Hokkaido Mathematical Journal Vol. 11 (1982) p. 106-110

# Solvability of some groups

By Masahiko MIYAMOTO

(Received May 1, 1980)

**Introduction.** K. Nomura posed the following problem: Let $G$ be a finite group that has a large inner automorphism consition. Then, is $G$ a solvable group? Especially, we call a finite group $G$ to be an AI-group if $G$ satisfies the following: $N_G(A)/C_G(A)$ is isomorphic to the full automorphism of $A$ for every Abelian subgroup $A$ of $G$.

The purpose in the paper is to show the following theorem:

THEOREM A. *Let $G$ be a finite AI-group, then $G$ is solvable.*

To prove the theorem, we introduce the weaken condition. A finite group $G$ is called an $A_3I$-group if $G$ satisfies the following condition: For every Abelian $3'$-subgroup $A$ and a 3-subgroup $B$ of $C_G(A)$, $C_G(B)\cap N_G(A)/C_G(B)\cap C_G(A)\geqq O^3(\mathrm{Aut}(A))$. We will say that $C_G(B)$ covers $O^3(\mathrm{Aut}(A))$ if the condition holds.

Using the above notion, we will change the form of the theorem.

THEOREM B. *The following hold*:

a) *All AI-groups are $A_3I$-groups.*

b) *Every $A_3I$-group is solvable.*

Clearly, Theorem A is an immediate consequence of Theorem B. Most of our notation is standard and taken from [1]. All groups considered in this paper will be finite. Let $G$ be a group. Then $F(G)$ denotes the Fitting subgroup of $G$.

**2. Preliminary lemmas.** In this section, we will search the properties of $A_3$I-groups.

LEMMA 2.1. *Let $G$ be a finite $A_3I$-group and $B$ be a 3-subgroup of $G$. Then the following hold*;

a) *$i(G)=1$, the conjugate class of involutions of $G$ is one.*

b) *$Z(P)$ is of order $p$ for a Sylow $p$-subgroup $P$ of $G$ and every prime divisor $p$ of the order of $G$, $(p\neq 3)$.*

c) *$C_G(B)$ is an $A_3I$-group.*

d) *$G/O_3(G)$ is an $A_3I$-group.*

e) *$O^3(G)$ is an $A_3I$-group.*

PROOF. By the definition of $A_3I$-groups, we easily get the above statements.

LEMMA 2.2. *Let $G$ be a finite solvable $A_3I$-group. Assume that $O_3(G)=1$ and $O^3(G)=G$. Then one of the following holds;*

a) *$G/F(G)\cong Z/2Z$ and $F(G)$ is cyclic of odd prime order.*

b) *$G$ is isomorphic to $Z/2Z$ or the quaternion group of order 8.*

c) *A Sylow 2-subgroup of $G$ is isomorphic to $Q_8$ and moreover, one of the following holds;*

i) *$G\trianglerighteq G_1\cong Z/4Z\times Z/pZ$ and $G/G_1\cong Z/2Z$,*

ii) *$G\trianglerighteq G_1\cong Z/2Z\times Z/qZ\times Z/pZ$ and $G/G_1\cong Z/2Z\times Z/2Z$.*

PROOF. If $F(G)$ is of odd order, then $F(G)$ is cyclic since $O^3(GL_2(p))$ is not solvable for $p\geq 5$. Therefore, $G/F(G)$ is Abelian and $|G|_2=2$. The desired statement a) follows from the easy calculation. Next, we assume $O_2(G)\neq 1$. By Lemma 2.1 a), every involution of $G$ is contained in $V=\Omega_1(Z(O_2(G)))$. Since $G$ is an $A_3$I-group, $G/C_G(V)\cong O^3(\mathrm{Aut}(V))$. If $m_2(G)=2$, then $G/C_G(V)$ is isomorphic to the symmetric group $S^3$ on 3 letters. Let $P$ be a Sylow 3-subgroup of $G$, then $N_G(P)$ is of even order, which contradicts $N_G(P)\cap V=1$. Clearly, the solvability of $G$ implies $m_2(G)=1$. In this case, a Sylow 2-subgroup of $G$ is isomorphic to $Q_8$ or $Z/2Z$, and we obtain the desired statement b) and c).

