),$$

$x=(x_1, \cdots, x_n)$, in a simply connected, bounded region $\Omega = G\times J \subset \boldsymbol{R}^{n+1}$, $n \geqq 2$, $J=\{t \mid 0<t\leqq T\}$, satisfying the conditions

$$(2\text{ a})\qquad u(x, 0) = 0,\ x \in \bar{G},$$

$$(2\text{ b})\qquad u\big|_{\partial G\times \bar{J}} = \phi(x, t),$$

under the following assumptions.

(A) $a_{ik} \in C^\alpha(\bar{G})$, a_i, a, $f \in C^\alpha(\bar{\Omega})$, $0<\alpha<1$,

(B) $\phi(x, 0)=0$, $\phi \in C^{2+\alpha}[\partial G \backslash E)\times \bar{J}] \cap C^0(\partial G\times \bar{J})$,

(C) $\omega(P)<\pi$ for all $P\in E$.

Here $E=\bigcup_i E_i$, where $E_1, \cdots, E_m$ are $(n-2)$-dimensional edges (the intersections of portions of hypersurfaces $\Gamma_1, \cdots, \Gamma_m$ constituting the boundary ∂G of G), and $\omega(P)$ is the angle between the images of the two Γ_j's corresponding to a point P: x^0 of E under the transformation of

$$a_{ik}(x^0)\, u^*_{x_i x_k} = 0$$

to canonical form.

Condition (C) means that the edges of the image of G are non-re-entrant. The question arises whether this restriction can be removed. This would be of practical importance, for the following reason.

It is well known that in physical and other applications, a great majority of boundary value or initial value problems are such that the given data or the boundaries of the domains have singularities (corners or edges); cf. [3], Chaps. V, VI, [4], [10], [13]. Not infrequently, some of those edges are re-entrant; for typical examples, see [5], Chap. 3, [7], Secs. 24.3-24.7, [8], Chap. 24, and [14], Chap. 8. In each such case, it is desirable to have knowledge about the kind of singularities of solutions and derivatives one has to expect, as a consequence of the singularities of the boundary.

The knowledge just mentioned is even more mandatory in finite difference, finite element or other numerical methods. For a general characterization of the problems, difficulties and theoretical and numerical results in this area, see G. E. Forsythe and W. R. Wasow [7], Sec. 23, and G. Strang and G. J. Fix [14], Chap. 8, and the given references; cf. also I. Babuška [2].

To improve convergence near singularities, for instance, by local "polar grids" (cf. [15]), reduction of mesh size, local use of series, or addition of suitable singular functions (cf. [14], p. 268), one must know the kind of possible singularities of solutions at edges. In this way, one can see that certain methods (*e.g.* [9]) are rather useless since they presuppose greater smoothness than would be achievable, whereas others (*e.g.* [11], [12]) do not extend to more than two dimensions since they are based on complex analysis. In all those investigations, re-entrant edges are worse than others and have attracted particular attention. For instance, the method in [11] fails to give best bounds in the re-entrant case. See also L. Fox [8], p. 304. Moreover, whereas for *exact* solutions, edges constitute a *local* smoothness problem, for approximate solutions, in the case of re-entrant edges, there is numerical evidence (obtained by Forsythe and others) which seems to indicate a *global* change of the order of magnitude of the discretization error.

All those facts point to a basically different situation for re-entrant edges.

In the present paper we shall indeed succeed in removing condition (C), that is, in extending our results obtained in [1] to the case of re-entrant edges. It is remarkable that this can be done simply by extending the previous method of proof, and that for this purpose we can use a barrier function which is even simpler than that used before.

In [1] we proved

THEOREM 1. *Let u be a bounded solution of the first boundary value problem* (1), (2) *in* Ω. *Assume that* (*A*), (*B*), (*C*) *hold. Then u, considered as a function of x, satisfies*

$$u\in C^{\mu}(\bar{\Omega}), \qquad \textit{where} \quad \mu=\begin{cases} 2 & \textit{if } \omega_0<\pi/2, \\ \dfrac{\pi}{\omega_0}-\varepsilon & \textit{otherwise}, \end{cases}$$

with $\omega_0=\max_{P\in E}\omega(P)$ *and arbitrarily small* $\varepsilon>0$.

