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Complex powers of a class of pseudodifferential operators and their applications

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§ 0. Introduction

Seeley [15] has defined complex powers of elliptic operators P on a compact C^{∞} manifold Ω without boundary and examined asymptotic behaviors of the eigenvalues. For hypoelliptic operators satisfying, what is called, strong (H) condition of Hörmander [6], Kumano-go and Tsutsumi [9] have constructed complex powers suitable for them.

In the present paper we shall discuss complex powers $\{P_z\}_{z\in C}$ of a class of pseudodifferential operators P on the manifold Ω. Here the operator P has a symbol which vanishes exactly of order M on the characteristic set Σ, that is, P belongs to $OPL^{m,M}(\Omega;\Sigma)$ which is defined by Sjöstrand [16]. Then a condition of hypoellipticity of P with loss of $M/2$ derivatives is well known (see Boutet de Monvel [1], Boutet de Monvel-Grigis-Helffer [2] and Helffer [5]). Moreover, we shall develop asymptotic behaviors of the eigenvalues of P on the further hypotheses that P is self-adjoint and semibounded from below. For this purpose we have to construct two kinds of complex powers of P and use more convenient one for each situation.

For $M=2$, Menikoff-Sjöstrand [10], [11], [12], Sjöstrand [17] and Iwasaki [8] have studied asymptotic behaviors under various assumptions on Σ and P. In particular [12] and [17] have treated more general non-semibounded cases. Their methods are based on the construction of the heat kernel and an application of Karamata's Tauberian theorem. For general M, see also Mohamed [13]. However our method is essentially due to the theory of complex analysis (c. f. Smagin [18]). In order to carry out this, we shall study the first singularity of the trace of P_z. In elliptic case, Trace (P_z) has an extension to a meromorphic function in z in C with only simple poles ([15]). But in our case, even the first singularity is able to have a pole of second order. Accordingly we have to extend Ikehara's Tauberian theorem (see Wiener [19]).

The plan of this paper is as follows. In § 1 we give the precise definition of the operator mentioned above and a main theorem (Theorem 1.2).

In § 2 taking applications of Theorem 1. 2 in § 4 and § 5 into consideration, we construct two kinds of parametrices of $P-\zeta$ for some $\zeta\in\boldsymbol{C}$. In § 3 we construct two kinds of complex powers of P corresponding to them respectively. In § 4 we give a theorem on the first singularity of the trace of P_z. In § 5 we study asymptotic behaviors of the eigenvalues using the results in § 4 and give an example.

We shall use the notations and results of pseudodifferential operators, for which we refer to [1], [2], Duistermaat-Hörmander [4] and Hörmander [7].

§ 1. Definitions and the main theorem

Let Ω be a compact C^∞ manifold without boundary of dimension n and Σ be a closed conic submanifold of codimension d in the cotangent bundle minus the zero section $T^*\Omega\backslash 0$.

DEFINITION 1. 1. *Let m be a real number and M be a non-negative integer. The space $OPL^{m,M}(\Omega\,;\,\Sigma)$ is the set of all pseudodifferential operators $P\in L^m(\Omega)$ (see [6]) that for every local coordinate neighborhood $V\subset\Omega$, P has a symbol $\sigma(P)=p$ of the form:*

$$p(x,\xi)\sim\sum_{j=0}^{\infty}p_{m-j/2}(x,\xi)\,, \tag{1.1}$$

where $\sigma_{m-j/2}(P)=p_{m-j/2}(x,\xi)$ are elements of $C^\infty(R^n\times(R^n\backslash 0))$ and positively-homogeneous of degree $m-j/2$ in ξ (j integral) and satisfy:

(1. 2) *For every $K\Subset V$, there exists a constant $C_K>0$ such that*

$$\frac{|p_{m-j/2}(x,\xi)|}{|\xi|^{m-j/2}}\leq C_K d_\Sigma(x,\xi)^{M-j}\,,\qquad j=0,1,\cdots,M$$

and

$$\frac{|p_m(x,\xi)|}{|\xi|^m}\geq C_K d_\Sigma(x,\xi)^M \tag{1.3}$$

for $(x,\xi)\in K\times(R^n\backslash 0)$ and $|\xi|\geq 1$.
Here

$$d_\Sigma(x,\xi)=\inf_{(x',\xi')\in\Sigma}\left(|x'-x|+\left|\xi'-\frac{\xi}{|\xi|}\right|\right)$$

is the distance from $\left(x,\dfrac{\xi}{|\xi|}\right)$ to Σ. Note that d_Σ is a positively-homogeneous function of degree 0 in ξ.

The class of symbols satisfying (1. 1), (1. 2) *and* (1. 3) *in an open conic set U in $T^*\Omega\backslash 0$ is denoted by $SL^{m,M}(U;\Sigma)$.*

We describe the following hypotheses (H. 1)$\sim$(H. 3).

(H. 1) There exists a fixed proper closed convex cone Γ in $\boldsymbol{C}$ such that

$$p_m(x, \xi)\in\Gamma \qquad \text{for all} \quad (x, \xi)\in T^*\Omega\backslash 0 .$$

For every $\rho\in\Sigma$, we define a differential operator with polynomial coefficients on $\boldsymbol{R}^n$ (c. f. [2]):

$$\sigma_\rho(P)(y, D_y)=\sum_{j=0}^{M}\sum_{|\alpha+\beta|=M-j}\frac{1}{\alpha!\,\beta!}\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(\frac{\partial}{\partial\xi}\right)^{\beta}p_{m-j/2}(\rho)\,y^\alpha D_y^\beta . \tag{1.4}$$

(H. 2) There exists a ray $l=\{\zeta=\lambda e^{i\theta_0};\ \lambda\geq 0\}\subset-\Gamma$ such that for every ζ in the ray, $\sigma_\rho(P)(y, D_y)-\zeta$ is an isomorphism from $\mathcal{S}$ onto $\mathcal{S}$ where $\mathcal{S}$ denotes the space of rapidly decreasing functions.

For every $\rho_0\in\Sigma$, we can choose a conic neighborhood U_{ρ_0} of ρ_0 and a local coordinate system in U_{ρ_0}:

$$u=(u_1, u_2, \cdots, u_d)\,, \qquad v=(v_1, v_2, \cdots, v_{2n-d})$$

where u_i and v_j are C^∞ positively-homogeneous of degree 1 such that $\Sigma\cap U_{\rho_0}$ is defined by $u_1=u_2=\cdots=u_d=0$. If we choose pseudodifferential operators $U_1, U_2, \cdots, U_d$ of order 1 with symbols $\sigma(U_j)=u_j$, we can write (in U_{ρ_0})

$$P=\sum_{|\alpha|\leq M}A_\alpha(x, D_x)\,U(x, D_x)^\alpha \tag{1.5}$$

where A_α are classical pseudodifferential operators of order $m-(M+|\alpha|)/2$. If we define

$$\check{p}=\sum_{|\alpha|\leq M}a_\alpha(\rho)\,u^\alpha$$

where $\rho=(0, v(x, \xi))$ and a_α are the principal symbols of A_α, we have

$$p-\check{p}\in SL^{m,M+1} . \tag{1.6}$$

Note that $\check{p}$ is uniquely determined modulo $SL^{m,M+1}$ and

$$\sigma_\rho(P)(y, D_y)=\sum_{|\alpha|\leq M}a_\alpha(\rho)\left(\sigma_\rho(U)(y, D_y)\right)^\alpha .$$

If we write $\check{p}=\sum_{j=0}^{M}\check{p}_{m-j/2}$, we can define a function on $N_\rho\Sigma=T_\rho(T^*\Omega\backslash 0)/T_\rho\Sigma$ by the following formula:

For every $X\in N_\rho\Sigma$, $\tilde{p}(\rho, X)=\sum_{j=0}^{M}\frac{1}{(M-j)!}\tilde{X}^{M-j}\check{p}_{m-j/2}(\rho)$ where $\tilde{X}$ designs an extension of X to a neighborhood of ρ.

(H. 3) $\tilde{p}(\rho, X)\in\Gamma\backslash\{0\}$ for every $\rho\in\Sigma$ and $X\in N_\rho\Sigma$.

Note that under the conditions (H. 1)$\sim$(H. 3), P is hypoelliptic with loss of $M/2$ derivatives (see [2]), that is, for any distribution f, $Pf\in H^s(\Omega)$

implies $f \in H^{s+m-M/2}(\Omega)$ where $H^s(\Omega)$ is the Sobolev space.

Let $\mathscr{H}^{m-M/2}(\Omega;\Sigma)$ be $\bigcap_N S^{m-N,M-2N}(\Omega;\Sigma)$, which abbreviately is written by $\mathscr{H}^{m-M/2}$. Then our main theorem is as follows.

THEOREM 1.2. *Assume that $P \in OPL^{m,M}(\Omega;\Sigma)$ satisfies the hypotheses* (H. 1)~(H. 3) *and $m > M/2$. Then we can define complex powers $\{P_z\}_{z\in C}$ of P in the following sense:*

(i) $P_z \in OPS^{m\mathscr{R}e z, M\mathscr{R}e z}(\Omega;\Sigma)$,

(ii) $P_1 \equiv P$, $P_0 \equiv I$ *(modulo* $OP\mathscr{H}^{m-M/2}(\Omega;\Sigma)$),

(iii) $P_{z_1}P_{z_2} \equiv P_{z_1+z_2}$ *(modulo analytic functions of z_1 and z_2 with values in $OP\mathscr{H}^{m'-k'/2}(\Omega;\Sigma)$ for any m' and k' such that $m' > m\mathscr{R}e(z_1+z_2)$ and $m'-k'/2 > (m-M/2)\,\mathscr{R}e(z_1+z_2)$,*

(iv) *For any real s_0, $\sigma(P_z)(x,\xi)$ is an analytic function of z on $\{z;\ \mathscr{R}e < s_0\}$ with values in $S^{ms_0,Ms_0}(\Omega;\Sigma)$.*

REMARK 1.3. If we put

$$P'_z = P_z + z(P-P_1) + (1-z)(I-P_0),$$

then $\{P'_z\}_{z\in C}$ satisfy (i), (ii), (iv) and (ii)' $P'_1 = P$, $P'_0 = I$.

