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A congruence between modular forms of half-integral weight

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Introduction

In [7], Shimura showed a natural correspondence between modular forms of integral weight and those of half-integral weight. On the other hand, primitive modular forms have congruences, as discussed by Doi-Ohta [1], which are closely connected with the special values of the zeta functions associated with these forms (Doi-Hida [2] and Hida [3], [4]). Thus it is natural to ask whether these congruences of primitive forms of integral weight induce the same congruences of the corresponding forms of half-integral weight. The purpose of this paper is to show an affirmative example to the following problem of Hida:

For primitive forms F, $G\in S(2k, 2N)$ *with a congruence* $F\equiv G \bmod \mathfrak{p}$, *can one find corresponding eigenfunctions* f, $g\in S((2k+1)/2, 4N)$ *with* $\mathfrak{p}$-*integral Fourier coefficients such that* $f\equiv g \bmod \mathfrak{p}$ *and* $f\not\equiv 0 \bmod \mathfrak{p}$?
Here, $\mathfrak{p}$ *is a prime ideal of* $\bar{\boldsymbol{Q}}$ *and the congruence "*$f\equiv g \bmod \mathfrak{p}$*" means that all Fourier coefficients of* $f-g$ *vanish modulo* $\mathfrak{p}$.

The converse statement is trivial, that is, the congruence $f\equiv g \bmod \mathfrak{p}$ and $f\not\equiv 0 \bmod \mathfrak{p}$ implies the congruence $F\equiv G \bmod \mathfrak{p}$ (see § 1).

We note here that the Fourier coefficients of the cusp form f are closely connected with the special values of a certain zeta function associated with F (Waldspurger [8] and Kohnen-Zagier [5]).

§ 1. The precise statement of the problem

For a positive integer N, put

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in SL_2(\boldsymbol{Z})\,|\,c\equiv 0 \bmod N\right\},$$

$$\mathfrak{H}=\left\{z\in\boldsymbol{C}\,|\,\mathrm{Im}(z)>0\right\}.$$

Further we put

$$\theta(z)=\sum_{n=-\infty}^{\infty}e(n^2z),$$

$$j(\gamma, z)=\theta\left(\frac{az+b}{cz+d}\right)/\theta(z) \qquad \text{for } \gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_0(4),$$

where $e(z)=\exp(2\pi iz)$ $(z\in\mathfrak{H})$.
Given a non-negative element κ of $2^{-1}\boldsymbol{Z}$ and a Dirichlet character χ modulo N, we denote by $S(\kappa, N, \chi)$ the space of cusp forms of weight κ satisfying

$$f\left(\frac{az+b}{cz+d}\right)=\begin{cases}\chi(d)f(z)(cz+d)^{\kappa} & \text{if } \kappa\in\boldsymbol{Z},\\ \chi(d)f(z)j(\gamma, z)^{2\kappa} & \text{if } \kappa\notin\boldsymbol{Z},\end{cases}$$

for every $\gamma=\begin{pmatrix} a & b \\ c & d\end{pmatrix}\in\Gamma_0(N)$.

We assume $4|N$ if $\kappa\notin\boldsymbol{Z}$. We write simply $S(\kappa, N)$ for $S(\kappa, N, \chi)$ if $\chi=id.$. Every element f of $S(\kappa, N, \chi)$ has a Fourier expansion $f(z)=\sum_{n=1}^{\infty}a(n)e(nz)$ with $a(n)\in\boldsymbol{C}$.

Let $f(z)=\sum_{n=1}^{\infty}a(n)e(nz)$ and $g(z)=\sum_{n=1}^{\infty}b(n)e(nz)$ be elements of $S(\kappa, N, \chi)$ with algebraic $a(n)$ and $b(n)$ for all n. Then we call that f and g are congruent modulo a prime ideal $\mathfrak{p}$ of $\bar{\boldsymbol{Q}}$ and write $f\equiv g \bmod \mathfrak{p}$ if $a(n)\equiv b(n) \bmod \mathfrak{p}$ for all n.

