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The Lorentz-Poincaré metric on the upper half-space and its extension

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Let G_n be the matrix group consisting of all $n\times n$ matrices of the form

$$(1)\qquad g=\begin{bmatrix} a & & & b_1 \\ & a & & b_2 \\ & & \ddots & \vdots \\ & & a & b_{n-1} \\ 0 & \cdots & 0 & 1 \end{bmatrix},\ \text{where } a>0,\ b_1, \cdots, b_{n-1}\in\boldsymbol{R}.$$

It is a Lie group of type $\mathfrak{S}$ in the sense of [5] (also [4]), and, as such, it admits a left-invariant Lorentz metric with any prescribed constant k as its constant sectional curvature (Theorem 1, [5]). If we consider a diffeomorphism of G_b onto the upper half-space U_n given by

$$g\in G_n \longrightarrow (b_1, \cdots, b_{n-1}, a)\in U_n,$$

then the left translations on G_n

$$\begin{bmatrix} x_n & & & x_1 \\ & x_n & & x_2 \\ & & \ddots & \vdots \\ & & x_n & x_{n-1} \\ 0 & \cdots & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} a & & & b_1 \\ & a & & b_2 \\ & & \ddots & \vdots \\ & & a & b_{n-1} \\ 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_n & & & x_1 \\ & x_n & & x_2 \\ & & \ddots & \vdots \\ & & x_n & x_{n-1} \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

correspond to the action of G_n on U_n by

$$(2)\qquad (x_1, \cdots, x_{n-1}, x_n) \longrightarrow (a\,x_1+b_1, \cdots, a\,x_{n-1}+b_{n-1}, a\,x_n).$$

The Lorentz metric on U_n

$$(3)\qquad ds^2=(dx_1^2+\cdots+dx_{n-1}^2-dx_n^2)/x_n^2,$$

is invariant by the action (2) of G_n and corresponds to a left-invariant Lorentz matric on the group G_n of constant sectional curvature 1. The Lorentz metric (3) is, apparently, an analogue of the well-known Riemannian metric, due to Poincaré, on the space U_n which has constant sectional curvature -1.

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In this note we discuss the geometry of the metric (3), to be called the *Lorentz-Poincaré metric* on U_n. We find that this metric is not geodesically complete. Can we, then, extend it to a geodesically complete Lorentzian manifold? We answer this question affirmatively by isometrically imbedding U_n into the de Sitter space S_1^n; the imbedding is actually equivariant relative to an isomorphism of the largest connected isometry group of U_n into the proper Lorentz group $SO^+(1, n)$.

I wish to thank John Beem and Steven Harris for discussing my perliminary draft on the metric (3) and its geodesic behaviors. J. Beem called my attention to his work with Busemann [2] and Busemann [1], which mention the metric (3) as an example in their axiomatic approach to Lorentzian differential geometry. S. Harris suggested a way of imbedding U_2 into S_1^2 by using families of null geodesics that cover U_2.

1. Geodesics of the Lorentz-Poincaré metric.

On the upper half-space $U_n=\{(x_1, \cdots, x_n)\,;\, x_n>0\}$, let X_i be the coordinate vector fields: $X_i=\partial/\partial x_i$, $1\leq i\leq n$. The Levi-Civita connection for the metric (3) is described by

$$\nabla_{X_i} X_j=0 \qquad \text{for } i\neq j,\ 1\leq i,\ j\leq n-1\,;$$

$$\nabla_{X_i} X_i=-X_n/x_n \qquad \text{for } 1\leq i\leq n\,;$$

$$\nabla_{X_i} X_n=\nabla_{X_n} X_i=-X_i/x_n \qquad \text{for } 1\leq i\leq nn-1\,.$$

From these we may calculate the curvature tensor R as follows. If i, j, k are distinct among $1, \cdots, n-1$, then

$$R(X_i, X_j)\, X_k=R(X_i, X_k)\, X_n=R(X_i, X_n)\, X_j=0$$

$$R(X_i, X_n)\, X_i=R(X_i, X_n)\, X_n=-X_i/x_n^2\,.$$

Thus

$$R(X, Y)\, Z=\langle Y, Z\rangle X-\langle X, Z\rangle Y$$

for any tangent vectors X, Y and Z (where, of course, $\langle\,,\rangle$ denotes the inner product by the metric). This means that our metric has constant sectional curvature 1.