**3. Proof of the statement a) in Theorem B.** Let $G$ be an AI-group but not $A_3$I-group. Namely, there is an Abelian $3'$-subgroup $A$ and a 3-subgroup $B$ in $C_G(A)$ such that $C_G(B)$ does not cover $O^3(\mathrm{Aut}(A))$. Clearly, we can choose $B$ to be a Sylow 3-subgroup of $C_G(A)$. Let $B_0$ be a maximal normal Abelian subgroup of $B$, then $N_C(A)/C_C(A)\cong\mathrm{Aut}(A)$, where $C=C_G(B_0)$ First, we assert that $B_0$ is a Sylow 3-subgroup of $C_G(B_0\times A)$. In fact, this follows from the maximality of $B_0$. Next, since $B$ is a Sylow 3-subgroup of $BC_G(B_0)\cap C_G(A)$, $N_G(B)\cap C_G(B_0)$ induces $O^3(\mathrm{Aut}(A))$ by the Frattini argument. Since $B_0$ is a self centralizer subgroup of $B$, we have that $C_G(B)$ induces $O^3(\mathrm{Aut}(A))$, a contradiction.

This completes the proof of the statement a).

**4. Proof of the statement b) in Theorem B.** Let $G$ be a minimal counterexample. Clearly, we have $O_3(G)=1$ and $O^{2'}(G)=G$. For a 3-subgroup $B$ of $G$, since $C_G(B)$ is an $A_3$I-group by Lemma 2.1, the minimality of $G$ means that $m_2(C_G(B))\leq 1$ by Lemma 2.2. Therefore, we have $m_2(\mathrm{G})\leq 3$. First, we assume $m_2(G)=3$. Let $V$ be a subgroup of type $(2, 2, 2)$.

LEMMA 4.1. *$C_G(V)$ is a 2-group.*

PROOF. As we showed, since $m_2(C_G(B)\leq 1$ for a 3-subgroup $B$, we have that $C_G(V)$ is a $3'$-group and $N_G(V)/C_G(V)$ is isomorphic to $\mathrm{GL}(3, 2)$. Since $G$ is an $A_3$I-group, every odd Sylow subgroup of $C_G(V)$ is cyclic, which means that $[C_G(V), a]\subseteq O_2(C_G(V)$ for an element $a$ of order 3 in $N_G(V)$. Let $R$ be a Sylow $p$-subgroup of $N_G(V)$. Since $C_G(a)$ is isomorphic to one of the groups in the list of Lemma 2.2, we get that $N_{C_G(V)}(R)$ is of even order, which contradicts $N_G(R)\cap V=1$.

LEMMA 4.2. $C_G(V)=V$.

PROOF. Suppose false and set $U=C_G(V)$. Since $m_2(C_G(a))=1$ and $C_{C_G(V)}(a)$ is a subgroup of $Q_8$ by Lemma 2.2, we obtain that $U/V$ is isomorphic to one of the following:

a) $U/V\cong(2, 2, 2)$;

b) $U/V\cong(2, 2, 2)\times(2, 2, 2)$; and

c) $U\supseteq U_1\supseteq V$ such that $U/U_1\cong(2, 2, 2)$ and $U_1/V\cong(2, 2, 2)$.

In any case, $U$ has a maximal normal Abelian subgroup $U_0\supseteq V$ such that $N_G(U_0)$ does not cover $C_G(\Omega_1(U_0))\cap O^3(\mathrm{Aut}(U_0))$, a contradiction.

Let $A$ be a subgroup of $V$ with $A\cong(2, 2)$.