Here $S^{m,k}(\Omega;\Sigma)$ denotes the symbol class of [1, p. 591] i.e. $a \in S^{m,k}(\Omega;\Sigma)$ means that a is in $C^\infty(T^*\Omega\backslash 0)$ and for any vector fields $X_1, X_2, \cdots, X_p, Y_1, Y_2, \cdots, Y_q$ with smooth coefficients on $T^*\Omega\backslash 0$, positively-homogeneous of degree 0, the X_j being tangent to Σ,

$$|X_1X_2\cdots X_pY_1Y_2\cdots Y_q a| \lesssim r^m \rho_\Sigma^{k-q}$$

where r is a positively-homogeneous function of degree 1 such that it is equal to 1 on the cosphere bundle and $\rho_\Sigma = (d_\Sigma^2 + r^{-1})^{1/2}$. Here we use the notation $f \lesssim g$ for C^∞ positive functions f, g on $T^*\Omega\backslash 0$, if for any subcone $U \subset T^*\Omega\backslash 0$ with compact basis and $\varepsilon > 0$, there exists a constant C such that

$$f \le Cg \quad \text{in } U \quad \text{when } r > \varepsilon.$$

Moreover we write $f \approx g$ if $f \lesssim g$ and $g \lesssim f$ (see also [1, p. 590]). Denote by $OPS^{m,k}(\Omega;\Sigma)$ the set of pseudodifferential operators corresponding to the symbols in $S^{m,k}(\Omega;\Sigma)$. Then we remark that if M is a non-negative integer, we have $OPL^{m,M}(\Omega;\Sigma) \subset OPS^{m,M}(\Omega;\Sigma)$.

§ 2. Construction of parametrices

In this section we shall introduce the operators defined by [2] and construct parametrices of $P-\zeta$ $(\zeta \in l)$. There exists a unique differential operator on $N_\rho\Sigma$:

(2. 1) $$P_\Sigma = \sum_{|\alpha+\beta|\le M} a_{\alpha\beta}(\rho)\, u^\alpha D_u^\beta$$

where $a_{\alpha\beta}$ are positively-homogeneous of degree $m-(M+|\alpha|-|\beta|)/2$ such that

$$(p\#q)^\wedge = P_\Sigma \check{q}$$

for every $q\in SL^{m',M'}$. Here $\#$ means the composition of the symbols. In view of [2], if we put a matrix $A=(A_{jk}(\rho))_{j,k=1,2,\cdots,d}$ where $A_{jk}(\rho)=\sum_{s=1}^{n}\frac{\partial u_j}{\partial\xi_s}(\rho)\frac{\partial u_k}{\partial x_s}(\rho)$ are positively-homogeneous of degree 1, we have that for every $q\in S^{m',M'}$

(2. 2) $$(p\#q)-\sum_\beta \frac{i^{|\beta|}}{\beta!} D_u^\beta \check{p}(AD_u)^\beta q \in S^{m+m',M+M'+1}.$$

Now we shall construct a parametrix of $P-\zeta$ for every $\zeta\in l=\{\zeta=\lambda e^{i\theta_0};\ \lambda\ge 0\}$. (H. 1) ensures that we can define, for every $\zeta\in l$,

$$q_\zeta'(x,\xi)=\left(p_m(x,\xi)-e^{i\theta_0}|\xi|^{m-M/2}-\zeta\right)^{-1}.$$

PROPOSITION 2. 1. (i) *q_ζ' is analytic in ζ on l with values in $S^{-m,-M}$.*

(ii) *For any multi-indices α, β, $D_x^\alpha D_\xi^\beta q_\zeta'$ is a linear combination of the form*

$$(q_\zeta')^{k+1} h_k\ (0\le k\le |\alpha|+|\beta|)$$

where $h_k\in S^{mk-|\beta|,Mk-|\alpha+\beta|}$ are independent of ζ. In particular there exists a constant C (independent of ζ) such that

$$\left|q_\zeta'(x,\xi)\right| \le C(|\zeta|+r^m\rho_\Sigma{}^M)^{-1}.$$

(iii) $(p-\zeta)\#q_\zeta'-1=r_\zeta'\in S^{-1/2,-1}$. *Here r_ζ' is of the form $q_\zeta' r_\zeta''$ and r_ζ'' is analytic on l such that for any multi-indices α, β, we have with a constant $C_{\alpha\beta}$ (independent of ζ)*

$$|D_x^\alpha D_\xi^\beta r_\zeta''| \le C_{\alpha\beta} r^{m-1/2}\rho_\Sigma{}^{M-1}.$$

This proposition follows easily from the symbol calculus.

Next we shall construct a parametrix near Σ. Under the hypothesis (H. 2), for any $\rho\in\Sigma$ and $\zeta\in l$, $\tilde{p}(\rho,X)-\zeta$ has an inverse $\tilde{q}_\zeta(\rho,X)$ in the following sense (see [2]) : $\tilde{q}_\zeta$ satisfies

(2. 3) $$\sum_\beta \frac{i^{|\beta|}}{\beta!} D_X^\beta\left(\tilde{p}(\rho,X)-\zeta\right)\left(A(\rho)\,r(\rho)^{-2}D_X\right)^\beta \tilde{q}_\zeta(\rho,X)=1$$

If we identify X with $u/r(\rho)$ and define $q_\zeta(\rho,u)=q(\zeta;\rho,u)=\tilde{q}_\zeta(\rho,X)$, we have

PROPOSITION 2. 2. *With the above notations, we have*

(i) $\tilde{q}_\zeta$ *is quasi-homogeneous of degree* $-(m-M/2)$ *in the sense:*

$$\tilde{q}_{\lambda^{m-M/2}\zeta}(\lambda\rho, \lambda^{-1/2}X) = \lambda^{-(m-M/2)}\tilde{q}_\zeta(\rho, X).$$

(ii) q_ζ *is an analytic function on* l *with values in* $S^{-m,-M}$ *such that for any multi-indices* α, β, *we have with a constant* $C_{\alpha\beta}>0$ (*independent of* ζ)

$$|D_u^\alpha D_\rho^\beta q_\zeta| \le C_{\alpha\beta}(r^m \rho_\Sigma{}^M + |\zeta|)^{-1} r^{-(|\alpha|+|\beta|)} \rho_\Sigma{}^{-|\alpha|}$$

where $r=r(\rho)$, $\rho_\Sigma = \dfrac{|u|}{r(\rho)} + r(\rho)^{-1/2}$.

(iii) $q_\zeta(\rho, u) = (\check{p}(\rho, u)-\zeta)^{-1}$ *modulo analytic functions on* l *with values in* $S^{-m-1/2,-M-1}$.

PROOF. Since

$$\tilde{p}(\rho, X)-\zeta = \sum_{|\alpha|\le M} a_\alpha(\rho)\, r(\rho)^{|\alpha|} X^\alpha - \zeta,$$

it is quasi-homogeneous of degree $m-M/2$. Thus by the uniqueness of the inverse, (i) and the analyticity in (ii) are clear. From (2. 3) we have

$$1 = \left(\check{p}(\rho, u)-\zeta\right) q_\zeta(\rho, u) + \sum_{|\alpha|\le M} \sum_{\substack{|\beta|\ge 1\\ \beta\le\alpha}} \binom{\alpha}{\beta} \left(a_\alpha(\rho)\, u^{\alpha-\beta}\right) \left(A(\rho)\, D_u\right)^\beta q_\zeta. \tag{2. 4}$$

Here if we note that the sum in the right hand side belongs to the set of analytic functions with values in $S^{-1,-2}$, we can solve (2. 4) asymptotically. Thus let $q_\zeta \sim \sum_{k=0}^{\infty} q_{\zeta,k}$ modulo $\mathscr{X}^{-(m-M/2)}$ where $q_{\zeta,k} \in S^{-m-k,-M-2k}$, then we see from (H. 3) that $q_{\zeta,0} = (\check{p}(\rho, u)-\zeta)^{-1}$ and for $k\ge 1$, $q_{\zeta,k}$ is a linear combination of the from $(\check{p}(\rho, u)-\zeta)^{-(l+1)} r_{k,l}$ where $r_{k,l} \in S^{lm-k,lM-2k}$ $(2\le l\le 2k)$ are independent of ζ. So there exists $q_\zeta^0 \in S^{-m,-M}$ uniformly in ζ such that

$$\sum_{|\alpha|\le M} \sum_{\beta\le\alpha} \frac{i^{|\beta|}}{\beta!} D_u^\beta \left(a_\alpha(\rho)\, u^\alpha - \zeta\right) \left(A(\rho)\, D_u\right)^\beta q_\zeta^0 - 1 = h_\zeta$$

where $h_\zeta \in \mathscr{X}^0$ uniformly in ζ and $|\zeta|\, h_\zeta \in \mathscr{X}^{m-M/2}$. Again using (H. 2) we obtain $h_\zeta^0 \in \mathscr{X}^{-(m-M/2)}$ so that

$$\sum_{|\alpha|\le M} \sum_{\beta\le\alpha} \frac{i^{|\beta|}}{\beta!} D_u^\beta \left(a_\alpha(\rho)\, u^\alpha - \zeta\right) \left(A(\rho)\, D_u\right)^\beta h_\zeta^0 = h_\zeta$$

and $|\zeta|\, h_\zeta^0 \in \mathscr{X}^0$. Thus we see that (ii) and (iii) hold.