The Christofel symbols are given by

$$(4')\qquad \begin{aligned}&\Gamma_{ii}^n=\Gamma_{ni}^i=\Gamma_{in}^i=\Gamma_{nn}^n=-1/x_n,\ 1\leq i\leq n-1\,;\\ &\text{other } \Gamma_{ij}^k=0\,.\end{aligned}$$

The equations for a geodesic $x^i(t)$, with t as affine parameter, are

$$(5)\qquad \begin{aligned} d^2x_i/dt^2 &= 2(dx_i/dt)(dx_n/dt)/x_n \\ d^2x_n/dt^2 &= \left\{\sum_{i=1}^{n-1}(dx_i/dt)^2+(dx_n/dt)^2\right\}/x_n \end{aligned}$$

Let $(dx_i/dt)\ (0)=c_i$, $1\leq i\leq n$, so that the initial tangent vector of the geodesic is given by $(c_1, \cdots, c_{n-1}, c_n)$. By an appropriate rotation of the first $n-1$ variables (which is an isometry of the metric) we may assume that $c_2=\cdots=c_{n-1}=0$. From the equations (5) it follows that $x_2(t), \cdots, x_{n-1}(t)$ are constant functions in this case. This argument reduces the study of the geodesic behaviors of U_n to the case $n=2$.

For $n=2$, we write x, y instead of x_1, x_2. The equations (5) are now

$$(5')\qquad \begin{aligned} d^2x/dt^2 &= 2\,(dx/dt)\,(dy/dt)/y \\ d^2y/dt^2 &= \left\{(dx/dt)^2+(dy/dt)^2\right\}/y\,. \end{aligned}$$

We shall find the solutions of (5′). Denoting d/dt by prime $'$, we have $(x'/y^2)'=0$. Thus $x'/y^2=c$ (constant). We have also

$$(y'/y)' = x'^2/y^2 = cx'$$

and thus

$$(6)\qquad y'/y = cx+c_1 \qquad (c_1: \text{constant})\,.$$

Case I: $c=0$. We get $x'=0$ so that $x=b$ (constant). From (6) we have $y'=c_1y$ so that $y=ae^{c_1t}$, where a is a constant >0. The geodesic is thus a vertical line parametrized by

$$(7)\qquad x=b,\ y=ae^{c_1t}\,.$$

This is a time-like geodesic defined for all values of its affine parameter t (complete in both directions). See Fig. 1.

Case II: $c\neq 0$. From $x'/y^2=c$ and (6), we get $y'/x'=(cx+c_1)/cy$ so that $cyy'=cxx'+c_1x'$. Thus

$$(1/2)\,cy^2=(1/2)\,cx^2+c_1x+c_2 \qquad (c_2: \text{constant})]\,.$$

Then

$$y^2=x^2+2c_1x/c+(c_1/c)^2+(2c_2/c-c_1^2/c^2)\,.$$

Depending on whether the last term is 0, >0 or <0, we have

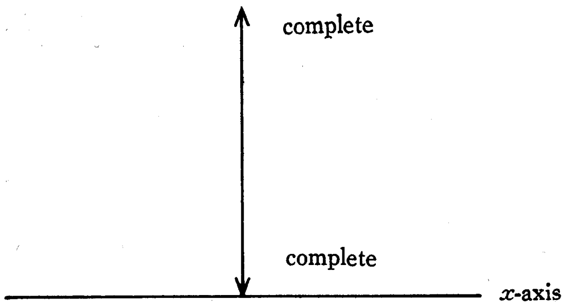
$$(8)\qquad y^2=(x-b)^2 \qquad (b: \text{constant})$$

