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On the non-existence of smooth actions of complex symplectic group on cohomology quaternion projective spaces

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0. Introduction

We have studied actions of non-compact classical Lie groups $\boldsymbol{SL}(n, \boldsymbol{R})$ and $\boldsymbol{SL}(n, \boldsymbol{C})$, in the previous papers [4], [5]. It seems to be important to consider the restricted actions of maximal compact groups. In this paper, we shall study smooth actions of complex symplectic group $\boldsymbol{Sp}(n, \boldsymbol{C})$ and its maximal compact group $\boldsymbol{Sp}(n)$ on rational cohomology quaternion projective spaces. We shall show the following result.

THEOREM. *Suppose $n \geqq 5$ and $m \leqq 2n-2$. Then $\boldsymbol{Sp}(n, \boldsymbol{C})$ does not act smoothly and non-trivially on any rational cohomology quaternion projective m-space.*

By a rational cohomology quaternion projective m-space we mean a closed orientable smooth manifold whose cohomology ring with rational coefficients is isomorphic to that of the quaternion projective m-space.

1. Certain subgroups of $\boldsymbol{Sp}(n, \boldsymbol{C})$

Let $\boldsymbol{GL}(m, \boldsymbol{C})$ and $\boldsymbol{U}(m)$ denote the group of regular matrices of degree m with complex coefficients and the group of unitary matrices of degree m, respectively. Let I_n denote the unit matrix of degree n, and we put

$$J_n = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Define $\boldsymbol{Sp}(n, \boldsymbol{C}) = \{A \in \boldsymbol{GL}(2n, \boldsymbol{C}) : {}^tAJ_nA = J_n\}$ and $\boldsymbol{Sp}(n) = \boldsymbol{Sp}(n, \boldsymbol{C}) \cap \boldsymbol{U}(2n)$. Then $\boldsymbol{Sp}(n, \boldsymbol{C})$ and $\boldsymbol{Sp}(n)$ are connected closed subgroups of $\boldsymbol{GL}(2n, \boldsymbol{C})$.

As usual, we regard $M_m(\boldsymbol{C})$ with the bracket operation $[A, B] = AB - BA$ as the Lie algebra of $\boldsymbol{GL}(m, \boldsymbol{C})$. Let $\mathfrak{sp}(n, \boldsymbol{C})$ and $\mathfrak{sp}(n)$ denote the Lie subalgebras of $M_{2n}(\boldsymbol{C})$, considered as a real Lie algebra, corresponding to the subgroups $\boldsymbol{Sp}(n, \boldsymbol{C})$ and $\boldsymbol{Sp}(n)$, respectively. Then

$$\mathfrak{sp}(n, \boldsymbol{C})=\left\{X\in M_{2n}(\boldsymbol{C}) : {}^tXJ_n=-J_nX\right\},$$

$$\mathfrak{sp}(n)=\left\{X\in M_{2n}(C) : {}^tXJ_n=-J_nX,\ {}^tX+\bar{X}=0\right\}.$$

We can describe more explicitly as follows.

$$\mathfrak{sp}(n, \boldsymbol{C})=\left\{\begin{pmatrix} X & Z \\ Y & -{}^tX\end{pmatrix} : {}^tY=Y,\ {}^tZ=Z;\ X, Y, Z\in M_n(\boldsymbol{C})\right\},$$

$$\mathfrak{sp}(n)=\left\{\begin{pmatrix} X & -\bar{Y} \\ Y & \bar{X}\end{pmatrix} : {}^tY=Y,\ {}^tX+\bar{X}=0;\ X, Y\in M_n(\boldsymbol{C})\right\}.$$

Put

$$\mathfrak{h}(n)=\left\{\begin{pmatrix} X & \bar{Y} \\ Y & -\bar{X}\end{pmatrix} : {}^tY=Y,\ {}^tX=\bar{X};\ X, Y\in M_n(\boldsymbol{C})\right\}.$$

Let Ad: $\boldsymbol{Sp}(n, \boldsymbol{C})\rightarrow \boldsymbol{GL}(\mathfrak{sp}(n, \boldsymbol{C}))$ be the adjoint representation defined by $Ad(A)\,X=AXA^{-1}$ for $A\in\boldsymbol{Sp}(n, \boldsymbol{C})$, $X\in\mathfrak{sp}(n, \boldsymbol{C})$. Then $\mathfrak{sp}(n)$ and $\mathfrak{h}(n)$ are $Ad(\boldsymbol{Sp}(n))$-invariant real vector subspaces of $\mathfrak{sp}(n, \boldsymbol{C})$, the correspondence of $M\in\mathfrak{sp}(n)$ into $\sqrt{-1}\,M\in\mathfrak{h}(n)$ is an $Ad(\boldsymbol{Sp}(n))$-equivariant isomorphism, and