REMARK 2. 3. By the quasi-homogeneity of $\tilde{q}_\zeta$ and (H. 3), we can extend $\tilde{q}_\zeta$ analytically to $\{\zeta\,;\, \tilde{p}(\rho, X)\ne\zeta\}$ for all (ρ, X).

Define a pseudodifferential operator Q_ζ with the symbol:

$$\sigma(Q_\zeta) = \begin{cases} q_\zeta & \text{in a conic neighborhood of } \Sigma, \\ q'_\zeta & \text{outside a conic neighborhood of } \Sigma. \end{cases}$$

Here we use a standard partition of unity $\{\psi_k(x, \xi)\}_{k\in K}$ such that ψ_k are homogeneous of degree 0 and if $\operatorname{supp}\psi_k \cap \Sigma \neq \phi$, $q_\zeta(\rho, u)$ is constructed in $\operatorname{supp}\psi_k$. Then by [1] and [2], we have

$$(P-\zeta)\, Q_\zeta - I = R_\zeta^{(1)},\ (P-\zeta)\, q'_\zeta(x, D_x) - I = R_\zeta^{(2)}$$

where $\sigma(R_\zeta^{(1)})$ is an analytic function with values in $S^{0,1}$ in a conic neighborhood of Σ and in $S^{-1/2,0}$ otherwise and where $\sigma(R_\zeta^{(2)})$ in $S^{-1/2,-1}$ uniformly in ζ (c. f. [9]). Then we construct two parametrices of $P-\zeta$ as follows. If we put

$$Q_{\zeta,0}^{(1)} = Q_\zeta - q'_\zeta(x, D_x)\, R_\zeta^{(1)},\ Q_{\zeta,0}^{(2)} = q'_\zeta(x, D_x) - Q_\zeta R_\zeta^{(2)}$$

then we have

$$(P-\zeta)\, Q_{\zeta,0}^{(1)} - I = -R'_\zeta \in OPS^{-1/2,0}.$$

If we put $Q_{\zeta,j}^{(1)} = Q_{\zeta,0}^{(1)}(R')^j \in OPS^{-m-j/2,-M}$ $j=0, 1, \cdots$, we have

$$(P-\zeta)\left(\sum_{j=0}^{N-1} Q_{\zeta,j}^{(1)}\right) - I \in OPS^{-N/2,0}.$$

Thus we can construct a parametrix $\tilde{Q}_\zeta^{(1)}$ of $P-\zeta$ such that $\sigma(\tilde{Q}_\zeta^{(1)}) - \sum_{j=0}^{N-1} \sigma(Q_{\zeta,j}^{(1)})$ is analytic function on l with values in $S^{-m-N/2,-M}$ for every N. Similarly we can also construct an another parametrix $\tilde{Q}_\zeta^{(2)}$ by using $Q_{\zeta,0}^{(2)}$.

§ 3. Construction of complex powers

In this section we shall construct complex powers $\{P_z^{(i)}\}_{z\in C}$, $i=1, 2$ of P. Let $\tilde{Q}_\zeta^{(i)}$ be the parametrices constructed in § 2 of $P-\zeta$ $(\zeta \in l)$ and let γ be a curve beginning at ∞, passing along l to a circle $|z| = \varepsilon_0$, then clockwise about the circle, and back to ∞ along l. If we choose ε_0 sufficiently small, we may assume that $\sigma(\tilde{Q}_\zeta^{(i)})$ are analytic on $l \cup \{|z| \le \varepsilon_0\}$. Then we define operators $P_{(z)}^{(i)}$ with symbols $\sigma(P_{(z)}^{(i)})$ by the formula:

$$\sigma(P_{(z)}^{(i)})\,(x, \xi) = \frac{-1}{2\pi i}\int_\gamma \zeta^z \sigma(\tilde{Q}_\zeta^{(i)})\,(x, \xi)\, d\zeta, \qquad i = 1, 2. \tag{3.1}$$

When $\mathcal{R}e\, z<0$, we see easily from § 2 that the integrals are absolutely convergent.

PROPOSITION 3.1. *Let $\mathcal{R}e\, z<0$. Then we have*

(i) $\sigma(P_{(z)}^{(i)}) \in S^{m\mathcal{R}e z, M\mathcal{R}e z}$ *and*

$$\sigma(P_{(z)}^{(1)}) = \begin{cases} \mu(z\,;\,\rho, u)+r(z\,;\,\rho, u) & \text{in a conic neighborhood of } \Sigma \\ (p_m - e^{i\theta_0}|\xi|^{m-M/2})^z & \text{outside a conic neighborhood of } \Sigma \end{cases}$$

modulo analytic functions on $\{\mathcal{R}e z<0\}$ *with values in* $S^{m\mathcal{R}ez-1/2, M\mathcal{R}ez}$ *uniformly in wider sense in* z. *Here* $r(z\,;\,\rho, u)=r_1(z\,;\,\rho, u)\ r_2(\rho, u)$, r_1 *is an analytic function on* $\{\mathcal{R}ez<0\}$ *with values in* $S^{m\mathcal{R}ez-m, M\mathcal{R}ez-M}$ *uniformly in* z *and* $r_2\in S^{m,M+1}$. *Moreover*

$$\mu(z\,;\,\rho, u)=\frac{-1}{2\pi i}\int_\gamma \zeta^z q_\zeta(\rho, u)\, d\zeta\,. \tag{3.2}$$

On the other hand

$$\sigma(P_{(z)}^{(2)})=(p_m-e^{i\theta_0}|\xi|^{m-M/2})^z$$

modulo analytic functions on $\{\mathcal{R}ez<0\}$ *with values in* $S^{m\mathcal{R}ez-1/2, M\mathcal{R}ez-1}$ *uniformly in wider sense in* z.

(ii) *For every* k,

$$\frac{d^k}{dz^k}\sigma(P_{(z)}^{(i)})=\frac{-1}{2\pi i}\int_\gamma (\log\zeta)^k\,\zeta^k\sigma(\tilde{Q}_\zeta^{(i)})\,d\zeta\,.$$

(iii) *Let* $\mathcal{R}ez_0<0$ *and* $m'>m\mathcal{R}ez_0$, $m'-k'/2>(m-M/2)\,\mathcal{R}ez_0$. *Then* $\sigma(P_{(z)}^{(i)})$ *are analytic on a neighborhood of* z_0 *with values in* $S^{m',k'}$.

Proof. For brevity we construct only in the case $i=1$ and drop out the index i. Let $Q_{\zeta,j}$ $(j=0, 1, \cdots)$ be the operators defined in § 2. In a conic neighborhood of Σ, $\sigma(Q_{\zeta,j})$ is of the form $(\check{p}(\rho, u)-\zeta)^{-1}r_j$ where $r_j\in S^{-j/2,0}$ uniformly in ζ. Thus we have

$$\begin{aligned} I_j(z\,;\,\rho, u) &= \frac{-1}{2\pi i}\int_\gamma \zeta^z\sigma(Q_{\zeta,j})\,(\rho, u)\,d\zeta \\ &= \frac{-1}{2\pi i}\int_\gamma \zeta^z\big(\check{p}(\rho, u)-\zeta\big)^{-1}r_j(\zeta\,;\,\rho, u)\,d\zeta\,. \end{aligned}$$

By (H. 3) and quasi-homogeneity of $\check{p}(\rho, u)$,

$$\left|\check{p}(\rho, u)\right|\geq Cr^m\rho_\Sigma{}^M\,.$$

Moreover (H. 3) implies that

$$\left|\check{p}(\rho, u)-\zeta\right|\gtrsim r^m\rho_\Sigma{}^M+|\zeta|$$

for all $\zeta\in l\cup\left\{|\zeta|\leq\frac{C}{2}r^m\rho_\Sigma{}^M\right\}$. Let γ' be a curve replaced the circle $|\zeta|=\varepsilon_0$ in γ with the circle $|\zeta|=\frac{C}{2}r^m\rho_\Sigma{}^M$. By Remark 2. 3, we may replace γ with γ' where $\gamma'=C_1+C_2+C_3$ such that

$$C_1: \zeta = -se^{i\theta_0} \qquad \text{if} \quad \frac{C}{2} r^m \rho_\Sigma^M \le s \le +\infty\,,$$

$$C_2: \zeta = \frac{C}{2} r^m \rho_\Sigma^M e^{-i\theta} \qquad \text{if} \quad \theta_0 \le \theta \le \theta_0 + 2\pi \quad \text{and}$$

$$C_3: \zeta = se^{i\theta_0} \qquad \text{if} \quad \frac{C}{2} r^m \rho_\Sigma^M \le s \le +\infty\,.$$

Put

$$I_{j,k} = \frac{-1}{2\pi i} \int_{C_k} \zeta^z \left(\check{p}(\rho, u) - \zeta\right)^{-1} r_j(\zeta\,;\, \rho, u)\, d\zeta\,, \qquad k = 1, 2, 3\,.$$