LEMMA 4.3. *$C_G(A)$ is a 2-subgroup.*

PROOF. Suppose false. By Lemma 2.2, we get that $C_G(A)$ is a $3'$-subgroup and $N_G(A)/C_G(A)$ is isomorphic to the Symmetric group $S^3$ on 3 letters, which means that every Sylow subgroup of $C_G(A)$ of odd order is cyclic. Therefore, for an element $a$ of $N_G(A)$ of order 3, we obtain that $[C_G(A), a]$ is a 2-subgroup. Let $P$ be a Sylow $p$-subgroup of $C_G(a)\cap C_G(A)$, then since $C_G(a)$ is a solvable $A_3$I-group and $m_2(C_G(P))=2$, we have that $|C_{C_G(A)}(a)|_2=2$. Since an odd element of $C_G(a)\cap C_G(A)$ centralizes $C_G(a)\cap O^2(C_G(A))$, we have $C_G(a)\cap O^2(C_G(A))=1$ as we showed in Lemma 4.2. Therefore, we get that the nilpotency class of $O_2(C_G(A))$ is at most 2 by Theorem 8.1 in [2]. Since $N_G(V)/V\cong\mathrm{GL}(3, 2)$ and $C_G(a)\cap O^2(C_G(A))=1$, we get $Z(O_2(C_G(A))=A$. Furthermore, since $C_G(V)=V$, we have $|O_2(C_G(A))/A|\leq 2^4$, which means that $O_2(C_G(A))/A$ is an elementary Abelian. If $P$ acts on $O_2(C_G(A)/A$ faithfully, we get $p=5$ since we already got $C_G(a)\cap O^2(C_G(A))=1$, which contradicts $\mathrm{Aut}(P)\cong Z/4Z$. Therefore, $P$ centralizes $O_2(C_G(A))$. Let $U_1$ be a maximal normal Abelian subgroup of $O_2(C_G(A))$, then $N_G(U_1)\cap C_G(A\times P)$ covers $O^3(\mathrm{Aut}(U_1)\cap C_G(A))$ by the definition of $S_3$.I-group, which contradicts $C_G(P)\cap C_G(A)=P\times O_2(C_G(A))$ by Lemma 2.2.

LEMMA 4.4. $N_G(A)\subseteq N_G(V)$.

PROOF. Since $N_G(V)$ covers $\mathrm{Aut}(A)$, it is sufficient to show $C_G(A)\subseteq$

$N_G(V)$. Let $a$ be a 3-element of $N_G(A)$ and $v$ be an element in $V-A$. We may assume that $a$ is contained in $N_G(V)$. Then, we have $C_U(a)=\langle \bar{v}\rangle$ where $U=C_G(A)/A$. Furthermore, since $C_U(v)=N_G(V)\cap C_G(A)$, we obtain that $C_U(v)$ is of order 8. But, since $a$ acts on $Z(U)^{\#}$ fixed point free, we have $C_U(v)=\langle \bar{v}\rangle \times Z(U)$, which implies $v \not\in U'$ and that $a$ acts on $[C_G(A), a]$ fixed point free. Set $W=[C_G(A), a]$. By Theorem 8.1 in [2], we get $cl(W)\leq 2$ and so we have $Z(W)=A$ since $G$ is an $A_3$.I-group. Since $a$ acts on $W$ fixed point free and $C_U(v)$ is of order 8, we obtain that $W/A$ is of order 16. In this case, we can check that there is an Abelian subgroup $U_1$ of $W$ containing $A$ such that $N_G(U_1)$ does not cover $O^3(\mathrm{Aut}(U_1) \cap C_G(A))$, a contradiction.

LEMMA 4.5. *$V=\Omega_1(S)$ for a Sylow 2-subgroup $S$ of $G$.*

PROOF. Suppose false, then there is an involution $i$ in $N_S(V)-V$ for a Sylow 2-subgroup $S$ containing $V$. Let $A=C_V(i)$, then we have $m_2(A)=2$ and $N_G(A)/V\cong S^4$, the symmetric group on 4 letters. By Lemma 4.4, we also get $N_G(A)\subseteq N_G(\langle A, i\rangle)$. However, $a$ does not normalize $\langle A, i\rangle$, where $a$ is a 3-element in $N_G(V)\subset N_G(A)$, by the structure of $\mathrm{GL}(3, 2)$, a contradiction.