We shall now obtain the extension of this result to the re-entrant case. We shall refer to [1] for those parts of the proof which remain practically unchanged. In particular, to avoid misunderstandings, we point to the fact that our general "strategy" is similar to that in [1], that is, in the special setting (below) we introduce an additional condition, and later we define suitable auxiliary functions which also satisfy that additional condition if the solution satisfies the others. Thus that additional condition is *not* imposed on the solution itself. Furthermore, all this is done so that we can later return to the solution without losing boundedness or smoothness properties.

THEOREM 2. *The conclusion of Theorem* 1 *continues to hold without assumption* (*C*).

PROOF. Since the case $\omega_0<\pi$ was considered in [1], we can assume that $\omega_0\geqq\pi$. We first consider the special case of a cylindrical sector

$$G=\left\{(r,\theta,x')\,\middle|\, r<\sigma,\ \beta<\theta<\beta+\omega_0,\ |x_i|<\sigma \text{ if } i>2\right\},$$

where $\sigma>0$, $\beta>0$ and sufficiently small, and

$$x_1=r\cos\theta\,,\qquad x_2=r\sin\theta\,,\qquad x'=(x_3,\cdots,x_n)\,.$$

The case of an arbitrary Ω will then follow by the mapping used in [1]. Accordingly, we first introduce notations corresponding to our present special setting:

Π_1 and Π_2 denote the two portions of the hyperplanes

$$x_2=x_1\tan\beta\,,\qquad x_2=x_1\tan(\beta+\omega_0)$$

bounding G laterally,

$$N_k=\left\{x\,|\,x\in G,\ |x|<k\right\},\qquad 0<k\leqq\sigma,$$

$$S_k=\partial N_k\cap(\Pi_1\cup\Pi_2)\,,$$

$$R_k=\Pi_1\cap\Pi_2\cap\bar{N}_k\,.$$

Then we state our modified problem

(3) $\qquad Lu=f$ in $N_\sigma\times J$,

(4 a) $\qquad u\big|_{t=0}=0$,

(4 b) $\qquad u\big|_{S_\sigma}=\phi(x,t)$,

and assumptions:

(A′) $a_{ik}\in C^\alpha(\bar{N}_\sigma)$, $a_{ik}(0)=\delta_{ik}$ for $i,\ k=1,2$, a_{ik} $(i>2$ or $k>2)$, a_i, a, $f\in C^\alpha(\bar{N}_\sigma\times J)$,

(B′) $\phi\in C^{2+\alpha}\big((S_\sigma\setminus R_\sigma)\times J\big)\cap C^0(S_\sigma\times J)$, $\phi(x,0)=0$.

Note that the condition $a_{ik}(0)=\delta_{ik}$, i, $k=1, 2$, is a consequence of the transformation which we use (as in [1]) in order that the new setting correspond to the setting in Theorem 1.

In the proof we shall also use the following assumption.

$$(\mathrm{B}^*) \qquad \psi\Big|_{R_\sigma}=0 .$$

This assumption is *not* a restriction of generality, because the function u^* defined by

$$u^*(x, t)=u(x_1, x_2, x', t)-\psi(0, 0, x', t)$$

satisfies all the conditions of the theorem, and the function $\psi^*=u^*|_{\partial G}$ satisfies all the conditions which ψ satisfies, as well as assumption (B*).

We prove that under assumptions (A′), (B′) and (B*), for a bounded solution u of (3), (4) there exists a number $c\in(0, \sigma)$ such that

$$(5) \qquad u\in C^\mu(\bar{N}_c\times J),\ \mu=\pi/\omega_0-\varepsilon \qquad \text{with arbitrarily small } \varepsilon>0 .$$

To prove (5) we first derive a bound

$$(6) \qquad \Big|u(x, t)\Big|\leqq Kr^\mu \qquad \text{with } \mu \text{ as in (5)}$$

in $\bar{N}_{r_0}\times J$, where $r_0<\sigma$ is suitable. Without restriction we assume that u is zero outside a hypersphere of radius r_0 about 0. We can accomplish this if we replace u by a function w, where $w=\xi u$, with $\xi\in C^\infty$, $\xi(|x|)=1$ when $|x|\leqq r_0$ and $\xi(|x|)=0$ when $|x|\geqq 2r_0$. Note that then $w=u$ when $|x|\leqq r_0$ and $w=0$ when $|x|\geqq 2r_0$. For the simplicity of writing we shall use u and r_0 rather than w and $2r_0$. Let