or

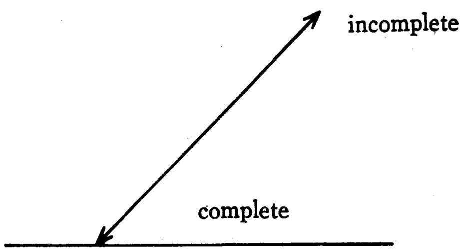
$$(9)\qquad y^2=(x-b)^2+a^2 \qquad (a>0: \text{constant})$$

Figures

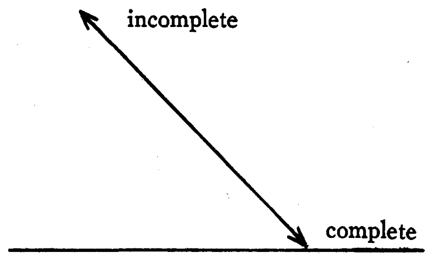
1. time-like



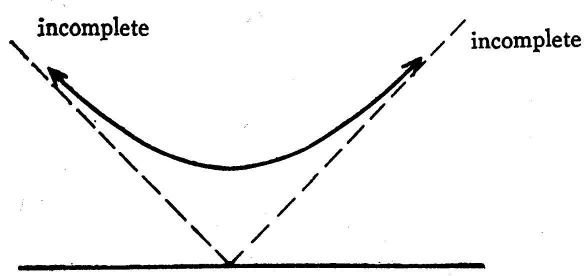
2 *a*. null



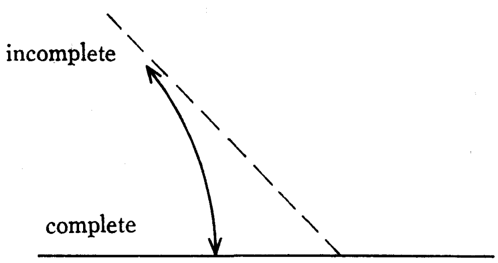
2 *b*. null



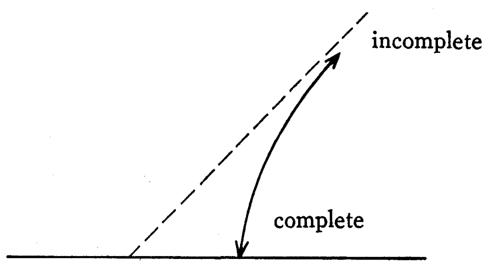
3. space-like



4 *a*. time-like



4 *b*. time-like



or

(10) $$y^2+a^2=(x-b)^2.$$

(8) gives rise to the lines $y=x-b$ and $y=-x+b$. The first line is a null geodesic

(8a) $$x(t)=x_0-y_0+y_0^2/(y_0-t),\ y(t)=y_0^2/(y_0-t),$$

where t is an affine parameter (i. e. these functions are solutions of the equations (5')). Observe that this geodesic with initial tangent vector (1, 1) at the initial point (x_0, y_0) for $t=0$) is defined for all $t<y_0$. It is complete for $t\to-\infty$ but incomplete in the other direction. See Fig. 2 *a*. The second line is a null geodesic

(8b) $$x(t)=x_0+y_0-y_0^2/(y_0+t),\ y(t)=y_0^2/(y_0+t),$$

where t is an affine parameter. This geodesic is complete in the downward direction $(t\to\infty)$ and incomplete in the other direction. See Fig. 2 *b*.

