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Mean curvature for certain p-planes in Sasakian manifolds

Dedicated to the memory of Professor Yoshie Katsurada

By Shigeyoshi FUJIMURA

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Introduction

Let (M, g) be an m-dimensional Riemannian manifold with a metric tensor g. We denote by $K(X, Y)$ the sectional curvature for a 2-plane spanned by tangent vectors X and Y at $x\in M$, and by π a p-plane at $x\in M$. Let $\{e_1, \cdots, e_m\}$ be an orthonormal base of the tangent space at $x\in M$ such that $\{e_1, \cdots, e_p\}$ spans π, which is called an adapted base for π. S. Tachibana [7] defined the mean curvature $\rho(\pi)$ for π by

$$\rho(\pi)=\frac{1}{p(m-p)}\sum_{a=1}^{p}\sum_{b=p+1}^{m}K(e_a, e_b),$$

which is independent of the choice of adapted bases for π, and proved the following:

THEOREM A (S. Tachibana, [7]). *In a Riemannian manifold (M, g) of dimension $m>2$, if the mean curvature for p-plane is independent of the choice of p-planes at each point, then*

(i) *for $p=1$ or $m-1$, (M, g) is an Einstein space,*

(ii) *for $2\leqq p\leqq m-2$ and $2p\neq m$, (M, g) is of constant curvature,*

(iii) *for $2p=m$, (M, g) is conformally flat.*

The converse is true.

Taking holomorphic $2q$-planes or antiholomorphic p-planes instead of p-planes, analogous results in Kählerian manifolds have been obtained.

THEOREM B (S. Tachibana [8] and S. Tanno [9]). *In a Kählerian manifold (M, g, J) of dimension $2n\geqq 4$, if the mean curvature for $2q$-plane is independent of the choice of holomorphic $2q$-planes at each point, then*

(i) *for $1\leqq q\leqq n-1$ and $2q\neq n$, (M, g, J) is of constant holomorphic sectional curvature,*

(ii) *for $2q=n$, the Bochner curvature tensor vanishes.*

The converse is true.

normal pair $\{X, Y\}$, *then*

$$K(X', Y')=K(X, Y)$$

for the 2-plane spanned by any $X'\in C(X)$ and $Y'\in C(Y)$.

PROOF. Putting $Y_1=\alpha Y+\beta\phi Y$ $(\alpha^2+\beta^2=1)$, since $\{X, Y_1\}$ is a ϕ-antiholomorphic orthonormal pair, we have

$$K(X, Y_1)=K(X, \phi Y_1),$$

from which it follows that

THEOREM C (K. Iwasaki and N. Ogitsu [3]). *In a Kählerian manifold (M, g, J) of dimension $2n \geqq 4$, if the mean curvature for p-plane is independent of the choice of antiholomorphic p-planes at each point, then*

(i) *for $p=1$, (M, g, J) is an Einstein space,*

(ii) *for $2 \leqq p \leqq n-1$, (M, g, J) is of constant holomorphic sectional curvature,*

(iii) *for $p=n$, the Bochner curvature tensor vanishes.*

The converse is true.

L. Vanhecke ([10], [11]) generalized Theorems B and C, and the present author [2] obtained analogous results in quaternion Kählerian manifolds.

The main purpose of this paper is to prove analogous results in Sasakian manifolds.

§ 1. Sasakian manifolds ([1], [6]).

Let (M, ϕ, ξ, η, g) be a Sasakian manifold of dimension $2n+1 \geqq 5$, that is, a manifold M which admits a 1-form η, a vector ξ, a tensor ϕ of type (1, 1) and a metric tensor g satisfying

$$\eta(\xi)=1\,, \tag{1.1}$$

$$\begin{aligned}
K(X, Y) &= K(X', Y') = K(X', \phi Y') \\
&= \frac{1}{4}\big\{K(X, \phi X) + K(Y, \phi Y) + 2K(X, \phi Y) \\
&\quad +2R(X, \phi X, X, \phi Y) - 2R(X, \phi X, Y, \phi X) \\
&\quad +2R(X, \phi X, Y, \phi Y) + 2R(X, \phi Y, Y, \phi X) \\
&\quad -2R(X, \phi Y, Y, \phi Y) + 2R(Y, \phi X, Y, \phi Y)\big\}\,.
\end{aligned}$$

From this identity, (1.12), (1.13) and (1.15), we have Lemma 5.