$$\mathfrak{sp}(n, \boldsymbol{C})=\mathfrak{sp}(n)\oplus\mathfrak{h}(n)$$

as a direct sum of $Ad(\boldsymbol{Sp}(n))$-vector spaces. Define certain real vector subspaces of $\mathfrak{sp}(n, \boldsymbol{C})$ as follows:

$$\mathfrak{sp}(n-1, \boldsymbol{C})=\left\{\left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & X_{11} & 0 & X_{12} \\ \hline 0 & 0 & 0 & 0 \\ 0 & X_{21} & 0 & X_{22}\end{array}\right) : X_{ij}\in M_{n-1}(\boldsymbol{C})\right\},$$

$$\mathfrak{a}=\left\{\left(\begin{array}{cc|cc} 0 & -{}^tV & 0 & {}^tU \\ X & 0 & U & 0 \\ \hline 0 & {}^tY & 0 & -{}^tX \\ Y & 0 & V & 0\end{array}\right) : X, Y, U, V\in\boldsymbol{C}^{n-1}\right\},$$

$$\mathfrak{z}=\left\{\left(\begin{array}{cc|cc} \alpha & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 \\ \hline \beta & 0 & -\alpha & 0 \\ 0 & 0 & 0 & 0\end{array}\right) : \alpha, \beta, \gamma\in\boldsymbol{C}\right\},$$

$$\mathfrak{sp}(n-1)=\mathfrak{sp}(n-1, \boldsymbol{C})\cap\mathfrak{sp}(n),\quad \mathfrak{h}(n-1)=\mathfrak{sp}(n-1, \boldsymbol{C})\cap\mathfrak{h}(n).$$

Let $\boldsymbol{Sp}(n-1, \boldsymbol{C})$ and $\boldsymbol{Sp}(n-1)$ denote the connected subgroups of $\boldsymbol{Sp}(n, \boldsymbol{C})$ corresponding to the Lie subalgebras $\mathfrak{sp}(n-1, \boldsymbol{C})$ and $\mathfrak{sp}(n-1)$, respectively. Then

$$\mathfrak{sp}(n, \boldsymbol{C}) = \mathfrak{sp}(n-1) \oplus \mathfrak{h}(n-1) \oplus \mathfrak{a} \oplus \mathfrak{z}$$

as a direct sum of $Ad(\boldsymbol{Sp}(n-1))$-invariant vector spaces.

Denote by $\mathfrak{a}(a+jb, c+jd)$, the real vector subspace of $\mathfrak{a}$ consisting of all matrices of the form

$$\left(\begin{array}{cc|cc} 0 & * & 0 & * \\ Xa-\bar{Y}b & 0 & Xc-\bar{Y}d & 0 \\ \hline 0 & * & 0 & * \\ Ya+\bar{X}b & 0 & Yc+\bar{X}d & 0 \end{array}\right) : X, Y \in \boldsymbol{C}^{n-1}.$$

Here a, b, c, d are complex numbers and j is a quaternion such that $j^2 = -1$ and $ju = \bar{u}j$ for each complex number u. It is easy to see that $\mathfrak{a}(a+jb, c+jd)$ is $Ad(\boldsymbol{Sp}(n-1))$-invariant and each $Ad(\boldsymbol{Sp}(n-1))$-invariant proper subspace of $\mathfrak{a}$ is of the form $\mathfrak{a}(a+jb, c+jd)$. By definition, there is a relation

$$(1) \qquad \mathfrak{a}(q_0 q_1, q_0 q_2) = \mathfrak{a}(q_1, q_2) \quad \text{for} \quad q_r = a_r + jb_r \ \text{and} \ q_0 \neq 0 .$$

By the relation (1), we obtain the following relations:

$$(2) \qquad \begin{array}{lll} \mathfrak{a}(a+jb, c+jd) + \mathfrak{a}(a-jb, c-jd) = \mathfrak{a} & \text{if} & ad \neq bc, \\ \mathfrak{a}(a+jb, c+jd) = \mathfrak{a}(a-jb, c-jd) & \text{if} & ad = bc. \end{array}$$

Moreover we obtain the following relations by a routine work.

$$(3) \qquad \begin{array}{l} [\mathfrak{a}, \mathfrak{a}] = \mathfrak{sp}(n-1, \boldsymbol{C}) \oplus \mathfrak{z}, \\ \big[\mathfrak{h}(n-1), \mathfrak{a}(a+jb, c+jd)\big] = \mathfrak{a}(a-jb, c-jd), \\ \big[\mathfrak{a}(a+jb, c+jd), \mathfrak{a}(a+jb, c+jd)\big] = (ad-bc)\, \mathfrak{sp}(n-1) \oplus \mathfrak{z}', \end{array}$$

where $\mathfrak{z}'$ is a real vector subspace of $\mathfrak{z}$.