Then we have

$$\begin{aligned} |I_{j,1}| &\le \tilde{C}_1 r^{-j/2} \int_{\frac{C}{2} r^m \rho_\Sigma^M}^{\infty} s^{\mathcal{R}ez-1} ds \\ &\le \tilde{C}_2 r^{-j/2} \frac{1}{\mathcal{R}ez} \left[s^{\mathcal{R}ez}\right]_{\frac{C}{2} r^m \rho_\Sigma^M}^{\infty} \\ &\le \tilde{C}_3 r^{m\mathcal{R}ez - j/2} \rho_\Sigma^{M\mathcal{R}ez} \end{aligned}$$

where $\tilde{C}_1$, $\tilde{C}_2$ and $\tilde{C}_3$ are independent of x, ξ. Similarly we can estimate $I_{j,3}$. Moreover we have easily $|I_{j,2}| \le \tilde{C}_4 r^{m\mathcal{R}ez - j/2} \rho_\Sigma^{M\mathcal{R}ez}$. Since we can also estimate the derivatives of I_j, we see $I_j(z\,;\, \rho, u) \in S^{m\mathcal{R}ez - j/2, M\mathcal{R}ez}$. In particular, we have

$$\sigma(Q_{\zeta,0}) = q_\zeta + r_\zeta$$

where $r_\zeta = (p - \check{p})\, q'_\zeta(x, \xi)\, (\check{p} - \zeta)^{-1}$ modulo analytic functions on l with values in $S^{-m-1/2, -M}$. Therefore we have (i) in a conic neighborhood of Σ. Outside a conic neighborhood of Σ, $\sigma(Q_{\zeta,j})$ is of the form $q'_\zeta\ r_j$ where $r_j \in S^{-j/2}$ uniformly in ζ for all $j = 0, 1, \cdots$. Now we have with a constant $C' > 0$

$$\left|p_m(x, \xi) - e^{i\theta_0} |\xi|^{m - M/2}\right| \ge C' r^m \rho_\Sigma^M\,.$$

Then (H. 1) implies that for all $\zeta \in l \cup \left\{|\zeta| \le \frac{C'}{2} r^m \rho_\Sigma^M\right\}$,

$$\left|p_m(x, \xi) - e^{i\theta_0} |\xi|^{m - M/2} - \zeta\right| \gtrsim r^m \rho_\Sigma^M + |\zeta|\,.$$

Therefore by the same way as above we see that (i) holds. Finally (ii) follows from the fact that for any small $\varepsilon_1 > 0$, $|(\log \zeta)^k| \le C_{k,\varepsilon_1} |\zeta|^{\varepsilon_1}$ and (iii) is clear from (i).

PROPOSITION 3. 2. (i) *Let $\mathcal{R}ez_1 < 0$ and $\mathcal{R}ez_2 < 0$. Then we have*

$$P^{(i)}_{(z_1)} P^{(i)}_{(z_2)} \equiv P^{(i)}_{(z_1 + z_2)} \qquad i = 1, 2\,.$$

Here $\equiv$ *means that* $\sigma(P^{(i)}_{(z_1)}P^{(i)}_{(z_2)})-\sigma(P^{(i)}_{(z_1+z_2)})$ *are analytic functions in* z_1 *and* z_2 *with values in* $\mathcal{X}^{m'-k'/2}$ *for any* $m'>m\mathcal{Re}(z_1+z_2)$ *and* $m'-k'/2>(m-M/2)\mathcal{Re}(z_1+z_2)$.

(ii) *For any* $j>0$ *integer,* $P^{(i)}_{(-j)}\equiv(\tilde{Q}^{(i)})^j$ *where* $\tilde{Q}^{(i)}=\tilde{Q}^{(i)}_0$ *are the parametrices of* P.

PROOF. As in the proof of Proposition 3. 1, we prove only the case $i=1$ and drop out the index i. Since ζ^{-1} is a single valued function on γ,

$$\sigma(P_{(-1)})=-\frac{1}{2\pi i}\int_{|\zeta|=\varepsilon_0}\zeta^{-1}\sigma(\tilde{Q}_\zeta)(x,\xi)\,d\zeta\,.$$

Analyticity of $\sigma(\tilde{Q}_\zeta)$ on $|\zeta|\leq\varepsilon_0$ implies $\sigma(P_{(-1)})=\sigma(\tilde{Q})$. Therefore it suffices to prove (i). If we put

$$r_N(\zeta\,;\,x,\xi)=\sum_{j=0}^{N-1}\sigma(Q_{\zeta,j})\,,$$

then we have

$$\left|\left(\sigma(\tilde{Q}_\zeta)-r_N\right)^{(\alpha)}_{(\beta)}\right|\lesssim(|\zeta|+r^m\rho_\Sigma{}^M)^{-1}r^{-N/2-|\alpha|}\rho_\Sigma{}^{-(|\alpha|+|\beta|)}$$

uniformly in ζ. Since

$$\sigma(P_{(z_1)}P_{(z_2)})-\sum_{|\alpha|<N}\frac{1}{\alpha\,!}\sigma(P_{(z_1)})^{(\alpha)}D^\alpha_x\sigma(P_{(z_2)})$$

is an analytic function with values in $S^{m',k'}$ for any $m'>m(\mathcal{Re}z_1+\mathcal{Re}z_2)-N$ and $m'-k'/2>(m-M/2)\,\mathcal{Re}(z_1+z_2)$, we see that

$$T_1=\sigma(P_{(z_1)}P_{(z_2)})-T_2$$

where

$$T_2=\sum_{|\alpha|<N}\frac{1}{\alpha\,!}\frac{1}{(2\pi i)^2}\int_\gamma\int_{\gamma'}\zeta_1^{z_1}\zeta_2^{z_2}r_N(\zeta_1\,;\,x,\xi)^{(\alpha)}D^\alpha_x r_N(\zeta_2\,;\,x,\xi)\,d\zeta_2 d\zeta_1$$

is an analytic function in z_1 and z_2 with values in $S^{m',k'}$. Here we may assume that γ' is outside γ, but close to γ. In view of [9], if we define

$$\begin{aligned}K_N(\zeta_1,\zeta_2)&=r_N(\zeta_1\,;\,x,\xi)-r_N(\zeta_2\,;\,x,\xi)\\&\quad+(\zeta_2-\zeta_1)\left[\sum_{|\gamma|<N}\frac{1}{\gamma\,!}r_N(\zeta_1\,;\,x,\xi)^{(\gamma)}D^\gamma_x r_N(\zeta_2\,;\,x,\xi)\right],\end{aligned}$$

we have

$$\left|K_N(\zeta_1,\zeta_2)\right|\lesssim(|\zeta_1|+r^m\rho_\Sigma{}^M)^{-1}(|\zeta_2|+r^m\rho_\Sigma{}^M)^{-1}r^{m-N}\rho_\Sigma{}^{M-2N}\,.$$

Thus we have

$$T_2 = \frac{1}{(2\pi i)^2}\int_{\gamma}\int_{\gamma'} \zeta_1^{z_1}\zeta_2^{z_2}(\zeta_2-\zeta_1)^{-1}\Big[r_N(\zeta_1;\ x,\xi) - r_N(\zeta_2;\ x,\xi)\Big]\,d\zeta_2 d\zeta_1$$
$$+\frac{1}{(2\pi i)^2}\int_{\gamma}\int_{\gamma'} \zeta_1^{z_1}\zeta_2^{z_2}(\zeta_2-\zeta_1)^{-1}K_N(\zeta_1,\zeta_2)\,d\zeta_2 d\zeta_1\,.$$

Since

$$-\frac{1}{2\pi i}\int_{\gamma}\zeta_1^{z_1}(\zeta_2-\zeta_1)^{-1}d\zeta_1 = 0\,, \qquad -\frac{1}{2\pi i}\int_{\gamma'}\zeta_2^{z_2}(\zeta_2-\zeta_1)^{-1}d\zeta_2 = \zeta_1^{z_2}\,,$$

we have

$$T_2 - \left(-\frac{1}{2\pi i}\int_{\gamma}\zeta_1^{z_1+z_2}r_N(\zeta_1;\ x,\xi)\,d\zeta_1\right) \in S^{m\mathcal{R}e(z_1+z_2)-N,\,M\mathcal{R}e(z_1+z_2)-2N}\,.$$

On the other hand,

$$\sigma(P_{(z_1+z_2)}) - \frac{-1}{2\pi i}\int_{\gamma}\zeta_1^{z_1+z_2}r_N(\zeta_1;\ x,\xi)\,d\zeta_1$$

is analytic in z_1 and z_2 with values in $S^{m',k'}$. The proof is complete.