When $l\in\boldsymbol{Z}$, $F(z)=\sum_{n=1}^{\infty}A(n)e(nz)\in S(l, N, \chi)$ is called primitive if it satisfies the following conditions:

(i) F is a common eigenfunction of all Hecke operators $T(n)$ with $A(1)=1$;

(ii) For every positive integer M such that $M<N$, $M|N$ and χ is defined modulo M, we have $\langle F, H^t\rangle=0$ for all $H\in S(l, M, \chi)$ and all positive integers t with $tM|N$.

Here $H^t(z)=H(tz)$ and $\langle\ ,\ \rangle$ denotes the Petersson inner product on $S(l, N, \chi)$.

Now we fix a positive integer k. Let $F(z)=\sum_{n=1}^{\infty}A(n)e(nz)$ and $G(z)=\sum_{n=1}^{\infty}B(n)e(nz)$ be two primitive elements of $S(2k, N, \chi^2)$. Suppose that $f(z)=\sum_{n=1}^{\infty}a(n)e(nz)$ and $g(z)=\sum_{n=1}^{\infty}b(n)e(nz)$ are common eigenfunctions of Hecke operators $T(p^2)$ for all primes p of $S((2k+1)/2, N', \chi)$ for some level N' to which F and G correspond, respectively. *We keep this notation and these assumptions throughout this section.* Moreover, we assume that all $a(n)$ and $b(n)$ are algebraic. By Shimura [7, Theorem 1.9 and Corollary in p. 458], for every square-free positive integer t, we have

$$\sum_{n=1}^{\infty}a(tn^2)n^{-s}=a(t)\cdot\prod_p\left[1-\chi(p)\left(\frac{-1}{p}\right)^k\left(\frac{t}{p}\right)p^{k-1-s}\right]\cdot\sum_{n=1}^{\infty}A(n)n^{-s}.$$

Especially, for all primes p we have

$$a(t)\,A(p)=a(tp^2)+\chi(p)\left(\frac{-1}{p}\right)^k\left(\frac{t}{p}\right)p^{k-1}a(t)\,.$$

Consequently, if there exists a square-free positive integer t such that $a(t)=b(t)$, then we have

$$a(t)\left\{A(p)-B(p)\right\}=a(tp^2)-b(tp^2)\,.$$

From this we easily get the following

PROPOSITION. *Suppose* $f\equiv g \bmod \mathfrak{p}$ *with a prime ideal* $\mathfrak{p}$ *of* $\bar{\boldsymbol{Q}}$, *and further suppose that for some square-free positive integer* t,

$$a(t)=b(t),$$
$$a(t)\not\equiv 0 \bmod \mathfrak{p}.$$

Then we have $F\equiv G \bmod \mathfrak{p}$.

Thus we can naturally ask the following

PROBLEM. *Under the same notation as above, if* $F\equiv G \bmod \mathfrak{p}$ *with a prime ideal* $\mathfrak{p}$ *of* $\bar{\boldsymbol{Q}}$, *can one find* f *and* g *with the following properties?*:

(1) f *and* g *have* $\mathfrak{p}$*-integral Fourier coefficients*;

(2) *For some square-free positive integer* t,

$$a(t)=b(t),$$
$$a(t)\not\equiv 0 \bmod \mathfrak{p};$$

(3) $f\equiv g \bmod \mathfrak{p}$.

REMARK. Suppose the above conditions are satisfied by f and g for some square-free positive integer t. For another square-free t' such that $a(t')\not\equiv 0 \bmod \mathfrak{p}$, put $f'=b(t')\cdot f$ and $g'=a(t')\cdot g$. Then the above conditions are also satisfied by f' and g' for t'.

§ 2. An example

We take $S(8, 2\cdot 13)$ and $S(9/2, 4\cdot 13)$. We have $\dim_C S(8, 2\cdot 13)=\dim_C S(9/2, 4\cdot 13)=23$ and the number of the primitive forms in $S(8, 2\cdot 13)$ is 7. For an eigenvalue α of the Hecke operator $T(3)$, we denote by $F(\alpha)$ the primitive form F of $S(8, 2\cdot 13)$ such that $F|T(3)=\alpha\cdot F$. As the table (I) shows, $F(\alpha)$ is uniquely determined under this condition.