Using the theorem of K. Ogiue [5] and Lemma 5, we can obtain

THEOREM 1. *Let (M, ϕ, ξ, η, g) be a Sasakian manifold of dimension $2n+1 \geqq 7$. If the sectional curvature $K(X, Y)$ is independent of the choice of the ϕ-antiholomorphic orthonormal pair $\{X, Y\}$ at each point, then (M, ϕ, ξ, η, g) is of constant ϕ-holomorphic sectional curvature.*

§ 2. Contact Bochner curvature tensor.

The contact Bochner curvature tensor of a Sasakian manifold (M, ϕ, ξ, η, g) of dimension $2n+1$ is defined by

$$\begin{aligned}
&B(X, Y, Z, W) \\
&\quad = R(X, Y, Z, W) + U(X, Z)\, g(\phi Y, \phi W) - U(Y, Z)\, g(\phi X, \phi W) \\
&\quad + U(Y, W)\, g(\phi X, \phi Z) - U(X, W)\, g(\phi Y, \phi Z) + V(X, Z)\, g(\phi Y, W) \\
&\quad - V(Y, Z)\, g(\phi X, W) + V(Y, W)\, g(\phi X, Z) - V(X, W)\, g(\phi Y, Z) \\
&\quad + 2V(X, Y)\, g(\phi Z, W) + 2V(Z, W)\, g(\phi X, Y)
\end{aligned}$$

where

$$U(X, Y) = \frac{1}{2(n+2)}\Big\{R_1(X, Y) - \frac{S-6n-8}{4(n+1)}\, g(X, Y)$$

(1. 9) $$R(X, \phi X, Y, \phi Y) = R(X, Y, X, Y) + R(X, \phi Y, X, \phi Y) + 2g(X, X)\, g(Y, Y) - 2g(X, Y)^2 - 2g(X, \phi Y)^2,$$

(1. 10) $$K(X, \xi) = 1 \text{ for } X \neq 0,$$

(1. 11) $$R_1(\phi X, \phi Y) = R_1(X, Y).$$

If an orthonormal pair $\{X, Y\}$ at $x \in M$ satisfies

$$g(X, \phi Y) = g(X, \xi) = g(Y, \xi) = 0,$$

$\{X, Y\}$ will be called a ϕ-antiholomorphic orthonormal pair, and a 2-plane $C(X)$ spanned by X and ϕX orthogonal to ξ will be called a ϕ-holomorphic section determined by X. Then from Lemmas 1 and 2, we have

LEMMA 3. *For any ϕ-antiholomorphic orthonormal pair $\{X, Y\}$,*

(1. 12) $$R(X, \phi X, X, \phi Y) + R(X, \phi X, \phi X, Y) = 0,$$

(1. 13) $$R(X, \phi Y, Y, \phi X) + R(X, \phi Y, \phi Y, X) = 1,$$

(1. 14) $$R(X, Y, \phi Y, \phi X) = K(X, Y) - 1,$$

(1. 15) $$R(X, \phi X, Y, \phi Y) = -K(X, Y) - K(X, \phi Y) + 2.$$

LEMMA 4. *If $K(X, Y) = K(X, \phi Y)$ for any ϕ-antiholomorphic orthonormal pair $\{X, Y\}$, then*

$$K(X', Y') = K(X, Y)$$

for the 2-plane spanned by any $X' \in C(X)$ and $Y' \in C(Y)$.

PROOF. Putting $Y_1 = \alpha Y + \beta \phi Y$ $(\alpha^2 + \beta^2 = 1)$, since $\{X, Y_1\}$ is a ϕ-antiholomorphic orthonormal pair, we have

$$K(X, Y_1) = K(X, \phi Y_1),$$

from which it follows that

(1. 16) $$R(X, Y, \phi Y, X) = R(X, Y, Y, \phi X) = 0.$$

From (1. 8), (1. 13), (1. 14) and (1. 16), we have this lemma.