The equation (9) is a branch of a hyperbola $(y>0)$. We may first parametrize it by

(9') $$x(u)=b+a\sinh u,\ y(u)=a\cosh u.$$

The tangent vector $(dx/du, dy/du)$ has length $1/\cosh u$, and the arc-length parameter t (measured from the point $(x(u_0), y(u_0))$ is given by

$$t(u)=\int_{u_0}^{u} du/\cosh u=\sin^{-1}(\tanh u)-\sin^{-1}(\tanh u_0).$$

This space-like geodesic with affine parameter t is incomplete in both directions, because $t(u)\to\pm\pi/2-\sin^{-1}(\tanh u_0)$ as $u\to\pm\infty$. See Fig. 3.

The equation (10) gives two half-branches of hyperbolas $(y>0)$. We may first parametrize them by

(10') $$x(u)=b\pm a\cosh u,\ y(u)=a\sinh u,\ u>0.$$

The tangent vector $(dx/du, dy/du)$ is time-like with length $1/\sinh u$. The proper-time parameter t measured from $u=u_0>0$ for this time-like geodesic is given by

$$t(u)=\int_{u_0}^{u} du/\sinh u=\log(\tanh u/2)-\log(\tanh u_0/2).$$

The geodesic is complete as it approaches the x-axis, since $t(u)\to-\infty$ as $u\to 0$. It is incomplete in the other direction since $t(u)\to-\log(\tanh u_0/2)$ as $u\to\infty$. See Fig. 4 *a*, *b*.

2. Full isometry group.

We may now determine the full isometry group $I(U_n)$ of the space U_n with metric (3). Since the group G_n acts transitively on U_n, so does $I(U_n)$. Let $p_0=(0, \cdots, 0, 1)$, and we find the isotropy group at p_0. Suppose f is an isometry fixing p_0. Then the differential f_* at p_0 maps the tangent vector $(X_n)_{p_0}$ into $\pm(X_n)_{p_0}$, because the time-like geodesic $(0, \cdots, 0, e^t)$, which is complete in both directions, must be mapped by f into itself. Therefore f_* induces a rotation in the span of $(X_i)_{p_0}$, $1 \leq i \leq n-1$. This rotation is induced by an isometry of U_n of the form

$$f_A:\ (x_1, \cdots, x_{n-1}, x_n) \longrightarrow (y_1, \cdots, y_{n-1}, x_n)\,,$$

where

$$y_i=\sum_{j=1}^{n-1} a_{ij}x_j \qquad \text{with } A=[a_{ij}]\in O(n-1)\,.$$

Now if $f_*((X_n)_{p_0})=(X_n)_{p_0}$, then f must coincide with the transformation above. If $f_*((X_n)_{p_0})=-(X_n)_{p_0}$, then consider the null geodesic ray p_t with initial tangent vector $(1, 0, \cdots, 0, 1)$ which is defined for all $t<0$. The image of p_t by the isometry $f_A^{-1}f$ is the null geodesic ray through p with initial tangent vector $(1, 0, \cdots, \cdots, 0, -1)$ and is not defined for all $t<0$. This is a contradiction. We have just shown that f_* must map $(X_n)_{p_0}$ into itself and thus coincides with f_A.

The full isometry group $I(U_n)$ therefore consists of all matrices of the form

$$\begin{bmatrix} & aA & & \begin{matrix} b_1 \\ \vdots \\ b_{n-1} \end{matrix} \\ 0 & \cdots & 0 & 1 \end{bmatrix} \qquad \text{with } A\in O(n-1)\ a>0,\ b_1, \cdots, b_{n-1}\in \boldsymbol{R} \tag{11}$$

acting on U_n in the natural fashion. The identity component $I^0(U_n)$ consists of all such matrices with $A\in SO(n-1)$.

3. Isometric imbedding of U_n into S_1^n

We shall now give an isometric imbedding of U_n into the de Sitter space S_1^n. This space is the hypersurface

$$\left\{u=(u_0, u_1, \cdots, u_n)\,;\ \langle u, u\rangle \equiv -u_0^2+u_1^2+\cdots+u_n^2=1\right\}$$

in the Lorentz space L^{n+1} with its induced Lorentz metric of constant sectional curvature 1 [3], [6].