LEMMA 1.1. *Suppose $n \geqq 2$. Let $\mathfrak{g}$ be a proper real Lie subalgebra of $\mathfrak{sp}(n, \boldsymbol{C})$ which contains $\mathfrak{sp}(n-1)$. Then $\mathfrak{g}$ is one of the following up to conjugation*:

$$\begin{array}{ll} \mathfrak{sp}(n-1, \boldsymbol{C}) \oplus \mathfrak{a}(0, 1) \oplus \mathfrak{z}', & \mathfrak{sp}(n-1, \boldsymbol{C}) \oplus \mathfrak{z}', \\ \mathfrak{sp}(n-1) \oplus \mathfrak{a}(0, 1) \oplus \mathfrak{z}', & \mathfrak{sp}(n-1) \oplus \mathfrak{a}(1, j) \oplus \mathfrak{z}', \\ \mathfrak{sp}(n-1) \oplus \mathfrak{z}', & \end{array}$$

where $\mathfrak{z}'$ is a real vector subspace of $\mathfrak{z}$. In fact, there is an element M of *the centralizer of* $\boldsymbol{Sp}(n-1, \boldsymbol{C})$ *in* $\boldsymbol{Sp}(n, \boldsymbol{C})$ *such that* $Ad(M)\mathfrak{g}$ *coincides with one of the above.*

PROOF. Since $\mathfrak{g}$ contains $\mathfrak{sp}(n-1)$, $\mathfrak{g}$ is an $Ad(\boldsymbol{Sp}(n-1))$-invariant vector subspace of $\mathfrak{sp}(n, \boldsymbol{C})$. Hence we have

$$\mathfrak{g} = \mathfrak{sp}(n-1) \oplus \big(\mathfrak{g} \cap \mathfrak{h}(n-1)\big) \oplus (\mathfrak{g} \cap \mathfrak{a}) \oplus (\mathfrak{g} \cap \mathfrak{z})$$

as a direct sum of $Ad(\boldsymbol{Sp}(n-1))$-invariant vector subspaces. Since $\mathfrak{h}(n-1)$ is irreducible, we have $\mathfrak{g} \cap \mathfrak{h}(n-1) = 0$ or $\mathfrak{h}(n-1)$. Since $\mathfrak{g}$ is a proper Lie subalgebra of $\mathfrak{sp}(n, \boldsymbol{C})$, $\mathfrak{g}$ does not contain $\mathfrak{a}$ by (3), and hence $\mathfrak{g} \cap \mathfrak{a}$ is of the form $\mathfrak{a}(a+jb, c+jd)$. By a routine work from (1), (2) and (3), we see that $\mathfrak{g}$ is one of the following:

$$\mathfrak{sp}(n-1, \boldsymbol{C}) \oplus \mathfrak{a}(a, c) \oplus \mathfrak{z}' \ (a, c : \text{complex}), \quad \mathfrak{sp}(n-1, \boldsymbol{C}) \oplus \mathfrak{z}',$$
$$\mathfrak{sp}(n-1) \oplus \mathfrak{a}(a, c) \oplus \mathfrak{z}' \ (a, c : \text{complex}), \quad \mathfrak{sp}(n-1) \oplus \mathfrak{z}',$$
$$\mathfrak{sp}(n-1) \oplus \mathfrak{a}(a+jb, c+jd) \oplus \mathfrak{z}' \ (ad-bc=1).$$

Let a, b, c, d be complex numbers with $ad-bc=1$. Put

$$M\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\begin{array}{cc|cc} a & 0 & b & 0 \\ 0 & I_{n-1} & 0 & 0 \\ \hline c & 0 & d & 0 \\ 0 & 0 & 0 & I_{n-1} \end{array}\right).$$

Then $M\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element of the centralizer of $\boldsymbol{Sp}(n-1, \boldsymbol{C})$ in $\boldsymbol{Sp}(n, \boldsymbol{C})$, and

$$Ad\left(M\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\mathfrak{a}(1, j) = \mathfrak{a}(d-jc, -b+ja),$$
$$Ad\left(M\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)\mathfrak{a}(0, 1) = \mathfrak{a}(-c, a).$$

Thus we have the desired result. q. e. d.

Put

$$L = \left(\begin{array}{cc|c} 1 & 0 & \multirow{2}{*}{0} \\ 1 & 1 & \\ \hline \multicolumn{2}{c|}{0} & I_{n-2} \end{array}\right), \quad K = \begin{pmatrix} L & 0 \\ 0 & {}^{t}L^{-1} \end{pmatrix}.$$

Then K is an element of $\boldsymbol{Sp}(n, \boldsymbol{C})$.

LEMMA 1. 2. *Assume that* $\mathfrak{g}$ *is contained in one of the following*:

$$\mathfrak{sp}(n-1, C) \oplus \mathfrak{z}, \ \mathfrak{sp}(n-1) \oplus \mathfrak{a}(0, 1) \oplus \mathfrak{z}, \ \mathfrak{sp}(n-1) \oplus \mathfrak{a}(1, j) \oplus \mathfrak{z}.$$

Then $\mathfrak{sp}(n) \cap Ad(K)\,\mathfrak{g}$ *is contained in* $\mathfrak{sp}(2) \oplus \mathfrak{sp}(n-2)$.