LEMMA 5. *For any ϕ-anitholomorphic orthonormal pair $\{X, Y\}$, $K(X, Y) = K(X, \phi Y)$ holds if and only if*

$$8K(X, Y) = K(X, \phi X) + K(Y, \phi Y) + 6.$$

PROOF. Since $X' = \dfrac{X+Y}{\sqrt{2}}$ and $Y' = \dfrac{X-Y}{\sqrt{2}}$ constitute a ϕ-antiholomorphic orthonormal pair, from Lemma 4 we have

$$
\begin{aligned}
K(X, Y) &= K(X', Y') = K(X', \phi Y') \\
&= \frac{1}{4}\Big\{K(X, \phi X) + K(Y, \phi Y) + 2K(X, \phi Y) \\
&\quad + 2R(X, \phi X, X, \phi Y) - 2R(X, \phi X, Y, \phi X) \\
&\quad + 2R(X, \phi X, Y, \phi Y) + 2R(X, \phi Y, Y, \phi X) \\
&\quad - 2R(X, \phi Y, Y, \phi Y) + 2R(Y, \phi X, Y, \phi Y)\Big\}.
\end{aligned}
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From this identity, (1.12), (1.13) and (1.15), we have Lemma 5.

Using the theorem of K. Ogiue [5] and Lemma 5, we can obtain

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The contact Bochner curvature tensor of a Sasakian manifold (M, ϕ, ξ, η, g) of dimension $2n+1$ is defined by

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\begin{aligned}
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&\quad + U(Y, W)\, g(\phi X, \phi Z) - U(X, W)\, g(\phi Y, \phi Z) + V(X, Z)\, g(\phi Y, W) \\
&\quad - V(Y, Z)\, g(\phi X, W) + V(Y, W)\, g(\phi X, Z) - V(X, W)\, g(\phi Y, Z) \\
&\quad + 2V(X, Y)\, g(\phi Z, W) + 2V(Z, W)\, g(\phi X, Y)
\end{aligned}
$$

where

$$
\begin{aligned}
U(X, Y) &= \frac{1}{2(n+2)}\Big\{R_1(X, Y) - \frac{S-6n-8}{4(n+1)}\, g(X, Y) \\
&\qquad + \frac{S+10n+8}{4(n+1)}\, \eta(X)\, \eta(Y)\Big\}, \\
V(X, Y) &= \frac{1}{2(n+2)}\Big\{R_1(\phi X, Y) - \frac{S+4n^2+6n}{4(n+1)}\, g(\phi X, Y)\Big\},
\end{aligned}
$$

and S denotes the scalar curvature ([4]).

Since $g(X, \xi) = \eta(X)$, $\eta\phi = 0$ and $R_1(X, \xi) = 2n\eta(X)$, from (1.1) we have

$$
U(X, \xi) = \eta(X), \quad V(X, \xi) = 0 .
$$

From these identities, (1.3) and (1.5), we see

LEMMA 6. *For any vectors X, Y and Z, $B(\xi, X, Y, Z)$ vanishes.*

Assume that $K(X, Y) = K(X, \phi Y)$ for any ϕ-antiholomorphic orthonormal pair $\{X, Y\}$, and let $\{e_1, \cdots, e_n, \phi e_1, \cdots, \phi e_n, \xi\}$ be an orthonormal base of a tangent space $T_x(M)$ at $x \in M$ such that $X = e_1$ for a unit vector X orthogonal to ξ. Then, from (1.10), (1.11) and Lemma 5, we have

$$\begin{aligned} R_1(X, X) &= K(X, \phi X) + K(X, \xi) + 2\sum_{a=2}^{n} K(X, e_a) \\ &= \frac{n+2}{4} K(X, \phi X) + \frac{1}{4}\sum_{a=1}^{n} K(e_a, \phi e_a) + \frac{3n-1}{2}, \end{aligned}$$

$$\begin{aligned} S &= \sum_{a=1}^{n} \left\{ R_1(e_a, e_a) + R_1(\phi e_a, \phi e_a) \right\} + R_1(\xi, \xi) \\ &= (n+1)\sum_{a=1}^{n} K(e_a, \phi e_a) + n(3n+1). \end{aligned}$$