Table (I). Fourier coefficients of the primitive forms of $S(8, 2\cdot 13)$

n	$F(-87)$	$F\left(\frac{87+\sqrt{2305}}{2}\right)$	$F\left(\frac{87-\sqrt{2305}}{2}\right)$	$F(-27)$	$F(-39)$	$F(-6+7\sqrt{105})$	$F(-6-7\sqrt{105})$
2	2^3	2^3	2^3	2^3	-2^3	-2^3	-2^3
3	-87	$\frac{87+\sqrt{2305}}{2}$	$\frac{87-\sqrt{2305}}{2}$	-27	-39	$-6+7\sqrt{105}$	$-6-7\sqrt{105}$
5	321	$\frac{215+5\sqrt{2305}}{2}$	$\frac{215-5\sqrt{2305}}{2}$	-245	385	$-73+36\sqrt{105}$	$-73-36\sqrt{105}$
7	-181	$\frac{705-49\sqrt{2305}}{2}$	$\frac{705+49\sqrt{2305}}{2}$	-587	-293	$-890+27\sqrt{105}$	$-890-27\sqrt{105}$
11	7782	$\frac{614-190\sqrt{2305}}{2}$	$\frac{614+190\sqrt{2305}}{2}$	-3874	-5402	$5452+90\sqrt{105}$	$5452-90\sqrt{105}$
13	13^3	13^3	13^3	-13^3	13^3	-13^3	-13^3

Now we can find a prime ideal $\mathfrak{p}$ of $\bar{\boldsymbol{Q}}$ such that

$$F(-87) \equiv F\left(\frac{87+\sqrt{2305}}{2}\right) \bmod \mathfrak{p}, \qquad \mathfrak{p}|433.$$

Put $h=\sum_{n=1}^{\infty}\mathrm{tr}(T(n))\,e(nz)$, where $\mathrm{tr}(T(n))$ denotes the trace of the Hecke operator $T(n)$ on $S(4, 4\cdot 13)$. We calculate $\mathrm{tr}(T(n))$ by the Eichler-Selberg trace formula. Then by Shimura [6, Remark 3. 46], we have $h\in S(4, 4\cdot 13)$. Further for every positive integer m and every prime p or $p=1$, put

$$h(m, p)=\left[(h|T(m))\cdot\theta\right]|T(p^2),$$

where $T(m)$ denotes the Hecke operator on $S(4, 4\cdot 13)$ and $T(p^2)$ denotes the Hecke operator on $S(9/2, 4\cdot 13)$. Then $h(m, p)$ belongs to $S(9/2, 4\cdot 13)$ and we can show through the explicit computation that the space $S(9/2, 4\cdot 13)$ is spanned by all $h(m, p)$ with the following (m, p):

(1, 1), (1, 2), (1, 3), (1, 13), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1), (4, 2), (5, 1), (5, 2), (6, 1), (7, 1), (8, 1), (9, 1), (10, 1), (11, 1), (12, 1), (13, 1), (14, 1), (16, 1).

We write such forms as $h^{(1)}, \cdots, h^{(23)}$. *They have all rational integral Fourier coefficients.* Put

$$^t(h^{(1)}, \cdots, h^{(23)})=\sum_{n=1}^{\infty}\boldsymbol{c}(n)\,e(nz), \qquad \boldsymbol{c}(n)\in\boldsymbol{C}^{23}.$$

Then we can show the 25 vectors $\boldsymbol{c}(1), \cdots, \boldsymbol{c}(25)$ span $\boldsymbol{C}^{23}$ and therefore $\{h^{(1)}, \cdots, h^{(23)}\}$ forms a basis of $S(9/2, 4\cdot 13)$. Thus every cusp form $f=\sum_{n=1}^{\infty}a(n)\,e(nz)$ of $S(9/2, 4\cdot 13)$ is uniquely determined by its Fourier coefficients