PROOF. Each element of $\mathfrak{sp}(n)$ is of the form

$$A = \begin{pmatrix} X & -\bar{Y} \\ Y & \bar{X} \end{pmatrix}, \quad {}^tX + \bar{X} = 0, \quad {}^tY = Y.$$

Then

$$K^{-1}AK = \begin{pmatrix} L^{-1}XL & -L^{-1}\bar{Y}{}^tL^{-1} \\ {}^tLYL & -{}^t(L^{-1}XL) \end{pmatrix}.$$

Since $\mathfrak{sp}(n) \cap Ad(K)\,\mathfrak{g} = \{A \in \mathfrak{sp}(n) : K^{-1}AK \in \mathfrak{g}\}$, we have the desired result by a routine work. q. e. d.

Let $L(n)$, $N(n)$ denote the subgroups of $Sp(n, C)$ consisting of all matrices of the form

$$\left(\begin{array}{cc|cc} 1 & * & * & * \\ 0 & X_{11} & * & X_{12} \\ \hline 0 & 0 & 1 & 0 \\ 0 & X_{21} & * & X_{22} \end{array}\right), \qquad \left(\begin{array}{cc|cc} * & * & * & * \\ 0 & X_{11} & * & X_{12} \\ \hline 0 & 0 & * & 0 \\ 0 & X_{21} & * & X_{22} \end{array}\right)$$

for $X_{ij} \in M_{n-1}(C)$, respectively.

REMARK. The standard $Sp(n, C)$ action on $C^{2n} - \{0\}$ is transitive and $L(n)$ is an isotropy group. The standard $Sp(n, C)$ action on the complex projective $(2n-1)$-space is transitive and $N(n)$ is an isotropy group. $N(n)$ is the normalizer of $L(n)$ in $Sp(n, C)$.

THEOREM 1.3. *Suppose* $n \geqq 4$. *Let* G *be a closed proper subgroup of* $Sp(n, C)$ *which contains* $Sp(n-1)$. *Assume that each isotropy group of the restricted* $Sp(n)$ *action on the homogeneous space* $Sp(n, C)/G$ *contains a subgroup conjugate to* $Sp(n-1)$. *Then* $L(n) \subset hGh^{-1} \subset N(n)$ *for an element* h *of the centralizer of* $Sp(n-1, C)$ *in* $Sp(n, C)$.

PROOF. Let $\mathfrak{g} = \mathrm{Lie}\, G$ be the Lie algebra of G. By the assumption that G contains $Sp(n-1)$, $\mathfrak{g}$ contains $\mathfrak{sp}(n-1)$, and hence there is an element h of the centralizer of $Sp(n-1, C)$ in $Sp(n, C)$ such that $Ad(h)\,\mathfrak{g}$ coincides with one of the Lie algebras listed in Lemma 1.1. By the second assumption on G, $\mathfrak{sp}(n) \cap Ad(K)\, Ad(h)\,\mathfrak{g}$ contains a subalgebra $Ad(h')\,\mathfrak{sp}(n-1)$ for some $h' \in Sp(n)$, and hence $Ad(h)\,\mathfrak{g} = \mathfrak{sp}(n-1, C) \oplus \mathfrak{a}(0, 1) \oplus \mathfrak{z}'$ for certain real vector subspace $\mathfrak{z}'$ of $\mathfrak{z}$, by Lemma 1.2. Let $\mathfrak{z}_0$, $\mathfrak{z}_1$ denote the subspaces of $\mathfrak{z}$ consisting of all matrices of the form

$$\left(\begin{array}{cc|cc} 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \quad \left(\begin{array}{cc|cc} * & 0 & * & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & * & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

respectively. We see that if $\mathfrak{sp}(n-1, \boldsymbol{C})\oplus\mathfrak{a}(0, 1)\oplus\mathfrak{z}'$ is a Lie algebra, then $\mathfrak{z}_0\subset\mathfrak{z}'\subset\mathfrak{z}_1$. On the other hand, it is easy to see that

$$\text{Lie } L(n) = \mathfrak{sp}(n-1, \boldsymbol{C})\oplus\mathfrak{a}(0, 1)\oplus\mathfrak{z}_0,$$

$$\text{Lie } N(n) = \mathfrak{sp}(n-1, \boldsymbol{C})\oplus\mathfrak{a}(0, 1)\oplus\mathfrak{z}_1.$$

Hence we obtain $L(n)\subset hG^0h^{-1}\subset N(n)$, where G^0 is the identity component of G. Since $N(n)/L(n)$ is isomorphic to the multiplicative group of non-zero complex numbers, we see that $hG^0h^{-1}=L(n)$, $N(n)$ or $hG^0h^{-1}/L(n)$ is isomorphic to the multiplicative group of positive real numbers or the circle group. For each case the normalizer of hG^0h^{-1} in $\boldsymbol{Sp}(n, \boldsymbol{C})$ coincides with $N(n)$, and hence $L(n)\subset hGh^{-1}\subset N(n)$. q. e. d.