Thus we get

$$R_1(X, X) = \frac{n+2}{4} K(X, \phi X) + \frac{S + 3n^2 + 3n - 2}{4(n+1)}. \tag{2.1}$$

From (1.11), (2.1) and Lemma 5, we have

LEMMA 7. *If $K(X, Y) = K(X, \phi Y)$ for any ϕ-antiholomorphic orthonormal pair $\{X, Y\}$, $B(X, Y, Y, X)$ and $B(X, \phi X, \phi X, X)$ vanish.*

LEMMA 8. *Assume that $K(Z, W) = K(Z, \phi W)$ for any ϕ-antiholomorphic orthonormal pair $\{Z, W\}$. If unit vectors X and Y orthogonal to ξ are mutually orthogonal and satisfy $Y \notin C(X)$, then $B(X, Y, Y, X)$ vanishes.*

PROOF. Let Y' be a unit vector given by $\alpha\{Y - g(Y, \phi X)\phi X\}$ $(\alpha > 0)$. Then we can take an orthonormal base $\{e_1, \cdots, e_n, \phi e_1, \cdots, \phi e_n, \xi\}$ such that $e_1 = X$ and $e_2 = Y'$. Thus, from (1.5), (1.7) and (1.8), we have

$$\begin{aligned} B_{121^*1} &= R_{2111^*} - \frac{2}{n+2} R_{21^*} \\ &= R_{2111^*} - \frac{2}{n+2}\left(\sum_{a \neq 2} R_{2aa1^*} - \sum_{a \neq 1} R_{1aa2^*} \right) \end{aligned} \tag{2.2}$$

where $e_{n+a} = \phi e_a = e_{a^*}$, $R_{hijk} = R(e_h, e_i, e_j, e_k)$ and $R_{ij} = R_1(e_i, e_j)$ $(h, i, j, k = 1, \cdots, 2n)$. For $n \geqq 3$ and $a \neq 1, 2$, using Lemma 5, we have

$$\begin{aligned} K\left(\frac{e_2 + e_{1^*}}{\sqrt{2}}, e_a \right) &= R_{2aa1^*} + \frac{1}{8} K_{aa^*} + \frac{1}{16}(K_{11^*} + K_{22^*}) + \frac{3}{4} \\ &= \frac{1}{8}\left\{ K\left(\frac{e_2 + e_{1^*}}{\sqrt{2}}, \frac{e_{2^*} - e_1}{\sqrt{2}} \right) + K_{aa^*} + 6 \right\} \end{aligned}$$

where $K_{ij} = K(e_i, e_j)$. From this identity, (1.8), (1.14), (1.15) and Lemma 5, we have

$$(2.3)\qquad R_{2aa1^*}=\frac{1}{16}\left\{2K\left(\frac{e_2+e_{1^*}}{\sqrt{2}},\ \frac{e_{2^*}-e_1}{\sqrt{2}}\right)-K_{11^*}-K_{22^*}\right\}$$
$$=\frac{1}{8}(R_{2111^*}-R_{1222^*}),$$

$$(2.4)\qquad R_{1aa2^*}=\frac{1}{8}(R_{1222^*}-R_{2111^*}).$$

Substituting (2.3) and (2.4) into (2.2), we have

$$(2.5)\qquad B(X, Y', \phi X, X)=\frac{1}{2}(R_{2111^*}+R_{1222^*}).$$

For $n=2$, we also have (2.5). On the other hand, since $K(X, Y'')=K(X, Y')$ for $Y''=\dfrac{Y'+\phi Y'}{\sqrt{2}}$ by virtue of Lemma 4, we have

$$(2.6)\qquad R(X, Y', X, \phi Y')=0.$$

Exchanging X and Y' in (2.6) for $\dfrac{X+Y'}{\sqrt{2}}$ and $\dfrac{X-Y'}{\sqrt{2}}$ respectively, from (2.6) we have

$$(2.7)\qquad R(X, Y', Y', \phi Y')+R(Y', X, X, \phi X)=0.$$

From (2.5) and (2.7), it follows that $B(X, Y', \phi X, X)$ vanishes. Thus, using Lemma 7, we see that $B(X, Y, Y, X)$ vanishes.