$a(1), \cdots, a(25)$. Explicit calculation using this basis shows that the eigen space of the $T(3^2)$ on $S(9/2, 4\cdot 13)$ with the eigenvalue α of the $T(3)$ in the table (I) is one-dimensional. Thus every nonzero cusp form belonging to this space must be a common eigenfunction of all Hecke operators $T(p^2)$. Let us pick one of these forms with 433-integral Fourier coefficients and denote it by $f(\alpha)$. The Fourier coefficients of $f(\alpha)$ of our choice are listed in the table (II). Thus $f(\alpha)$ corresponds to $F(\alpha)$ in the sence of Shimura [7]. As the table (II) shows, the second Fourier coefficients of $h_0=240\cdot f(-87)$ and $h_1=13\cdot f\left(\frac{87+\sqrt{2305}}{2}\right)$ coincide with each other. Let us take the prime ideal $\mathfrak{p}$ with $\mathfrak{p}|433$ and $F(-87)\equiv F\left(\frac{87+\sqrt{2305}}{2}\right) \bmod \mathfrak{p}$ as mentioned before. Then these forms h_0 and h_1 do not vanish modulo $\mathfrak{p}$. Now we will prove that

$$h_0 \equiv h_1 \bmod \mathfrak{p}.$$

Let $\boldsymbol{E}$ be the subspace of $S(9/2, 4\cdot 13)$ consisting of the eigenfunctions of $T(2^2)$ with an eigenvalue 8. Let $g_0=13649181\cdot h(1,1)$ and g the orthogonal projection of g_0 to $\boldsymbol{E}$. By expressing g explicitly as a linear combination of the above basis $\{h^{(i)}\}$, we find all the Fourier coefficients of g are 433-adic integers. Since the space $\boldsymbol{E}$ is spanned by $f(-87)$, $f\left(\frac{87+\sqrt{2305}}{2}\right)$, $f\left(\frac{87-\sqrt{2305}}{2}\right)$ and $f(-27)$, we can express

$$g=a\cdot f(-87)+b\cdot f\left(\frac{87+\sqrt{2305}}{2}\right)+c\cdot f\left(\frac{87-\sqrt{2305}}{2}\right)+d\cdot f(-27)$$

with a, b, c, $d\in\bar{\boldsymbol{Q}}$ listed below:

$$a=\frac{4307941508}{433},$$

$$b=\frac{2592113149717+107114327957\sqrt{2305}}{2^5\cdot 5\cdot 433\cdot 461},$$

$$c=\frac{2592113149717-107114327957\sqrt{2305}}{2^5\cdot 5\cdot 433\cdot 461},$$

$$d=-34193796.$$

Now a and b are not $\mathfrak{p}$-integral, while c and d are prime to $\mathfrak{p}$, because our prime 433 is decomposed into the product of two different prime factors in $\boldsymbol{Q}(\sqrt{2305})$ and we have

$$N(2592113149717+107114327957\sqrt{2305})$$
$$=2^9\cdot3^2\cdot7^2\cdot13^2\cdot17^4\cdot173\cdot433\cdot461\cdot179243.$$

Therefore, there exist two rational integers a' and b' which are prime to $\mathfrak{p}$ and

$$a'\cdot f(-87)\equiv b'\cdot f\left(\frac{87+\sqrt{2305}}{2}\right) \bmod \mathfrak{p}.$$

By comparing the second Fourier coefficients of these forms, we have $13\cdot a' \equiv 240\cdot b' \bmod \mathfrak{p}$. Consequently we conclude $h_0\equiv h_1 \bmod \mathfrak{p}$.

The idea of the proof is based on the discussion of [2].

§ 3. Miscellaneous remarks

(1) The Fourier coefficients of $F\left(\frac{87+\sqrt{2305}}{2}\right)$ and $F(-27)$ coincide modulo $\mathfrak{p}$, $\mathfrak{p}|13$, within the limit of the table (I). However, the table (II) shows that $f\left(\frac{87+\sqrt{2305}}{2}\right)$ and $f(-27)$ can never be congruent modulo $\mathfrak{p}$ in the sence of the Problem in § 1. (Observe the third Fourier coefficients of these forms.) It seems that the Problem is negative for the congruence divisor $\mathfrak{p}$ which divides the level N.