2. Smooth $Sp(n)$ actions

First we prepare the following two lemmas which are proved by a standard method (cf. [1], [5]).

LEMMA 2.1. *Suppose $n\geqq 5$. Let G be a closed connected proper subgroup of $\boldsymbol{Sp}(n)$ such that* $\dim \boldsymbol{Sp}(n)/G\leqq 8n-8$. *Then G coincides with $\boldsymbol{Sp}(n-i)\times K$ $(i=1, 2)$ up to an inner automorphism of $\boldsymbol{Sp}(n)$, or $n=5$ and G is isomorphic to $\boldsymbol{U}(5)$ or $\boldsymbol{SU}(5)$. Here K is a closed connected subgroup of $\boldsymbol{Sp}(i)$.*

LEMMA 2.2. *Suppose $r\geqq 4$ and $k\leqq 8r-6$. Then an orthogonal non-trivial representation of $\boldsymbol{Sp}(r)$ of degree k is equivalent to $(\nu_r)_R\oplus\theta^{k-4r}$ by an inner automorphism of $\boldsymbol{O}(k)$. Here $(\nu_r)_R\colon \boldsymbol{Sp}(r)\to\boldsymbol{O}(4r)$ is the canonical inclusion, and θ^t is the trivial representation of degree t.*

REMARK. $\dim \boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-k)\times\boldsymbol{Sp}(k)=4k(n-k)$, $\dim \boldsymbol{Sp}(5)/\boldsymbol{U}(5)=30$, $\chi(\boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-k)\times\boldsymbol{Sp}(k))=\binom{n}{k}$, $\chi(\boldsymbol{Sp}(5)/\boldsymbol{U}(5))=32$, where $\chi(\ )$ denotes the Euler characteristic. The normalizer $N(\boldsymbol{U}(5))$ of $\boldsymbol{U}(5)$ in $\boldsymbol{Sp}(5)$ has just two connected components and its identity component coincides with $\boldsymbol{U}(5)$.

In the following, let M be a closed connected smooth manifold with a non-trivial smooth $\boldsymbol{Sp}(n)$ action, and suppose $n\geqq 5$ and $\dim M\leqq 8n-8$. Put

$$F_{(i)}=\left\{x\in M\colon \boldsymbol{Sp}(n-i)\subset\boldsymbol{Sp}(n)_x\subset\boldsymbol{Sp}(n-i)\times\boldsymbol{Sp}(i)\right\},$$

$$M_{(i)}=\boldsymbol{Sp}(n)\,F_{(i)}=\left\{gx:\ g\in \boldsymbol{Sp}(n),\ x\in F_{(i)}\right\}.$$

Here $\boldsymbol{Sp}(n)_x$ denotes the isotropy group at x.

PROPOSITION 2.3. *Suppose* $M=M_{(0)}\cup M_{(1)}\cup M_{(2)}$. *Then,* (a) *the fixed point set* $F(\boldsymbol{Sp}(n-k), M_{(i)})$ *of the restricted* $\boldsymbol{Sp}(n-k)$ *action on* $M_{(i)}$ *is empty for* $k<i\leqq n-i$, (b) *if* $M_{(0)}$ *is non-empty, then* $M_{(2)}$ *is empty.*

PROOF. To prove (a), suppose that $F(\boldsymbol{Sp}(n-k), M_{(i)})$ is non-empty. Then there are $x\in F_{(i)}$ and $g\in \boldsymbol{Sp}(n)$ such that $gx\in F(\boldsymbol{Sp}(n-k), M_{(i)})$, and hence

$$\boldsymbol{Sp}(n-k)\subset \boldsymbol{Sp}(n)_{gx}=g\boldsymbol{Sp}(n)_x g^{-1}\subset g\left(\boldsymbol{Sp}(n-i)\times \boldsymbol{Sp}(i)\right)g^{-1}.$$

Since $\boldsymbol{Sp}(n-k)$ is a simple Lie group, we obtain $n-k\leqq\max(n-i, i)$, and hence $k\geqq\min(i, n-i)$. Therefore, if $k<i\leqq n-i$, then $F(\boldsymbol{Sp}(n-k), M_{(i)})$ is empty. Next we show (b). Notice that $M_{(0)}$ is the fixed point set of the $\boldsymbol{Sp}(n)$ action on M. Let σ be the isotropy representation at $x\in M_{(0)}$. By Lemma 2.2, σ is equivalent to $(\nu_n)_R\oplus$*trivial*. Then $\boldsymbol{Sp}(n-1)$ is a principal isotropy group, and hence $M_{(2)}$ is empty by (a). q. e. d.

PROPOSITION 2.4. *Suppose* $M=M_{(1)}\cup M_{(2)}$. *If* $M_{(1)}$ *and* $M_{(2)}$ *are non-empty, then* $F_{(1)}$ *is a finite set and* $\dim M=8n-8$.