Next for any vectors X, $Y\in T_x(M)$, we can put

$$X=\alpha X'+\beta\xi,\quad Y=\alpha' Y'+\beta'\xi$$

for certain unit vectors X', $Y'\in T_x(M)$ orthogonal to ξ. From Lemma 6 we have

$$B(X, Y, Y, X)=\alpha^2\alpha'^2 B(X', Y', Y', X').$$

Since we can put $Y'=\gamma X'+\delta Y''$ for a certain unit vector $Y''\in T_x(M)$ such that Y'' is orthogonal to X', using Lemma 8, we see that $B(X', Y'', Y'', X')$ vanishes, and therefore $B(X, Y, Y, X)$ vanishes. Thus we can see that B vanishes, using Lemma 6 and the following algebraic properties:

$$B(X, Y, Z, W)=-B(Y, X, Z, W)=B(Z, W, X, Y),$$
$$B(X, Y, Z, W)+B(X, Z, W, Y)+B(X, W, Y, Z)=0$$

for any vectors X, Y, Z and W.

Conversely, if B vanishes, for any ϕ-antiholomorphic orthonormal pair $\{X, Y\}$ we have

$$K(X, Y)=\frac{1}{2(n+2)}\left\{R_1(X, X)+R_1(Y, Y)-\frac{S-6n-8}{2(n+1)}\right\}$$
$$=K(X, \phi Y).$$

Therefore, together with Lemma 5 we can obtain

THEOREM 2. *In a Sasakian manifold of dimension* $2n+1\geqq 5$, *the following conditions are equivalent to each other*:

(i) *For any* ϕ-*antiholomorphic orthonormal pair* $\{X, Y\}$,

$$K(X, Y)=K(X, \phi Y).$$

(ii) *For any* ϕ-*antiholomorphic orthonormal pair* $\{X, Y\}$,

$$8K(X, Y)=K(X, \phi X)+K(Y, \phi Y)+6.$$

(iii) *The contact Bochner curvature tensor vanishes.*

§ 3. Mean curvature for ϕ-holomorphic $2q$-plane.

In a Sasakian manifold (M, ϕ, ξ, η, g) of dimension $2n+1$, if a $2q$-plane π in $T_x(\mathrm{M})$ satisfies $\eta(\pi)=\{0\}$ and $\phi\pi\subset\pi$, such a $2q$-plane π will be called the ϕ-holomorphic $2q$-plane. Then we can take an orthonormal base $\{e_1, \cdots, e_n, \phi e_1, \cdots, \phi e_n, \xi\}$ of $T_x(M)$ such that π is spanned by $\{e_1, \cdots, e_q, \phi e_1, \cdots, \phi e_q\}$, and from (1. 8), (1. 10) and (1. 15), the mean curvature $\rho(\pi)$ for π is given by

$$\text{(3. 1)} \qquad q(2n-2q+1)\,\rho(\pi)=q+\sum_{a=1}^{q}\sum_{b=q+1}^{n}(K_{ab}+K_{ab^*})$$
$$=q(2n-2q+1)+\sum_{a=1}^{q}\sum_{b=q+1}^{n}R_{aa^*b^*b}.$$

Assume that the mean curvature for $2q$-plane is independent of the choice of ϕ-holomorphic $2q$-planes.