(2) From the table (II), we observe that $f(-87)$, $f\left(\frac{87+\sqrt{2305}}{2}\right)$ and $f(-39)$ have the following property:

(+) *The n-th Fourier coefficient vanishes whenever* $\left(\frac{n}{13}\right)=1$.

On the other hand, for $f(-27)$ and $f(-6+7\sqrt{105})$, we see that:

(−) *The n-th Fourier coefficient vanishes whenever* $\left(\frac{n}{13}\right)=-1$.

In fact, it can be shown by the result of Waldspurger [8, Théorème 1] that $f(\alpha)$ *has the property* $(\pm)$ *according as the parity of the eigenvalue* $\pm13^3$ *of the* $T(13)$ *for* $F(\alpha)$. Moreover, the analogous assertion also holds for the other prime factor 2 of the level $4\cdot13$:

The n-th Fourier coefficient of $f(\alpha)$ *vanishes whenever* $n\equiv1 \bmod 8$ *or* $n\equiv5 \bmod 8$, *according as the parity of the eigenvalue* $\pm2^3$ *of the* $T(2)$ *for* $F(\alpha)$.

We calculated the Fourier coefficients of $f(\alpha)$ up to 500 (by HITAC M-200H, Hokkaido University Computing Center), and here we list these coefficients 1 to 100:

Table (II).

n	$f(-87)$	$f((87+\sqrt{2305})/2)$	$f(-27)$	$f(-39)$	$f(-6+7\sqrt{105})$
1	0	0	0	0	8
2	13	240	0	1	0
3	0	0	1	0	$-167+13\sqrt{105}$
4	0	0	0	0	-64
5	76	$-650+10\sqrt{2305}$	0	0	0
6	-29	$-20+52\sqrt{2305}$	0	15	0
7	47	$1820-28\sqrt{2305}$	0	-9	0
8	104	1920	0	-8	0
9	0	0	0	0	$-264+56\sqrt{105}$
10	0	0	5	0	$505-99\sqrt{105}$
11	-352	$-4000+128\sqrt{2305}$	0	-32	0
12	0	0	8	0	$1336-104\sqrt{105}$
13	52	$3310+82\sqrt{2305}$	-8	0	0
14	0	0	13	0	$-1799+237\sqrt{105}$
15	-331	$7300-260\sqrt{2305}$	0	45	0
16	0	0	0	0	512
17	0	0	0	0	$3592-384\sqrt{105}$
18	-780	$16920+120\sqrt{2305}$	0	-12	0
19	188	$-16640-416\sqrt{2305}$	0	60	0
20	608	$-5200+80\sqrt{2305}$	0	0	0
21	644	$-18330-102\sqrt{2305}$	0	0	0
22	0	0	-12	0	$-1132+36\sqrt{105}$
23	0	0	-44	0	$-8332+708\sqrt{105}$
24	-232	$-160+416\sqrt{2305}$	0	-120	0
25	0	0	0	0	$-1584+288\sqrt{105}$
26	1612	$-1840+224\sqrt{2305}$	7	-52	$-5421+351\sqrt{105}$
27	0	0	-27	0	$10557-1247\sqrt{105}$
28	376	$14560-224\sqrt{2305}$	0	72	0
29	0	0	-40	0	0
30	0	0	65	0	$573+321\sqrt{105}$
31	-1444	$-57480-504\sqrt{2305}$	0	12	0
32	832	15360	0	64	0
33	0	0	0	24	0