We define f: $U_n \to S_1^n$ *by*

$$f(x_1, \cdots, x_{n-1}, x_n) = (u_0, u_1, \cdots, u_n)\,,$$

where

$$(12)\qquad \begin{cases} u_0 = (1 + x_1^2 + \cdots + x_{n-1}^2 - x_n^2)/2x_n \\ u_i = -x_i/x_n,\ 1 \leq i \leq n-1\,, \\ u_n = (1 - x_1^2 - \cdots - x_{n-1}^2 + x_n^2)/2x_n\,. \end{cases}$$

It is straightforward to verify that f is an isometric imbedding of U_n into S_1^n. The image $f(U_n)$ is the open submanifold

$$V_n = \{u \in S_1^n\,;\ u_0 + u_n > 0\}\,.$$

We now define an isomorphism h of the group G_n into the proper Lorentz group $SO^+(1, n)$, which is the largest connected group of isometries of S_1^n. Note that $SO^+(1, n)$ consists of all Lorentz-orthogonal matrices with determinant 1 whose first column vectors are future-pointing time-like unit vectors $a = (a_0, a_1, \cdots, a_n)$, $\langle a, a \rangle = -1$, $a_0 > 0$. In order to define h, let us observe the following about the Lie algebras of G_n and $SO^+(1, n)$. In the Lie algebra $\mathfrak{g}$ of G_n, let

$$(13)\qquad X_i = \begin{bmatrix} & & 0 \\ & & \vdots \\ & 0 & 1 \\ & & \vdots \\ 0 \cdots 0 & & 0 \end{bmatrix},\ 1 \leq i \leq n-1,\ X_n = \begin{bmatrix} -1 & & & 0 \\ & \ddots & & \vdots \\ & & -1 & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}$$

Then $X_1, \cdots, X_{n-1}, X_n$ form a basis of $\mathfrak{g}$ such that

$$[X_i, X_j] = 0 \qquad \text{for } 1 \leq i, j \leq n-1$$

$$[X_i, X_n] = X_i \qquad \text{for } 1 \leq i \leq n-1\,.$$

Similarly, in the Lie algebra $\mathfrak{o}(1, n)$ of $SO^+(1, n)$ let

$$Y_i = \begin{bmatrix} 0 & \cdots & -1 & \cdots & 0 \\ \vdots & & & & \vdots \\ -1 & & 0 & & -1 \\ \vdots & & & & \vdots \\ 0 & & 1 & & 0 \end{bmatrix},\ 1 \leq i \leq n-1$$

$$Y_n = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$$

Then $Y_1, \cdots, Y_{n-1}, Y_n$ satisfy the same commutation conditions

$$[Y_i, Y_j]=0 \qquad \text{for } 1\leq i,\ j\leq n-1$$

$$[Y_i, Y_n]=Y_i \qquad \text{for } 1\leq i\leq n-1$$

and generate a Lie subalgebra of $\mathfrak{o}(1, n)$ which is isomorphic to $\mathfrak{g}$.

The isomorphism h_* of $\mathfrak{g}$ into $\mathfrak{o}(1, n)$ mapping X_i into Y_i for $1\leq i\leq n$ gives rise to a homomorphism h of the Lie group G_n into $SO^+(1, n)$ which maps

$$\exp(sX_n)=\begin{bmatrix} e^{-s} & & \\ & \ddots & \\ & & e^{-s}\end{bmatrix} \text{ into } \exp(sY_n)=\begin{bmatrix} \cosh s & \cdots & & \sinh s \\ & 1 & & \\ \vdots & & \ddots & \vdots \\ & & 1 & \\ \sinh s & \cdots & & \cosh s\end{bmatrix}$$

and

$$\exp(tX_i)=\begin{bmatrix} 1 & & & 0 \\ & \ddots & & \vdots \\ & & \ddots & t \\ & & & \vdots \\ & & 1 & 0 \\ 0 & \cdots & 0 & 1\end{bmatrix} \text{ into } \exp(tY_i)=\begin{bmatrix} 1+t^2/2 & \cdots & -t & \cdots & t^2/2 \\ \vdots & 1 & & & \vdots \\ -t & & \ddots & & -t \\ \vdots & & & 1 & \vdots \\ -t^2/2 & \cdots & t & \cdots & 1-t^2/2\end{bmatrix}$$

for each i, $1\leq i\leq n-1$.