Proof of Theorem 1.2. For $i=1, 2$, we set

$$P_z^{(i)} = \begin{cases} P_{(z)}^{(i)} & \text{if } \mathcal{R}ez<0\,, \\ P^k P_{(z-k)}^{(i)} & \text{if } k \text{ is an integer such that } -1\leq \mathcal{R}ez-k<0\,. \end{cases} \tag{3.3}$$

Then we shall show that Theorem 1.2 is valid for each $\{P_z^{(i)}\}$ $i=1, 2$. Let $\{P_z\}$ be one of them. First we have

$$P_1 = P^2P_{(-1)} \equiv P^2\tilde{Q} \equiv P \quad\text{and}\quad P_0 = PP_{(-1)} \equiv P\tilde{Q} \equiv I\,.$$

If $\mathcal{R}ez<0$, $P_{z-1} = P_{(z)}P_{(-1)} \equiv P_z\tilde{Q} \equiv P_{(-1)}P_{(z)} \equiv \tilde{Q}P_z$. Thus P_z commutes with $\tilde{Q}$ and therefore $P_zP \equiv PP_z$ if $\mathcal{R}ez<0$. Consequently if $-1\leq\mathcal{R}ez_1-k_1<0$ and $-1\leq\mathcal{R}ez_2-k_2<0$, we have

$$P_{z_1}P_{z_2} = P^{k_1}P_{(z_1-k_1)}P^{k_2}P_{(z_2-k_2)}$$
$$\equiv P^{k_1+k_2}P_{((z_1+z_2)-(k_1+k_2))}\,.$$

When $-1\leq\mathcal{R}e(z_1+z_2)-(k_1+k_2)<0$, it is equal to $P_{z_1+z_2}$. When $-2\leq \mathcal{R}e(z_1+z_2)-(k_1+k_2)<-1$, we have

$$P_{(z_1+z_2)-(k_1+k_2)} \equiv \tilde{Q}P_{(z_1+z_2)-(k_1+k_2)+1}\,.$$

So it is equal to $P_{z_1+z_2}$. Thus Theorem 1.2 follows from Proposition 3.1, 3.2 and (3.3).

§ 4. The first singularity of the trace of P_z

In this and next section, we assume that, as in § 1, Σ and P satisfy

(H. 1)~(H. 3) with $\Gamma=$ non-negative real line and Ω has a fixed positive C^{∞} density $d\Omega$.

The following definitions of the densities are due to [13]. For every $\rho\in\Sigma$ we define the Lebesgue measure dX_{ρ} on $N_{\rho}\Sigma$ by:

$$\int_{M\text{-}Hess\, p_m(\rho)(X)<1} dX_{\rho}=1$$

where $M\text{-}Hess\, p_m(\rho)(X)=\frac{1}{M!}(\tilde{X}^M p_m)(\rho)$ and $\tilde{X}$ is an extension of X to a neighborhood of ρ. Note that dX_{ρ} is positively-homogeneous of degree md/M in the sense: If for every $\rho\in\Sigma$, $f_{\rho}(X)$ is defined on $N_{\rho}\Sigma$,

$$\int_{N_{\lambda\rho}\Sigma} f_{\lambda\rho}(X)\, dX_{\lambda\rho}=\lambda^{md/M}\int_{N_{\rho}\Sigma} f_{\rho}(X)\, dX_{\rho}.$$

In a conic neighborhood of Σ, we choose a local coordinates (u, v) so that (u, v) is as in the beginning in § 2 and $dx\,d\xi=r(\rho)^{-n}dv\,du$ $(\rho=(0, v(x,\xi)))$. Define a positive C^{∞} density $d\rho$ on Σ by

$$d\rho=\left\{\int_{\sum_{|\alpha|=M} a_{\alpha}(\rho)u^{\alpha}<1} du\right\} r(\rho)^{-n}\, dv|_{\Sigma}.$$

Then $d\rho$ is homogeneous of degree $(Mn-md)/M$ in the same sense as above and we have $dx\,d\xi=dX_{\rho}\,d\rho$.

According to Schwartz' kernel theorem, each pseudodifferential operator P has a distribution kernel $K(x, y)\, d\Omega_y$ on $\Omega\times\Omega$:

$$\langle Pu, v\rangle=\langle K, u\otimes v\rangle \qquad \text{for all} \quad u,\ v\in C^{\infty}(\Omega)$$

where $u\otimes v(x, y)=u(x)\, v(y)$.

In the present section, we investigate the first singularity of the trace:

$$\mathrm{Trace}\,(P_z^{(i)})=\int_{\Omega} K_z^{(i)}(x, x)\, d\Omega_x \qquad i=1, 2$$

where $K_z^{(i)}(x, y)\, d\Omega_y$ are the kernels of complex powers $P_z^{(i)}$. Then we have

THEOREM 4.1. (i) Trace $(P_z^{(i)})$ *is analytic on*

$$\left\{z\,;\ \mathcal{R}e\, z<\min\left(-\frac{n}{m},\ -\frac{2n-d}{2m-M}\right)\right\}.$$

(ii) *There are three cases on the first singularity*:

(I) *If* $md>Mn$, Trace $(P_z^{(2)})$ *has the first singularity which is a pole of order* 1 *at* $z=-\frac{n}{m}$ *and the residue is equal to*

$$(4.1)\qquad -\frac{1}{m}(2\pi)^{-n}\int_{S^*\Omega} P_m(\omega)^{-\frac{n}{m}}d\tilde{\omega}$$

where $S^*\Omega$ *is the cosphere bundle and* $d\tilde{\omega}$ *is a density on* $S^*\Omega$ *defined by* $dx\, d\xi = r^{n-1}\, dr\, d\tilde{\omega}$.

(II) *If* $md=Mn$, Trace$(P_z^{(1)})$ *has the first singularity which is a pole of order* 2 *at* $z=-\frac{n}{m}\left(=-\frac{2n-d}{2m-M}\right)$. *The coefficient of* $\left(z+\frac{n}{m}\right)^{-2}$ *in the Laurent expansion is equal to*

$$(4.2)\qquad \frac{1}{M(m-M/2)}(2\pi)^{-n}\int_{\Sigma\cap S^*\Omega}\int_{SN_\omega\Sigma}\Big(\sum_{|\alpha|=M}a_\alpha(\omega)\,X^\alpha\Big)^{-\frac{n}{m}}dX_\omega\, d\omega$$

where $SN_\omega\Sigma=\{X\in N_\omega\Sigma\,;\ M\text{-}Hess\, p_m(\omega)(X)=1\}$ *and* $d\omega$ *is a density on* $\Sigma\cap S^*\Omega$ *defined by* $d\rho=r^{-1}\, dr\, d\omega$. *Note that* $d\rho$ *is homogeneous of degree* 0 *in this case.*

(III) *If* $md<Mn$, Trace$(P_z^{(1)})$ *has the first singularity which is a pole of order* 1 *at* $z=-\frac{2n-d}{2m-M}$ *and the residue is equal to*

$$(4.3)\qquad -\frac{1}{(m-M/2)}(2\pi)^{-n}\int_{\Sigma\cap S^*\Omega}\int_{N_\omega\Sigma}\mu\Big(-\frac{2n-d}{2m-M}\,;\ \omega,\ X\Big)dX_\omega\, d\omega\,.$$

Here $\mu(z\,;\ \rho,\ X)$ *is defined by*:

$$\mu(z\,;\ \rho,\ X)=\frac{-1}{2\pi i}\int_\gamma \zeta^z\tilde{q}_\zeta(\rho,\ X)\, d\zeta\,.$$

For the proof we have to use the following

LEMMA 4.2. *Let* p_z *be an analytic function on* $\{\mathcal{R}e\, z<0\}$ *with values in* $S^{m\mathcal{R}e z-j,M\mathcal{R}e z-k}$ *uniformly in* z *and* $m>M/2$. *Define*

$$F(z)=(2\pi)^{-n}\iint_{T^*\Omega}p_z(x,\ \xi)\, dx\, d\xi\,.$$

(i) *Let the support of* p_z *be outside a conic neighborhood of* Σ. *Then* $F(z)$ *is analytic on* $\left\{\mathcal{R}e\, z<-\frac{n}{m}+\frac{j}{m}\right\}$.

(ii) *With the notation of* § 2, *we put*

$$E_1=\{(x,\ \xi)\,;\ |u|\leq\varepsilon|v|\leq|u|^2\}\quad and\quad E_2=\{(x,\ \xi)\,;\ \varepsilon|v|\geq|u|^2\}$$

for a small $\varepsilon>0$.

(ii. 1) *Let the support of* p_z be in E_1. *When* $md>Mn$, $F(z)$ *is analytic on* $\left\{\mathcal{R}e\, z<-\frac{n}{m}\right\}$ *if* $j=k=0$ *and on* $\left\{\mathcal{R}e\, z\leq-\frac{n}{m}\right\}$ *if* $2j=k>0$ *or* $j>0,\ k=0$.

When $md\leq Mn$, $F(z)$ *is analytic on* $\left\{\mathscr{R}ez\leq-\dfrac{2n-d}{2m-M}\right\}$ *if* $2j>k\geq 0$ *and on* $\left\{\mathscr{R}ez<-\dfrac{2n-d}{2m-M}\right\}$ *if* $2j=k\geq 0$. *For* $j=0$, $k=-1$, $F(z)$ *is analytic on* $\left\{\mathscr{R}ez\leq-\dfrac{2n-d}{2m-M}\right\}$ *when* $md<Mn$.

(ii. 2) *Let the support of* p_z *be in* E_2. *Then* $F(z)$ *is analytic on* $\left\{\mathscr{R}ez<-\dfrac{2n-d}{2m-M}+\dfrac{2j-k}{2m-M}\right\}$.