Table (II) Continued

n	$f(-87)$	$f((87+\sqrt{2305})/2)$	$f(-27)$	$f(-39)$	$f(-6+7\sqrt{105})$
34	-1843	$13160+728\sqrt{2305}$	0	129	0
35	0	0	-105	0	$-8321+939\sqrt{105}$
36	0	0	0	0	$2112-448\sqrt{105}$
37	124	$16610-1378\sqrt{2305}$	0	0	0
38	0	0	114	0	$19658-2478\sqrt{105}$
39	-715	$13980-732\sqrt{2305}$	-44	-195	$-1612+260\sqrt{105}$
40	0	0	40	0	$-4040+792\sqrt{105}$
41	0	0	0	-280	0
42	0	0	77	0	$-943+469\sqrt{105}$
43	0	0	-5	0	$10947-225\sqrt{105}$
44	-2816	$-32000+1024\sqrt{2305}$	0	256	0
45	-4560	$-34300+380\sqrt{2305}$	0	0	0
46	-422	$119080-104\sqrt{2305}$	0	-270	0
47	1849	$-58340+100\sqrt{2305}$	0	129	0
48	0	0	64	0	$-10688+832\sqrt{105}$
49	0	0	0	0	$-9864+216\sqrt{105}$
50	5798	$55800+600\sqrt{2305}$	0	510	0
51	0	0	31	0	$2855+307\sqrt{105}$
52	416	$26480+656\sqrt{2305}$	-64	0	0
53	0	0	40	0	0
54	2523	$59060+2252\sqrt{2305}$	0	-585	0
55	0	0	-40	0	$632-1512\sqrt{105}$
56	0	0	104	0	$14392-1896\sqrt{105}$
57	0	0	0	480	0
58	1242	$-147400+2792\sqrt{2305}$	0	18	0
59	13652	$1040-880\sqrt{2305}$	0	-556	0
60	-2648	$58400-2080\sqrt{2305}$	0	-360	0
61	0	0	-104	0	0
62	0	0	-10	0	$-43330+3894\sqrt{105}$
63	-5358	$-2240+448\sqrt{2305}$	0	594	0
64	0	0	0	0	-4096
65	0	0	0	520	$-11024+1560\sqrt{105}$
66	0	0	90	0	$21346-182\sqrt{105}$

Table (II) Continued

n	$f(-87)$	$f((87+\sqrt{2305})/2)$	$f(-27)$	$f(-39)$	$f(-6+7\sqrt{105})$
67	4392	$-64240+848\sqrt{2305}$	0	-24	0
68	0	0	0	0	$-28736+3072\sqrt{105}$
69	0	0	-8	0	0
70	-875	$-139900-4420\sqrt{2305}$	0	105	0
71	-17681	$168460-2060\sqrt{2305}$	0	55	0
72	-6240	$135360+960\sqrt{2305}$	0	96	0
73	0	0	0	-120	0
74	0	0	-157	0	$17823-1701\sqrt{105}$
75	0	0	-120	0	$40456-5336\sqrt{105}$
76	1504	$-133120-3328\sqrt{2305}$	0	-480	0
77	0	0	8	0	0
78	6318	$40600-1592\sqrt{2305}$	77	-234	$28249-2899\sqrt{105}$
79	0	0	-12	0	$-7564+2052\sqrt{105}$
80	4864	$-41600+640\sqrt{2305}$	0	0	0
81	0	0	0	0	$25248-2184\sqrt{105}$
82	0	0	-286	0	$31658-3150\sqrt{105}$
83	756	$-84960+2304\sqrt{2305}$	0	52	0
84	5152	$-146640-816\sqrt{2305}$	0	0	0
85	2868	$-108550+3590\sqrt{2305}$	0	0	0
86	-28139	$-57460-1900\sqrt{2305}$	0	1049	0
87	0	0	536	0	$-35464+2968\sqrt{105}$
88	0	0	-96	0	$9056-288\sqrt{105}$
89	0	0	0	-1648	0
90	0	0	-270	0	$-89430+6802\sqrt{105}$
91	-6916	$207120-1584\sqrt{2305}$	35	-1092	$20683+1287\sqrt{105}$
92	0	0	-352	0	$66656-5664\sqrt{105}$
93	8424	$245160-3240\sqrt{2305}$	0	0	0
94	0	0	-505	0	$-8213-729\sqrt{105}$
95	0	0	140	0	$-50036+5628\sqrt{105}$
96	-1856	$-1280+3328\sqrt{2305}$	0	960	0
97	0	0	0	408	0
98	-6812	$2280-5880\sqrt{2305}$	0	-636	0
99	21120	$-134480+7024\sqrt{2305}$	0	384	0
100	0	0	0	0	$12672-2304\sqrt{105}$

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