PROOF. Fix $x\in F_{(1)}$. Let σ and ρ denote the slice representation at x and the isotropy representation of the orbit $\boldsymbol{Sp}(n)\,x$, respectively. Then the restriction $\sigma|\boldsymbol{Sp}(n-1)$ is equivalent to $(\nu_{n-1})_R\oplus$*trivial* by Lemma 2.2 and the assumption that $M_{(2)}$ is non-empty. On the other hand, we see that the restriction $\rho|\boldsymbol{Sp}(n-1)$ is equivalent to $(\nu_{n-1})_R\oplus$*trivial* by considering adjoint representations. Hence $(\sigma\oplus\rho)|\boldsymbol{Sp}(n-1)$ is equivalent to $2(\nu_{n-1})_R\oplus$*trivial*. The desired result follows immediately. q. e. d.

PROPOSITION 2.5. *Suppose* $M=M_{(0)}\cup M_{(1)}$. *Then there is a compact connected* $\boldsymbol{Sp}(1)$ *manifold* X *such that the* $\boldsymbol{Sp}(1)$ *action is free on the boundary* ∂X *and the* $\boldsymbol{Sp}(n)$ *manifold* M *is equivariantly diffeomorphic to* $\partial(\boldsymbol{D}^{4n}\times X)/\boldsymbol{Sp}(1)$. *Here* $\boldsymbol{Sp}(n)$ *acts naturally on* $\boldsymbol{D}^{4n}$ *and trivially on* X, *and* $\boldsymbol{Sp}(1)$ *acts on* $\boldsymbol{D}^{4n}$ *as right scalar multiplication.*

PROOF. Let U be a closed $\boldsymbol{Sp}(n)$ invariant tubular neighborhood of $M_{(0)}$ in M. Then U is regarded as a $4n$-disk bundle over $M_{(0)}$ with a smooth $\boldsymbol{Sp}(n)$ action as bundle isomorphisms. It follows from Lemma 2.2 that there is an equivariant decomposition:

$$U=\left(\boldsymbol{D}^{4n}\times F\left(\boldsymbol{Sp}(n-1), \partial U\right)\right)/\boldsymbol{Sp}(1),$$

where we regard $\boldsymbol{Sp}(1)=N(\boldsymbol{Sp}(n-1))/\boldsymbol{Sp}(n-1)$. Put $E=M-\operatorname{int} U$. Then there is an equivariant decomposition:

$$E=\left(\boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-1)\times F\left(\boldsymbol{Sp}(n-1), E\right)\right)\Big/\boldsymbol{Sp}(1).$$

Notice that $F(\boldsymbol{Sp}(n-1), \partial U)=\partial F(\boldsymbol{Sp}(n-1), E)$. Then we see that there is an equivariant decomposition:

$$M=\partial\left(\boldsymbol{D}^{4n}\times F\left(\boldsymbol{Sp}(n-1), E\right)\right)\Big/\boldsymbol{Sp}(1).$$

Here $X=F(\boldsymbol{Sp}(n-1), E)$ is a compact connected $\boldsymbol{Sp}(1)$ manifold. If $M_{(0)}$ is non-empty, then X has non-empty boundary on which $\boldsymbol{Sp}(1)$ acts freely.

q. e. d.

REMARK. T. Wada [6] has described explicitly about the equivariant decomposition of U. Proposition 2. 5 is proved in his paper.

THEOREM 2. 6. *Suppose* $5\leqq n\leqq m\leqq 2n-2$. *Let* M *be a rational cohomology quaternion projective m-space on which* $\boldsymbol{Sp}(n)$ *acts smoothly and non-trivially. Then there is a compact connected orientable smooth* $\boldsymbol{Sp}(1)$ *manifold* X *such that the* $\boldsymbol{Sp}(1)$ *action is free on the boundary* ∂X *and the* $\boldsymbol{Sp}(n)$ *manifold* M *is equivariantly diffeomorphic to* $\partial(\boldsymbol{D}^{4n}\times X)/\boldsymbol{Sp}(1)$. *Moreover* X *is rationally acyclic.*

PROOF. Suppose first $M=M_{(i)}$ $(i=1, 2)$. Then there is a fibration: $F_{(i)}\rightarrow M\rightarrow \boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-i)\times\boldsymbol{Sp}(i)$, and hence

$$m+1=\chi(M)=\chi(F_{(i)})\cdot\chi\left(\boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-i)\times\boldsymbol{Sp}(i)\right)\equiv 0 \mod\binom{n}{i}.$$