PROOF. By the hypothesis, there exists a constant $C>0$ (independent of z) such that

$$|p_z|\leq Cr^{m\mathscr{R}ez-j}\rho_\Sigma^{M\mathscr{R}ez-k}$$

and note that for every bounded set B,

$$(2\pi)^{-n}\int_B p_z(x,\xi)\,dx\,d\xi$$

is an entire function. In the case (i) $\rho_\Sigma\approx d_\Sigma+r^{-1/2}\gtrsim 1$ in $\operatorname{supp}p_z$ and hence $\rho_\Sigma\approx 1$. These facts imply

$$r^{m\mathscr{R}ez-j}\rho_\Sigma^{M\mathscr{R}ez-k}\approx r^{m\mathscr{R}ez-j}\,.$$

Therefore letting $\mathscr{R}ez\leq b$,

$$\int_{r\geq 1}r^{m\mathscr{R}ez-j}\rho_\Sigma^{M\mathscr{R}ez-k}dx\,d\xi\leq C'\int_1^\infty r^{mb-j+n-1}dr$$

for some constant C' (independent of z). Thus (i) holds.

In the case (ii. 1), we see $r\approx|v|$ and so $\rho_\Sigma\gtrsim\dfrac{|u|}{|v|}$. Therefore letting $a\leq\mathscr{R}ez\leq b$, we have for some $\delta>0$

$$\begin{aligned}&\int_{r\geq\delta}r^{m\mathscr{R}ez-j}\rho_\Sigma^{M\mathscr{R}ez-k}dx\,d\xi\\&\quad\leq C'\int_{1/\varepsilon}^\infty s^{(mb-Ma)-j+k+n-d-1}ds\int_{\sqrt{\varepsilon s}}^{\varepsilon s}t^{Ma-k+d-1}dt\\&\quad\leq\begin{cases}C''\displaystyle\int_{1/\varepsilon}^\infty s^{mb-j+n-1}\log s\,ds & \text{if}\quad Ma-k+d\geq 0\,,\\ C''\displaystyle\int_{1/\varepsilon}^\infty s^{(mb-Ma/2)-j+k/2+n-d/2-1}ds & \text{if}\quad Ma-k+d<0\end{cases}\end{aligned}$$

where C' and C'' are some constants independent of z. Thus (ii. 1) holds.

In the case (ii. 2), we see $r\approx|v|$ and $\rho_\Sigma\approx|v|^{-1/2}$. Since

$$r^{m\mathscr{R}ez-j}\rho_\Sigma^{M\mathscr{R}ez-k}\leq C|v|^{(m-M/2)\mathscr{R}ez-j+k/2}$$

where C is a constant (independent of z), we have for $\mathcal{R}e z\leq b$,

$$\int_{r\geq\delta} r^{m\mathcal{R}e z-j}\rho_{\Sigma}^{M\mathcal{R}e z-k}\,dx\,d\xi$$
$$\leq C'\int_{1/\varepsilon}^{\infty} s^{(m-M/2)b-j+k/2+n-d-1}ds\int_0^{\sqrt{\varepsilon s}} t^{d-1}dt$$
$$\leq C''\int_{1/\varepsilon}^{\infty} s^{(m-M/2)b-j+k/2+n-d/2-1}ds$$

where C' and C'' are some constants independent of z. Thus (ii. 2) holds.

PROOF OF THEOREM 4. 1.

Since (i) is clear from Lemma 4. 2, we shall prove (ii).

(I) *The case*: $md>Mn$ $\left(\text{therefore}-\dfrac{n}{m}<-\dfrac{2n-d}{2m-M}\right)$.

In this case we use $\sigma(P_z^{(2)})$. Then we can write

$$\sigma(P_z^{(2)})=\left(p_m(x,\xi)+|\xi|^{m-M/2}\right)^z+p_1(z;\ x,\xi)$$

where $p_1\in S^{m\mathcal{R}e z-1/2,M\mathcal{R}e z-1}$ uniformly in z. By Lemma 4. 2,

$$\int p_1(z;\ x,\xi)\,dx\,d\xi$$

is analytic on $\mathcal{R}e z\leq-\dfrac{n}{m}$. Therefore we may examine

$$(2\pi)^{-n}\int_{r\geq1}\left(p_m(x,\xi)+|\xi|^{m-M/2}\right)^z dx\,d\xi\,.$$

Since $(p_m(x,\xi)+|\xi|^{m-M/2})^z=p_m(x,\xi)^z+p_2(z;\ x,\xi)$ where $|p_2(z;\ x,\xi)|\leq C_\varepsilon|\xi|^{(m-M/2)\varepsilon}$ $p_m(x,\xi)^{z-\varepsilon}$ for a small $\varepsilon>0$, it is sufficient to consider

$$(2\pi)^{-n}\int_{r\geq1}p_m(x,\xi)^z dx\,d\xi\,.$$

Note that the integral is defined when $-\dfrac{d}{M}<\mathcal{R}e z<-\dfrac{n}{m}$. Since, for $\mathcal{R}e z<-\dfrac{n}{m}$,

$$(2\pi)^{-n}\int_{r\geq1}p_m(x,\xi)^z dx\,d\xi$$
$$=\int_1^{\infty} r^{mz+n-1}dr\,(2\pi)^{-n}\int_{S^*\Omega}p_m(\omega)^z d\tilde{\omega}$$
$$=\frac{-1}{mz+n}(2\pi)^{-n}\int_{S^*\Omega}p_m(\omega)^z d\tilde{\omega}\,,$$

we see that Trace (P_z) has the first singularity at $z=-\frac{n}{m}$ which is a pole of order 1 and the residue is equal to (4.1). The proof of that case is complete.

In the case $md\leq Mn$, we use $\sigma(P_z^{(1)})$. First we want to show that the integral

$$(2\pi)^{-n}\iint_{T^*\Omega}\sigma(P_z^{(1)})(x, \xi)\,dx\,d\xi$$

is analytic on $\left\{\mathscr{Re}\, z<-\frac{2n-d}{2m-M}\right\}$ and has a pole of order 1 or 2 at $z=-\frac{2n-d}{2m-M}$ as the first singularity if $md<Mn$ or $md=Mn$ respectively. For this purpose we say that a function $f(z;\ x, \xi)$ is negligible if

$$(2\pi)^{-n}\iint_{T^*\Omega}f(z;\ x, \xi)\,dx\,d\xi$$

is analytic on $\left\{\mathscr{Re}\, z<-\frac{2n-d}{2m-M}\right\}$ and is extended analytically to $\left\{\mathscr{Re}\, z\leq -\frac{2n-d}{2m-M}\right\}$ when $md<Mn$ or has at most a pole of order 1 at $z=-\frac{2n-d}{2m-M}$ as the first singularity when $md=Mn$. By Proposition 3.1 and Lemma 4.2, it is clear that

$$\sigma(P_z^{(1)})=\begin{cases}\mu(z;\ \rho, u)+r(z;\ \rho, u) & \text{in a conic neighborhood of } \Sigma,\\ (p_m+|\xi|^{m-M/2})^z & \text{outside a conic neighborhood of } \Sigma\end{cases}$$

modulo negligible terms.

In §2 we used the partition of unity $\{\psi_k(z, \xi)\}_{k\in K}$ such that ψ_k are homogeneous of degree 0 and if $\operatorname{supp}\psi_k\cap\Sigma\neq\phi$, $q_\zeta(\rho, u)$ in §2 is constructed in $\operatorname{supp}\psi_k$. When $\operatorname{supp}\psi_k\cap\Sigma=\phi$, by the same way as the case (I) we see that $(p_m+|\xi|^{m-M/2})^z\psi_k(x, \xi)$ is negligible. Let $\operatorname{supp}\psi_k\cap\Sigma\neq\phi$. Since $r(z;\rho, u)\in S^{m\mathscr{Re}\,z, M\mathscr{Re}\,z+1}\subset S^{m\mathscr{Re}\,z, M\mathscr{Re}\,z}$, Lemma 4.2 implies that $r(z;\rho, u)\ \psi_k$ are negligible when $md<Mn$. In the case $md=Mn$, we can write

$$\begin{aligned}r(z;\ \rho, u)&=\frac{-1}{2\pi i}\int_\gamma \zeta^z\frac{p-\check{p}}{(\check{p}-\zeta)(p_m+|\xi|^{m-M/2}-\zeta)}d\zeta\\&\equiv\frac{-1}{2\pi i}\int_\gamma \zeta^z\frac{p_m-\check{p}_m}{(\check{p}_m+|\xi|^{m-M/2}-\zeta)(p_m+|\xi|^{m-M/2}-\zeta)}d\zeta\\&=(p_m+|\xi|^{m-M/2})^z-(\check{p}_m+|\xi|^{m-M/2})^z\\&\equiv p_m^z-\check{p}_m^z \qquad \text{modulo negligible terms.}\end{aligned}$$

Since $p_m^z-\check{p}_m^z$ is homogeneous of degree m, we can write

$$\iint_{r\geq 1}(p_m{}^z-\tilde{p}_m{}^z)\,\psi_k\,dx\,d\xi=\int_1^\infty r^{mz+n-1}dr\int_{S^*\Omega}p_z^2(\omega)\,\psi_k(\omega)\,d\tilde{\omega}$$

where $|p_z^2(\omega)|\leq Cd_\Sigma(\omega)^{M\mathcal{R}ez+1}$. Since we may assume $d_\Sigma(\omega)\leq\delta$ for some $\delta>0$ in supp ψ_k, it is negligible. Moreover since $\psi_k(x,\xi)=\psi_{k|\Sigma}+\psi_k'(x,\xi)$ where $\psi_k'\in S^{0,1}$, by the same way as above $\mu(z;\ \rho, u)\ \psi_k'$ is negligible. Thus we may examine

$$I(z)=(2\pi)^{-n}\int_{\Sigma\cap\{r\geq 1\}}\int_{N_\rho\Sigma}\mu(z;\ \rho,\ X)\,dX_\rho d\rho\,.$$