LEMMA 4.6. $m_2(G)\neq 3$.

PROOF. As we showed, we got $V=\Omega_2(S)$ for a Sylow 2-subgroup $S$ of $G$. But, in this case, $V$ is a strongly closed Abelian subgroup in $S$ with respect, to $G$. Therefore, we can get a contradiction by the result of Goldschmidt [1].

Since $m_2(G)\leq 3$, we next assume $m_2(G)=2$.

LEMMA 4.7. *Let $S(G)$ be the unique maximal normal solvable subgroup of $G$. Then $S(G)$ is of odd order.*

PROOF. Suppose false and set $\bar{G}=G/O(G)$ and $V=\Omega_1(Z(O_2(\bar{G})))$. Since $i(\bar{G})=1$, every involution of $\bar{G}$ is contained in $V$ and so we get that $V$ is a four-group and $\bar{G}/C_{\bar{G}}(V)\cong S^3$, the symmetric group on 3 letters. Let $P$ be a Sylow 3-subgroup of $\bar{G}$, then $N_{\bar{G}}(P)$ is of even order, but $N_V(P)=1$, a contradiction.

Let$\bar{G}$ $=G/O(G)$, $\bar{E}=F^*(\bar{G})$, and $E$ be the inverseimage of $\bar{E}$ in $G$. Clearly $\bar{E}$ is a simple group.

LEMMA 4.8. $O(G)=1$.

PROOF. Suppose false and let $P$ be Sylow $p$-subgroup of $F(G)$. Then we can easily see that every element of $G$ of order $p$ is contained in $\Omega_1(Z(P))$. If $m(P)=1$, then we get that $P$ is a cyclic Sylow $p$-subgroup of $G$ and

$G/P$ is also an $A_3$.I-group, a contradiction. Since $m_2(G)=2$, we have $m(P)=2$ and $V=\Omega_1(Z(P))$ is of type $(p, p)$. Since $G/C_G(V)\cong O^3(\mathrm{GL}(2, p))$ and $O_2(O^3(\mathrm{GL}(2, p))\neq 1$ and $S(G)$ is of odd order, $C_G(V)$ is not solvable, in particular, $C_G(V)\supseteq E$, but $G/C_G(V)$ is not solvable, a contradiction.

LEMMA 4.9. $m_2(G)\neq 2$.

PROOF. As we showed, $E=F^*(G)$ is a simple group with $m_2(E)=2$. Then $E$ is of known type and isomorphic to one of the following:
$L_2(q)$, $L_3(q)$, $U_3(q)$, $q$ odd, $U_3(4)$, $A_7$, or $M_{11}$.
In any groups, we can easily check that they are not $A_3$.I-groups.

Last case is $m_2(G)=1$. Since $G$ is an $A_3$.I-group, a Sylow 2-subgroup of $G$ is isomorphic to $Q_8$. Let $i$ be an involution of $G$.

LEMMA 4.10. *$O(G)=1$, in particular, $G=C_G(i)$.*

PROOF. Let $P$ be a Sylow $p$-subgroup of $G$ for an odd prime $p$ $(\neq 3)$ Then we have $m(P)=1$, since $m_2(G)=1$. Therefore, $P$ is of order $p$. In particular, $F(O(G))$ is cyclic and a Hall subgroup of $G$. Therefore, we can easily see that $G/F(O(G))$ is also an $A_3$.I-group, a contradiction.

Finally, we have that $F^*(G)$ is a quasi-simple group. In particular, $F^*(G)$ is isomorphic to $SL_2(q)$ $q\equiv 3 \pmod 8$. We can check that $G$ is not an $A_3$.I-group for each $q$, a contradiction. This completes the proof of the statement b) of Theorem B.

## References

[1] D. M. GOLDSCHMIDT: 2-Fusion in Finite groups, Ann. Math. Vol. 99 (1974), 70–117.

[2] G. HIGMAN: Odd Characterization of Finite Simple Groups, preprint.

Department of Mathematics
Faculty of Science
Ehime University