(1) The case $n\geqq 3$. For a ϕ-holomorphic $2q$-plane π_1 spanned by $\{e_1', e_2, \cdots, e_q, \phi e_1', \phi e_2, \cdots, \phi e_q\}$, where $e_1'=\frac{e_1+e_{q+1}}{\sqrt{2}}$ and $e_{q+1}'=\frac{e_1-e_{q+1}}{\sqrt{2}}$, from (1. 8), (1. 13) and (3. 1), we have

$$\text{(3. 2)} \qquad q(2n-2q+1)\left\{\rho(\pi_1)-1\right\}$$
$$=\sum_{a=2}^{q}\sum_{b=q+2}^{n}R_{aa^*b^*b}+\frac{1}{4}\Big\{K_{11^*}+K_{(q+1)(q+1)^*}-4K_{1(q+1)^*}$$
$$+2R_{11^*(q+1)^*(q+1)}+2\Big\}+\frac{1}{2}\sum_{b=q+2}^{n}\left\{R_{11^*b^*b}+R_{(q+1)(q+1)^*b^*b}+2R_{1(q+1)^*b^*b}\right\}$$
$$+\frac{1}{2}\sum_{a=2}^{q}\left\{R_{aa^*1^*1}+R_{aa^*(q+1)^*(q+1)}-2R_{aa^*1^*(q+1)}\right\}.$$

Since $\rho(\pi)=\rho(\pi_1)$, from (3. 1) and (3. 2), we have

$$2\sum_{b=q+2}^{n} R_{1(q+1)^*b^*b}-2\sum_{a=2}^{q} R_{aa^*1^*(q+1)}-2K_{1(q+1)^*}$$
$$=R_{11^*(q+1)^*(q+1)}-\frac{1}{2}K_{11^*}-\frac{1}{2}K_{(q+1)(q+1)^*}-1$$
$$+\sum_{b=q+2}^{n}\left\{R_{11^*b^*b}-R_{(q+1)(q+1)^*b^*b}\right\}+\sum_{a=2}^{q}\left\{R_{aa^*(q+1)^*(q+1)}-R_{aa^*1^*1}\right\}. \tag{3. 3}$$

Similarly, for π and a ϕ-holomorphic $2q$-plane π_2 spanned by $\{e_1'', e_2, \cdots, e_q, \phi e_1'', \phi e_2, \cdots, \phi e_q\}$, where $e_1''=\dfrac{e_1+\phi e_{q+1}}{\sqrt{2}}$ and $e_{q+1}''=\dfrac{e_1-\phi e_{q+1}}{\sqrt{2}}$, we have

$$2\sum_{a=2}^{q} R_{aa^*(q+1)1}-2\sum_{b=q+2}^{n} R_{1(q+1)b^*b}-2K_{1(q+1)}$$
$$=R_{11^*(q+1)^*(q+1)}-\frac{1}{2}K_{11^*}-\frac{1}{2}K_{(q+1)(q+1)^*}-1$$
$$+\sum_{b=q+2}^{n}\left\{R_{11^*b^*b}-R_{(q+1)(q+1)^*b^*b}\right\}+\sum_{a=2}^{q}\left\{R_{aa^*(q+1)^*(q+1)}-R_{aa^*1^*1}\right\}. \tag{3. 4}$$

From (3. 3) and (3. 4), it follows that

$$K_{1(q+1)^*}-\sum_{b=q+2}^{n} R_{1(q+1)^*b^*b}+\sum_{a=2}^{q} R_{aa^*1^*(q+1)}$$
$$=K_{1(q+1)}+\sum_{b=q+2}^{n} R_{1(q+1)b^*b}-\sum_{a=2}^{q} R_{aa^*(q+1)1} \tag{3. 5}$$

Taking π_1 and a ϕ-holomorphic $2q$-plane spanned by $\{e_2, \cdots, e_q, e_{q+1}', \phi e_2, \cdots, \phi e_q, \phi e_{q+1}'\}$, we have

$$\sum_{a=2}^{q} R_{aa^*1^*(q+1)}=\sum_{b=q+2}^{n} R_{1(q+1)^*b^*b}. \tag{3. 6}$$