By the quasi-homogeneity of $\tilde{q}_\zeta(\rho, X)$, we have

$$\mu(z;\ \rho,\ X)=r^{(m-M/2)z}\mu(z;\ r^{-1}\rho, r^{1/2}X)\,.$$

Since dX_ρ and $d\rho$ are positively-homogeneous of degree md/M and $(Mn-md)/M$ respectively, we have, for $\mathcal{R}ez<-\dfrac{2n-d}{2m-M}$,

$$I(z)=\int_1^\infty r^{(m-M/2)z+md/M+(Mn-md)/M-d/2-1}dr(2\pi)^{-n}\int_{S^*\Sigma}\int_{N_\omega\Sigma}\mu(z;\ \omega,\ X)\,dX_\omega d\omega$$
$$=\frac{-1}{(m-M/2)\,z+n-d/2}(2\pi)^{-n}\int_{S^*\Sigma}\int_{N_\omega\Sigma}\mu(z;\ \omega,\ X)\,dX_\omega d\omega\,.$$

Next we consider

$$\int_{N_\omega\Sigma}\mu(z;\ \omega,\ X)\,dX_\omega\,.$$

If we define, for any $X\in N_\omega\Sigma$, $|X|_\omega=(M\text{-}Hess\,p_m(\omega)\,(X))^{1/M}$, we see that

$$\int_{|X|_\omega\leq 1}\mu(z;\ \omega,\ X)\,dX_\omega$$

is an entire function and put

$$\int_{|X|_\omega\geq 1}\mu(z;\ \omega,\ X)\,dX_\omega=\int_{|X|_\omega\geq 1}\Big(\Big(\sum_{|\alpha|=M}a_\alpha(\omega)\,X^\alpha\Big)^z+r_1(z;\ \omega,\ X)\Big)\,dX_\omega$$

where $r_1(z;\ \omega, X)=\mu(z;\ \omega, X)-(\sum_{|\alpha|=M}a_\alpha(\omega)\,X^\alpha)^z$. By (2. 3), we have

$$\tilde{q}_\zeta(\omega, X)-\big(\tilde{p}(\omega, X)-\zeta\big)^{-1}=\big(\tilde{p}(\omega, X)-\zeta\big)^{-1}$$
$$\Big(\sum_{|\beta|\geq 1}\frac{i^{|\beta|}}{\beta\,!}D_X^\beta\tilde{p}(\omega, X)\,\big(A(\omega)\,D_X\big)^\beta\tilde{q}_\zeta(\omega, X)\Big)\,.$$

Noting $\tilde{p}(\omega, X)\geq|X|_\omega^M$ and $|\tilde{q}_\zeta(\omega, X)|\leq|X|_\omega^{-M}$ for large $|X|_\omega$ uniformly in ζ, we see that $\mu(z;\ \omega, X)-\tilde{p}(\omega, X)^z=O(|X|_\omega^{M\mathcal{R}ez-1})$ as $|X|_\omega\to\infty$. Moreover since it is clear that $\tilde{p}(\omega, X)^z-(\sum_{|\alpha|=M}a_\alpha(\omega)\,X^\alpha)^z=O(|X|_\omega^{M\mathcal{R}ez-1})$ as $|X|_\omega\to\infty$, we see that $r_1(z;\ \omega, X)=O(|X|_\omega^{M\mathscr{R}ez-1})$ as $|X|_\omega\to\infty$. Therefore by the same way as Lemma 4.2 the integral of $r_1(z;\ \rho, X)$ is analytic on $\left\{\mathscr{R}ez\leq-\frac{d}{M}\right\}$. Thus we may consider, for $\mathscr{R}ez<-\frac{d}{M}$

$$I_1(z)=\int_{|X|_\omega\geq 1}\Big(\sum_{|\alpha|=M}a_\alpha(\omega)\,X^\alpha\Big)^z dX_\omega\,.$$

Let $\mathscr{R}ez<-\frac{d}{M}$. Since $I_1(z)$ is equal to

$$\int_1^\infty s^{Mz+d-1}ds\int_{SN_\omega\Sigma}\Big(\sum_{|\alpha|=M}a_\alpha(\omega)\,Y^\alpha\Big)^z dY_\omega$$
$$=\frac{1}{Mz+d}\int_{SN_\omega\Sigma}\Big(\sum_{|\alpha|=M}a_\alpha(\omega)\,Y^\alpha\Big)^z dY_\omega$$

where $SN_\omega\Sigma=\{X\in N_\omega\Sigma\,;\ |X|_\omega=1\}$, $I_1(z)$ is analytic on $\mathscr{R}ez<-\frac{d}{M}$ and has the first singularity at $z=-\frac{d}{M}$ which is a pole of order 1 and the residue is equal to

$$-\frac{1}{M}\int_{SN_\omega\Sigma}\Big(\sum_{|\alpha|=M}a_\alpha(\omega)\,Y^\alpha\Big)^{-\frac{d}{M}}dY_\omega\,.$$

Thus we have the case (II) and (III) as follows.

(II) *The case*: $md=Mn$ $\left(\text{therefore }-\frac{2n-d}{2m-M}=-\frac{n}{m}=-\frac{d}{M}\right)$. In this case we can write $I(z)$

$$=\frac{1}{(m-M/2)z+n-d/2}(2\pi)^{-n}\Big[\frac{1}{Mz+d}\int_{S^*\Sigma}\int_{SN_\omega\Sigma}\Big(\sum_{|\alpha|=M}a_\alpha(\omega)\,Y^\alpha\Big)^z dY_\omega$$
$$+F(z,\omega)\Big]d\omega$$

where $F(z,\omega)$ is analytic on $\left\{\mathscr{R}ez\leq-\frac{d}{M}\right\}$. Therefore $I(z)$ has the first singularity at $z=-\frac{n}{m}$ which is a pole of order 2 and the coefficient of $\left(z+\frac{n}{m}\right)^{-2}$ in the Laurent expansion of $I(z)$ is equal to (4.2).

(III) *The case*: $md<Mn$ $\left(\text{therefore }-\frac{2n-d}{2m-M}<-\frac{n}{m}\right)$. In this case since $-\frac{2n-d}{2m-M}<-\frac{d}{M}$, $I(z)$ has the first singularity at

$z=-\dfrac{2n-d}{2m-M}$ which is a pole of order 1 and the residue is equal to (4.3). The proof is complete.

§ 5. Asymptotic behaviors of the eigenvalues of P

In this section we assume that Σ and P satisfy (H. 1)$\sim$(H. 3) with $\Gamma=$ nonnegative real line as in § 4 and $m>M/2$. Moreover we assume:

(H. 4) P is formally self-adjoint, i.e. for every $u,\ v\in C^\infty(\Omega)$,

$$\int_\Omega Pu\bar{v}d\Omega=\int_\Omega u\overline{Pv}d\Omega$$

where $d\Omega$ is a fixed positive density on Ω.
Under (H. 1)$\sim$(H. 3) and (H. 4), P is hypoelliptic with loss of $M/2$ derivatives. Therefore we can regard P as an unbounded self-adjoint operator on $L^2(\Omega)$ with the domain $\{u\in L^2(\Omega)\,;\ Pu\in L^2(\Omega)\}$ and P has only eigenvalues of finite multiplicity whose limit point can be $\pm\infty$. Moreover we assume

(H. 5) P is semibounded from below.

Thus without loss of generality we may assume that the sequence of the eigenvalues is: $1\le\lambda_1\le\lambda_2<\cdots$, $\lim_{k\to\infty}\lambda_k=\infty$ with repetition according to multiplicity. Let $N(\lambda)$ be the number of eigenvalues $\le\lambda$, that is, $N(\lambda)=\sum_{\lambda_k\le\lambda}1$. It is well known that

$$\mathrm{Trace}\,(P_z^{(i)})=\int_\Omega K_z^{(i)}(x,x)\,d\Omega_x=\sum_{k=0}^{\infty}\lambda_k^z\qquad i=1,2\,.$$

Then we have the asymptotic formula for $N(\lambda)$.

THEOREM 5.1. (c.f. [13]) (I) *If $md>Mn$, then we have*

$$\lim_{\lambda\to\infty}N(\lambda)\,\lambda^{-\frac{n}{m}}=(2\pi)^{-n}\int_{p_m(x,\xi)\le1}dx\,d\xi\,.$$

(II) *If $md=Mn$, then we have*

$$\lim_{\lambda\to\infty}\frac{N(\lambda)\,\lambda^{-\frac{n}{m}}}{\log\lambda}=\frac{n}{m(n-d/2)}(2\pi)^{-n}\int_{S^*\Sigma}d\omega\,.$$

(III) *If $md<Mn$, then we have*

$$\lim_{\lambda\to\infty}N(\lambda)\,\lambda^{-\frac{2n-d}{2m-M}}=\frac{Mn-md}{M(n-d/2)}(2\pi)^{-n}\int_{\mu(\rho)\ge1}d\rho$$

where

$$\mu(\rho)=\int_{N_\rho\Sigma}\mathscr{R}e\,\mu\Big(-\frac{2n-d}{2m-M};\rho,X\Big)dX_\rho$$

and note that $\mu(\rho)$ *is homogeneous of degree* $(md-Mn)/M$.

For the proof of this theorem, we use the following lemma and proposition.