This contradicts the assumption: $5\leqq n<m+1<2n$. Suppose next $M=M_{(1)}\cup M_{(2)}$. Then we see from Proposition 2. 4 that $m=2n-2$ and the isotropy group at each point of $F_{(1)}$ coincides with $\boldsymbol{Sp}(n-1)\times\boldsymbol{Sp}(1)$. Let σ denote the slice representation at a point of $F_{(1)}$. Then σ is a non-trivial representation of degree $4n-4$, because $M_{(2)}$ is non-empty. We see that $\sigma|\boldsymbol{Sp}(n-1)=(\nu_{n-1})_R$ by Lemma 2. 2. Therefore the principal isotropy group is isomorphic to $\boldsymbol{Sp}(n-2)\times\boldsymbol{Sp}(1)$, and hence M has a codimension one orbit. Then M has a non-principal isotropy group $\boldsymbol{Sp}(n-i)\times K$ where K is a closed subgroup of $\boldsymbol{Sp}(i)$, and

$$2n-1=\chi(M)=\chi\left(\boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-1)\times\boldsymbol{Sp}(1)\right)+\chi\left(\boldsymbol{Sp}(n)/\boldsymbol{Sp}(n-i)\times K\right).$$

This follows from the fact that if M has a codimension one orbit, then M is a union of closed tubular neighborhoods of just two non-principal orbits (cf. [2], [3]). But there is not such a closed subgroup K. This is a contradiction. Suppose that $n=5$ and M has an isotropy group whose identity component is isomorphic to $\boldsymbol{SU}(5)$ or $\boldsymbol{U}(5)$. We see that $m=8$ and M

has an orbit of codimension 1 or 2. Then we have a contradiction by computing Euler characteristics. Hence we obtain $M=M_{(0)}\cup M_{(1)}=\partial(\boldsymbol{D}^{4n}\times X)/\boldsymbol{Sp}(1)$ by Proposition 2.5. Since M is orientable, we see that X is orientable. It remains to show that X is rationally acyclic. In the following, we consider the cohomology theory with rational coefficients. Since $(\boldsymbol{D}^{4n}\times \partial X)/\boldsymbol{Sp}(1)\to\partial X/\boldsymbol{Sp}(1)$ is an orientable $4n$-disk bundle, there is an isomorphism

$$H^i\big(M,(S^{4n-1}\times X)/\boldsymbol{Sp}(1)\big)\cong H^{i-4n}\big(\partial X/\boldsymbol{Sp}(1)\big).$$

Then we have

$$(*)\qquad H^i(M)\cong H^i\big((S^{4n-1}\times X)/\boldsymbol{Sp}(1)\big)\qquad\text{for}\quad i\leqq 4n-2.$$

Now we show that the Euler class $e(p)$ of the principal $\boldsymbol{Sp}(1)$ bundle p: $\partial(\boldsymbol{D}^{4n}\times X)\to M$ is non-zero in $H^4(M)$. Assume $e(p)=0$. Then the Euler class of the bundle $S^{4n-1}\times X\to(S^{4n-1}\times X)/\boldsymbol{Sp}(1)$ is zero, and hence there is an isomorphism

$$H^*(S^{4n-1})\otimes H^*(X)\cong H^*(S^3)\otimes H^*\big((S^{4n-1}\times X)/\boldsymbol{Sp}(1)\big)$$

as graded modules by a Gysin sequence. Therefore, rank $H^{4i}(X)=1$ for $0\leqq i<n\leqq m$ by (*) and the assumption that M is a rational cohomology quaternion projective m-space. Since X is a compact connected manifold with non-empty boundary, we see that dim $X>4n-4$. On the other hand, dim $X=4(m-n+1)\leqq 4n-4$. This is a contradiction. Therefore $e(p)\neq 0$ and hence $\partial(\boldsymbol{D}^{4n}\times X)$ is a rational homology $(4m+3)$-sphere by a Gysin sequence. By the Poincaré-Lefschetz duality for the compact orientable manifold $\boldsymbol{D}^{4n}\times X$ and the homology exact sequence for the pair $(\boldsymbol{D}^{4n}\times X, \partial(\boldsymbol{D}^{4n}\times X))$, we obtain $H^i(X)=0$ for $0<i\leqq 4n$. Hence X is rationally acyclic. q. e. d.

REMARK. This result is essentially due to T. Wada [6]. In particular, the second half of the above proof is the same as the proof of Theorem 2.1 [6].

3. Proof of main theorem

First we prepare the following result.

LEMMA 3.1. *Let X be a rationally acyclic compact orientable manifold. Suppose that $\boldsymbol{Sp}(1)$ acts smoothly on X and the $\boldsymbol{Sp}(1)$ action on the non-empty boundary ∂X is free. Then the fixed point set $F(\boldsymbol{U}(1), X)$ of the restricted $\boldsymbol{U}(1)$ action consists of just one point x, and the isotropy* *group* $\boldsymbol{Sp}(1)_x$ *coincides with* $\boldsymbol{Sp}(1)$ *or the normalizer* $N(\boldsymbol{U}(1))$ *of* $\boldsymbol{U}(1)$ *in* $\boldsymbol{Sp}(1)$.