And taking π_2 and a ϕ-holomorphic $2q$-plane spanned by $\{e_2, \cdots, e_q, e_{q+1}'', \phi e_2, \cdots, \phi e_q, \phi e_{q+1}''\}$, we have

$$\sum_{a=2}^{q} R_{aa^*(q+1)1}=\sum_{b=q+2}^{n} R_{1(q+1)b^*b}. \tag{3. 7}$$

From (3. 5), (3. 6) and (3. 7), it follows that

$$K_{1(q+1)^*}=K_{1(q+1)}.$$

Thus we can see that, for any ϕ-antiholomorphic orthonormal pair $\{X, Y\}$,

$$K(X, Y)=K(X, \phi Y).$$

Since we can use Lemma 5, from (3. 1) we have

$$(3.8)\qquad q(2n-2q+1)\,\rho(\pi) = \frac{q(3n-3q+2)}{2}+\frac{n-q}{4}\sum_{a=1}^{q} K_{aa^*}+\frac{q}{4}\sum_{b=q+1}^{n} K_{bb^*}.$$

Calculating the mean curvature for a ϕ-holomorphic $2q$-plane spanned by $\{e_2, \cdots, e_{q+1}, \phi e_2, \cdots, \phi e_{q+1}\}$ and using (3.8), we have

$$(n-2q)\,\{K_{11^*}-K_{(q+1)(q+1)^*}\}=0\,.$$

(2) The case $n=2$. For an arbitrary ϕ-antiholomorphic orthonormal pair $\{e_1, e_2\}$, we can take an orthonormal base $\{e_1, e_2, \phi e_1, \phi e_2, \xi\}$ of $T_x(M)$. Taking two ϕ-holomorphic 2-planes spanned by $\left\{\frac{e_1+e_2}{\sqrt{2}},\ \phi\frac{e_1+e_2}{\sqrt{2}}\right\}$ and $\left\{\frac{e_1+\phi e_2}{\sqrt{2}},\ \frac{\phi e_1-e_2}{\sqrt{2}}\right\}$, from (1.6), (1.8), (1.13) and (1.14), we have $K_{12}=K_{12^*}$.

Therefore, by virtue of the theorem of K. Ogiue ([5]) and Theorem 2, we can obtain

THEOREM 3. *In a Sasakian manifold (M, ϕ, ξ, η, g) of dimension $2n+1 \geqq 5$, if the mean curvature for $2q$-plane is independent of the choice of ϕ-holomorphic $2q$-planes at each point, then*

(i) *for $1\leqq q\leqq n-1$ and $2q\neq n$, (M, ϕ, ξ, η, g) is of constant ϕ-holomorphic sectional curvature,*

(ii) *for $2q=n$, the contact Bochner curvature tensor vanishes.*

The converse is true.

§ 4. Mean curvature for ϕ-antiholomorphic p-plane.

In a Sasakian manifold (M, ϕ, ξ, η, g) of dimension $2n+1$, if a p-plane π in $T_x(M)$ satisfies $\eta(\pi)=\{0\}$ and $\phi\pi$ is perpendicular to π, such a p-plane π will be called the ϕ-antiholomorphic p-plane. Then we can take an orthonormal base $\{e_1, \cdots, e_n, \phi e_1, \cdots, \phi e_n, \xi\}$ of $T_x(M)$ such that π is spanned by $\{e_1, \cdots, e_p\}$, and the mean curvature $\rho(\pi)$ for π is given by

$$p(2n-p+1)\,\rho(\pi)=p+\sum_{a=1}^{p}\left\{\sum_{b=p+1}^{n}(K_{ab}+K_{ab^*})+\sum_{b=1}^{p}K_{ab^*}\right\}.$$

Assume that the mean curvature for p-plane is independent of the choice of ϕ-antiholomorphic p-planes.