We shall show that the imbedding f: $U_n\to S_1^n$ is equivariant relative to $h: G_n\to SO^+(1, n)$, that is,

$$f(gp)=h(g)f(p) \qquad \text{for all } g\in G_n \text{ and } p\in U_n\,.$$

It suffices to prove this for $p=p_0=(0, \cdots, 1)$, for which $f(p_0)=(0, \cdots, 0, 1)\in S_1^n$. Now for g as in (1) with $a=e^{-s}$, we have

$$\begin{aligned} h(g)f(p_0) &= h(\exp b_{n-1}X_{n-1})\cdots h(\exp b_1X_1)\, h(\exp sX_n)f(p_0) \\ &= (\sinh s+Be^s/2,\ -b_1e^s, \cdots, -b_{n-1}e^s, \cosh s-Be^s/2)\,, \end{aligned}$$

where $B=b_1^2+\cdots+b_{n-1}^2$. On the other hand,

$$\begin{aligned} f(gp_0) &= f(b_1, \cdots, b_{n-1}, e^{-s}) \\ &= \left((1+B-e^{-2s})/2e^{-s},\ -b_1e^s, \cdots, -b_{n-1}e^s, (1-B+e^{-2s})/2e^{-s}\right) \end{aligned}$$

and hence $f(gp_0)=h(g)f(p_0)$.

We can now prove that h is an isomorphism. Suppose $h(g)=h(g')$ for g, $g\in G_n$. Then $h(g)f(p_0)=h(g')f(p_0)$. By adding the first and last coordinates of this point we find that g and g' have the same homothetic factor (that is a in (1)). The other coordinates of the point show that g and g' have the same translation part (that is, $b_1, \cdots, b_{n-1}$ in (1)). Thus $g=g'$.

We have shown that the imbedding f is equivariant relative to the isomorphism h of G_n into $SO^+(1, n)$. We can extend h to an isomorphism of the largest connected group $I^0(U_n)$ into $SO^+(1, n)$ in such a way that f remains equivariant. It is sufficient to define

$$h(g)=\begin{bmatrix}1 & 0 & 0\\ 0 & A & 0\\ 0 & 0 & 1\end{bmatrix}\in SO^+(1, n) \qquad \text{for } g=\begin{bmatrix}A & 0\\ 0 & 1\end{bmatrix}\in I^0(U_n),$$

where $A\in SO^+(n-1)$ as before.

Thus we state

THEOREM. *There is an isometric imbedding of the upper half-space U_n with Lorentz-Poincaré metric into the de Sitter space S_1^n which is equivariant relative to an isomorphism of the largest connected isometry group $I^0(U_n)$ into the proper Lorentz group $SO^+(1, n)$.*

A geodesic x_t in U_n is incomplete if and only if $f(x_t)$ reaches the boundary $f(U_n)=\{u\in S\ ;\ u_0+u_n=0\}$ for a finite value of the affine parameter t.

Finally, let us remark that the left-invariant Lorentz metric on the group G_n corresponding to the Lorentz-Poincaré metric is determined by the Lorentz inner product in the Lie algebra $\mathfrak{g}$ such that $\langle X_n, X_n\rangle=-1$, $\langle X_i, X_n\rangle=0$ and $\langle X_i, X_j\rangle=\delta_{ij}$ for $1\leq i, j\leq n-1$, where $X_1, \cdots, X_n$ are given in (13).

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