PROOF. Since $\boldsymbol{Sp}(1)$ acts freely on ∂X, each connected component of $F(\boldsymbol{U}(1), X)$ is a closed orientable manifold. On the other hand, $F(\boldsymbol{U}(1), X)$ is rationally acyclic by the Smith theorem. Therefore, $F(\boldsymbol{U}(1), X)$ consists of just one point x. The isotropy group $\boldsymbol{Sp}(1)_x$ coincides with $\boldsymbol{U}(1)$, $N(\boldsymbol{U}(1))$ or $\boldsymbol{Sp}(1)$. Suppose $\boldsymbol{Sp}(1)_x = \boldsymbol{U}(1)$. Then the subset $F(\boldsymbol{U}(1), \boldsymbol{Sp}(1)x)$ of $F(\boldsymbol{U}(1), X)$ consists of two points. This is a contradiction. q. e. d.

REMARK. $N(\boldsymbol{U}(1))/\boldsymbol{U}(1)$ is a cyclic group of order two. The standard $\boldsymbol{Sp}(n, \boldsymbol{C})$ action on the complex projective $(2n-1)$-space is transitive and $N(n)$ is an isotropy group. The restricted $\boldsymbol{Sp}(n)$ action is transitive and $\boldsymbol{Sp}(n) \cap N(n) = \boldsymbol{U}(1) \times \boldsymbol{Sp}(n-1)$. In particular,

$$\boldsymbol{Sp}(n, \boldsymbol{C}) = N(n) \cdot \boldsymbol{Sp}(n) = \left\{gh :\ g \in N(n),\ h \in \boldsymbol{Sp}(n)\right\}.$$

We shall prove now the main theorem stated in Introduction. Suppose $n \geqq 5$ and $m \leqq 2n-2$. Let $\boldsymbol{Sp}(n, \boldsymbol{C})$ act smoothly and non-trivially on a rational cohomology quaternion projective m-space M. Then the maximal compact group $\boldsymbol{Sp}(n)$ acts non-trivially on M. Suppose first $m < n$. Then we see that $m = n-1$ and $\boldsymbol{Sp}(n)$ acts transitively on M with the isotropy group $\boldsymbol{Sp}(1) \times \boldsymbol{Sp}(n-1)$ by Lemma 2. 1. Hence the $\boldsymbol{Sp}(n, \boldsymbol{C})$ action must be transitive. Since $\dim \boldsymbol{Sp}(n, \boldsymbol{C})/N(n) = 4n-2$, we get a contradiction by Theorem 1. 3. Suppose next $n \leqq m \leqq 2n-2$. From Theorem 2. 6 and Lemma 3. 1, we see that the difference $F(\boldsymbol{U}(1) \times \boldsymbol{Sp}(n-1), M) = F(\boldsymbol{Sp}(n), M)$ consists of just one point x and $\boldsymbol{Sp}(n)_x = K \times \boldsymbol{Sp}(n-1)$, where $K = N(\boldsymbol{U}(1))$ or $\boldsymbol{Sp}(1)$. Put $G = \boldsymbol{Sp}(n, \boldsymbol{C})_x$. Then G satisfies the condition of Theorem 1. 3, because $\boldsymbol{Sp}(n-1)$ is a principal isotropy group of the $\boldsymbol{Sp}(n)$ action on M. Hence $L(n) \subset hGh^{-1} \subset N(n)$ for some $h \in \boldsymbol{Sp}(n)$. Then

$$N\big(\boldsymbol{U}(1)\big) \times \boldsymbol{Sp}(n-1) \subset \boldsymbol{Sp}(n) \cap G \subset \boldsymbol{Sp}(n) \cap h^{-1}N(n)\, h\,,$$

$$\boldsymbol{Sp}(n) \cap h^{-1}N(n)\, h = h^{-1}\big(\boldsymbol{U}(1) \times \boldsymbol{Sp}(n-1)\big)\, h\,.$$

Therefore $h \in N(\boldsymbol{U}(1)) \times \boldsymbol{Sp}(n-1)$ and $N(\boldsymbol{U}(1)) \times \boldsymbol{Sp}(n-1) = \boldsymbol{U}(1) \times \boldsymbol{Sp}(n-1)$. This is a contradiction. Consequently, $\boldsymbol{Sp}(n, \boldsymbol{C})$ does not act smoothly and non-trivially on any rational cohomology quaternion projective m-space, for $n \geqq 5$ and $m \leqq 2n-2$.

REMARK. The group $\boldsymbol{GL}(n, \boldsymbol{H})$ of all regular matrices of degree n with quaternion coefficients acts naturally on the quaternion projective $(n-1)$-space $P_{n-1}(\boldsymbol{H})$. Since $\boldsymbol{Sp}(n, \boldsymbol{C})$ can be regard as a subgroup of $\boldsymbol{Gl}(2n, \boldsymbol{H})$, there is a smooth $\boldsymbol{Sp}(n, \boldsymbol{C})$ action on $P_{2n-1}(\boldsymbol{H})$.

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