(1) The case $p\geqq 2$. For a ϕ-antiholomorphic p-plane π_1 spanned by $\{\phi e_1, e_2, \cdots, e_p\}$, from (1.8) and (1.10) we have

$$p(2n-p+1)\,\rho(\pi_1) =p+K_{11^*}+\sum_{a=2}^{p}\left(2K_{1a}+\sum_{b=2}^{p}K_{ab^*}\right)+\sum_{a=1}^{p}\sum_{b=p+1}^{n}(K_{ab}+K_{ab^*})\,.$$

Since $\rho(\pi)=\rho(\pi_1)$, we see that

$$(4.1)\qquad \sum_{a=2}^{p}(K_{1a^*}-K_{1a})=0\,.$$

Similarly for ϕ-antiholomorphic p-planes spanned by $\{e_1, \phi e_2, e_3, \cdots, e_p\}$ and $\{\phi e_1, \phi e_2, e_3, \cdots, e_p\}$, we have

$$(4.2)\qquad K_{12}-K_{12^*}+\sum_{a=3}^{p}(K_{1a^*}-K_{1a})=0\,.$$

From (4.1) and (4.2), it follows that

$$(4.3)\qquad K_{12}=K_{12^*}\,.$$

Now let $\{X, Y\}$ be an arbitrary ϕ-antiholomorphic orthonormal pair. Then, since there exists an orthonormal base $\{e_1, \cdots, e_n, \phi e_1, \cdots, \phi e_n, \xi\}$ such that $e_1=X$ and $e_2=Y$, from (4.3) we see that

$$K(X, Y)=K(X, \phi Y)\,.$$

Thus, since we can use Lemma 5, we have

$$(4.4)\qquad \begin{aligned} &p(2n-p+1)\,\rho(\pi)\\ &=\frac{p(6n-3p+1)}{4}+\frac{n+3}{4}\sum_{a=1}^{p}K_{aa^*}+\frac{p}{4}\sum_{b=p+1}^{n}K_{bb^*}\end{aligned}$$

When $p<n$, similarly for a ϕ-antiholomorphic p-plane π_2 spanned by $\{e_2, \cdots, e_{p+1}\}$, we have

$$(4.5)\qquad \begin{aligned} &p(2n-p+1)\,\rho(\pi_2)\\ &=\frac{p(6n-3p+1)}{4}+\frac{n+3}{4}\sum_{a=2}^{p+1}K_{aa^*}+\frac{p}{4}\Big(K_{11^*}+\sum_{b=p+2}^{n}K_{bb^*}\Big)\,.\end{aligned}$$

From (4.4) and (4.5), we see that

$$K_{11^*}=K_{(p+1)(p+1)^*}\,.$$

(2) The case $p=1$. Let X and $\bar{\pi}$ be any unit vector orthogonal to ξ at $x\in M$ and a ϕ-antiholomorphic 1-plane spanned by X, respectively. Then we have

$$(4.6)\qquad R_1(X, X)=2n\rho(\bar{\pi})\,.$$

From the assumption and (4.6), we have

$$(4.7)\qquad R_1(Y, Z)=rg(Y, Z)$$

for any vectors Y and Z orthogonal to ξ, where $r=2n\rho(\bar{\pi})$. Since $R_1(Y, \xi)=2n\eta(Y)$, from (4.7) we have

$$R_1(Y, Z) = rg(Y, Z) + (2n-r)\,\eta(Y)\,\eta(Z)\,.$$

for any vectors Y and Z. Thus, from (1) and (2), we can obtain

THEOREM 4. *In a Sasakian manifold (M, ϕ, ξ, η, g) of dimension $2n+1 \geqq 5$, if the mean curvature for p-plane is independent of the choice of ϕ-antiholomorphic p-planes at each point, then*

(i) *for $p=1$, (M, ϕ, ξ, η, g) is an η-Einstein space,*

(ii) *for $2 \leqq p \leqq n-1$, (M, ϕ, ξ, η, g) is of constant ϕ-holomorphic sectional curvature,*

(iii) *for $p=n$, the contact Bochner curvature tensor vanishes.*

The converse is true.

Added in Proof. In "On vanishing contact Bochner curvature tensor" (Hokkaido Math. J., 9 (1980), 258-267), M. Seino proved a part of Theorem 2.

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Department of Mathematics
Ritsumeikan University