$$v(x)=-Kr^\mu \sin \lambda\theta ,$$

where $K>0$ is constant, $r^2=x_1^2+x_2^2$ and

$$\pi/\omega_0-\varepsilon=\mu<\lambda=\pi/(\omega_0+2\beta)$$

with sufficiently small $\beta>0$. For sufficiently small $r_0>0$ and sufficiently large K one can show that

$$Lv(x)\geqq f(x, t) \text{ in } \Omega_0=N_{r_0}\times J .$$

Hence in Ω_0,

$$L\Big(u(x, t)-v(x)\Big)\leqq 0 .$$

On the other hand, since $\sin \lambda\theta>\sin \lambda\beta$ for $\beta<\theta<\beta+\omega_0$, by taking K large we can make $u-v$ nonnegative on $\partial\Omega_0$. Hence by the maximum principle,

$$u(x, t)-v(x) \geqq 0 \qquad \text{in } \bar{\Omega}_0,$$

so that

$$u(x, t) \geqq -Kr^{\mu} \sin \lambda\theta \geqq -Kr^{\mu}.$$

The other part of (6) can be proved similarly. From (6) we obtain (5) by a Schauder type estimate as in [1]. Theorem 2 now follows by the mapping used in [1] which maps the general region in Theorems 1 and 2 onto the cylindrical sector. This completes the proof.

Furthermore, along the lines of [1] it is not difficult to prove

THEOREM 3. *For u as in Theorems* 1 *and* 2, *under assumptions* (A), (B),

$$\eta^{\delta} D_x u \in C^{\chi}(\bar{\Omega}),$$

where $\chi=\delta+\mu-1$, $1-\mu<\delta<2-\mu$, $0<\chi<1$, *and η is the distance from* (x, t) *to E.*

ACKNOWLEDGEMENT. We want to thank the Referee for two valuable suggestions which have helped to increase the lucidity of the presentation.

References

[1] A. AZZAM and E. KREYSZIG: On parabolic equations in n space variables and their solutions in regions with edges, Hokkaido Math. J. **9**, 140–154 (1980).

[2] I. BABUŠKA: Finite element method for domains with corners, Computing **6**, 264–273 (1970).

[3] H. S. CARSLAW and J. C. JAEGER: Conduction of Heat in Solids, 2nd ed., Clarendon Press, Oxford, 1959.

[4] R. COURANT and K. O. FRIEDRICHS: Supersonic Flow and Shock Waves, Interscience/Wiley, New York, 1948.

[5] E. R. G. ECKERT: Heat and Mass Transfer, McGraw-Hill, New York, 1959.

[6] G. FIX: Higher-order Rayleigh-Ritz approximations, J. Math. Mech. **18**, 645–657 (1969).

[7] G. E. FORSYTHE and W. R. WASOW: Finite-Difference Methods for Partial Differential Equations, Wiley, New York, 1960.

[8] L. FOX: Numerical Solution of Ordinary and Partial Differential Equations, Addison-Wesley, Reading, MA, 1962.

[9] S. GERSCHGORIN: Fehlerabschätzung für das Differenzenverfahren zur Lösung partieller Differentialgleichungen, Z. Angew. Math. Mech. **10**, 373–382 (1930).

[10] P. HARTMAN and A. WINTNER: On the solutions of the equation of heat conduction, Amer. J. Math. **72**, 367–395 (1950).

[11] P. LAASONEN: On the degree of convergence of discrete approximations for

the solutions of the Dirichlet problem, Ann. Acad. Sci. Fenn., Ser. AI **246,** 1–19 (1957).

[12] R. S. LEHMAN: Developments at an analytic corner of solutions of elliptic partial differential equations, J. Math. Mech. 8, 727–760 (1959).

[13] A. E. H. LOVE: A Treatise on the Mathematical Theory of Elasticity, 4th ed., Dover, New York, 1944.

[14] G. STRANG and G. J. FIX: An Analysis of the Finite Element Method, Prentice-Hall, Englewood Cliffs, N. J., 1973.

[15] E. A. VOLKOV: A mesh method for finite and infinite polygons and error bounds in terms of known quantities, Trudy Mat. Inst. Steklov **96,** 149–187 (1968) (Russian): translated in Proc. Steklov Inst. Math. No. **96,** 187–234 (1968).

Department of Mathematics
University of Windsor