LEMMA 5.2. *Let* $d\mu$ *be a measure on the right half axis in* $\boldsymbol{R}$ *defined by a non-negative monotone increasing function* μ *with* $\mu(0)=0$. *Assume that*

$$F(w)=\int_0^\infty e^{-wx}d\mu(x)$$

is convergent for $\mathscr{R}e\,w>1$ (*hence analytic*). *Moreover assume that there exist complex numbers* $A_1, A_2, \cdots, A_p$ *such that*

$$H(w)=F(w)-\sum_{j=1}^{p}\frac{A_j}{(w-1)^j}$$

is continuous on the closed half plane $\mathscr{R}e\,w\geq 1$. *Then we have*

$$\lim_{x\to\infty}\frac{\mu(x)}{x^{p-1}e^x}=\mathscr{R}e\,A_p\,.$$

Note that this lemma is an extension of Ikehara's Tauberian theorem which is treated the case $p=1$ (c.f. [19]). The proof is essentially based on Donoghue [3].

PROOF. Let $\mathscr{R}e\,w>1$. Then the integration by parts leads to

$$G(w)=\frac{1}{w}\int_0^\infty e^{-wx}d\mu(x)=\int_0^\infty e^{-wx}\mu(x)\,dx\,.$$

If we put

$$\frac{1}{w}=\sum_{k=0}^{p-1}(-1)^k(w-1)^k+g(w)\,,$$

we see that $g(w)$ has the zero of order p at $w=1$. Therefore we can write

$$G(w)=\sum_{j=1}^{p}\frac{A'_j}{(w-1)^j}+h(w)$$

where $h(w)$ is analytic on $\mathscr{R}e\,w>1$ and continuous on $\mathscr{R}e\,w\geq 1$ and $A'_p=A_p$. Next put $b(x)=e^{-x}\mu(x)$ and for $\varepsilon>0$,

$$a_\varepsilon(x)=\begin{cases}0 & \text{if}\quad x<0\,,\\ e^{-\varepsilon x} & \text{if}\quad x>0\,.\end{cases}$$

If we take $w=1+\varepsilon+i\xi$ (ξ real), then we have

$$G(w) = \int_0^\infty e^{-\varepsilon x} b(x)\, e^{-i\xi x} dx = (a_\varepsilon b)\hat{\ }(\xi)\,.$$

Here $\hat{\ }$ means the Fourier transformation. Since

$$\frac{(j-1)\,!}{(\varepsilon+i\xi)^j} = (x^{j-1} a_\varepsilon)\,(\xi)\,,$$

we have

$$(a_\varepsilon b)\hat{\ }(\xi) = \sum_{j=1}^{p} \frac{A'_j}{(j-1)\,!} (x^{j-1} a_\varepsilon)\hat{\ }(\xi) + h(1+\varepsilon+i\xi)\,.$$

Therefore by the definition of the Fourier transformation of $\mathscr{S}'$, for any $\phi \in \mathscr{S}$,

$$\int_0^\infty e^{-\varepsilon x} b(x)\, \hat{\phi}(x)\, dx$$
$$= \sum_{j=1}^{p} \frac{A'_j}{(j-1)\,!} \int_0^\infty x^{j-1} e^{-\varepsilon x} \hat{\phi}(x)\, dx + \int_{-\infty}^\infty h(1+\varepsilon+i\xi)\, \phi(\xi)\, d\xi\,.$$

Now select $\phi(\xi) \in C_0^\infty(\boldsymbol{R})$ such that $\hat{\phi}(x) \geq 0$ and $\int \hat{\phi}(x)\, dx = 1$, and then replace $\phi(\xi)$ in the above with $\phi(\xi) e^{iy\xi}$. Then we have

$$\int_0^\infty e^{-\varepsilon x} b(x)\, \hat{\phi}(x-y)\, dx$$
$$= \sum_{j=1}^{p} \frac{A'_j}{(j-1)\,!} \int_0^\infty x^{j-1} e^{-\varepsilon x} \hat{\phi}(x-y)\, dx + \int_{-\infty}^\infty h(1+\varepsilon+i\xi)\, \phi(\xi)\, e^{iy\xi} d\xi\,.$$

As $\varepsilon \to 0$, each integral on the right hand side converges to a finite limit because of the integrability of $\hat{\phi}$ and the continuity of h on $\mathscr{R}e\, w \geq 1$. Since the integral on the left is positive and increasing as $\varepsilon \to 0$, Beppo-Levi's theorem implies that the limit is integrable. If we take the real part in the above, then we have

$$\int_0^\infty b(x)\, \hat{\phi}(x-y)\, dx$$
$$= \mathscr{R}e \left[\sum_{j=1}^{p} \frac{A'_j}{(j-1)\,!} \int_0^\infty x^{j-1} \hat{\phi}(x-y)\, dx + \int_{-\infty}^\infty h(1+i\xi)\, \phi(\xi)\, e^{-iy\xi} d\xi \right].$$

As $y \to +\infty$, the last integral on the right hand side converges to 0 by the Riemann-Lebesgue lemma. Since

$$\int_0^\infty x^{j-1} \hat{\phi}(x-y)\, dx = \sum_{k=0}^{j-1} \binom{j-1}{k} y^{j-1-k} \int_{-y}^\infty x^k \hat{\phi}(x)\, dx\,,$$

$$\text{(A)} \qquad \lim_{y\to\infty} \frac{1}{y^{p-1}} \int_0^\infty b(x)\, \hat{\phi}(x-y)\, dx = \frac{\mathscr{R}e\, A_p}{(p-1)\,!} \int_{-\infty}^\infty \hat{\phi}(x)\, dx = \frac{\mathscr{R}e\, A_p}{(p-1)\,!}\,.$$

When $x>x'>0$, $b(x)\geq e^{x'-x}b(x')$. Therefore

$$\int_0^\infty b(x)\,\hat{\phi}(x-y)\,dx=\int_0^y b(x)\,\hat{\phi}(x-y)\,dx+\int_y^\infty b(x)\,\hat{\phi}(x-y)\,dx$$
$$\geq b(y)\int_y^\infty e^{-(x-y)}\,\hat{\phi}(x-y)\,dx=b(y)\int_0^\infty e^{-x}\hat{\phi}(x)\,dx\,.$$

Hence, from (A),

(B) $$\varlimsup_{y\to\infty}\frac{b(y)}{y^{p-1}}\leq\frac{\mathscr{Re}A_p}{\int_0^\infty e^{-x}\hat{\phi}(x)\,dx(p-1)\,!}\,.$$

Here for $\hat{\phi}(x)$ we substitute $\delta\hat{\phi}(\delta x-\sqrt{\delta})=\hat{\phi}(x)$, which is also a positive function in $\mathscr{S}$ with the integral equal to 1 and if $\delta\to 0$,

$$\int_0^\infty e^{-x}\hat{\phi}(x)\,dx$$

converges to 1. Then we have

$$\varlimsup_{y\to\infty}\frac{b(y)}{y^{p-1}}\leq\frac{\mathscr{Re}A_p}{(p-1)\,!}\,.$$

Next we decompose

$$\int_0^\infty b(x)\,\hat{\phi}(x-y)\,dx=\int_{-y}^\infty b(x+y)\,\hat{\phi}(x)\,dx=\int_{-y}^{-1}+\int_{-1}^0+\int_0^\infty=\sum_{k=1}^3 I_k(y)\,.$$

Since

$$\frac{1}{y^{p-1}}I_1(y)\leq\sup_{x>0}\frac{b(x)}{x^{p-1}}\int_{-y}^{-1}\left(\frac{x}{y}+1\right)^{p-1}\hat{\phi}(x)\,dx\,,$$
$$\frac{1}{y^{p-1}}I_2(y)\leq\frac{b(y)}{y^{p-1}}\int_{-1}^0 e^{-x}\hat{\phi}(x)\,dx$$

and

$$\frac{1}{y^{p-1}}I_3(y)\leq\sup_{x\geq y}\frac{b(x)}{x^{p-1}}\int_0^\infty\left(\frac{x}{y}+1\right)^{p-1}\hat{\phi}(x)\,dx\,,$$

from (A) and (B) we have

$$\frac{\mathscr{Re}A_p}{(p-1)\,!}\int_{-\infty}^\infty\hat{\phi}(x)\,dx$$
$$\leq\sup_{x>0}\frac{b(x)}{x^{p-1}}\int_{-\infty}^{-1}\hat{\phi}(x)\,dx+\varliminf_{y\to x}\frac{b(y)}{y^{p-1}}\int_{-1}^0 e^{-x}\hat{\phi}(x)\,dx$$
$$+\frac{\mathscr{Re}A_p}{(p-1)\,!}\int_0^\infty\hat{\phi}(x)\,dx\,.$$

Therefore

$$\frac{\mathscr{R}e A_p}{(p-1)!}\int_{-\infty}^{0}\hat{\phi}(x)\,dx$$
$$\leq \sup_{x>0}\frac{b(x)}{x^{p-1}}\int_{-\infty}^{-1}\hat{\phi}(x)\,dx+\overline{\lim_{y\to\infty}}\frac{b(y)}{y^{p-1}}\int_{-1}^{0}e^{-x}\hat{\phi}(x)\,dx\,.$$

Replacing $\hat{\phi}(x)$ with $\varepsilon\hat{\phi}(\varepsilon x)$ and letting $\varepsilon\to\infty$,

$$\frac{\mathscr{R}e A_p}{(p-1)!}\int_{-\infty}^{0}\hat{\phi}(x)\,dx\leq\overline{\lim_{y\to\infty}}\frac{b(y)}{y^{p-1}}\int_{-\infty}^{0}\hat{\phi}(x)\,dx\,.$$

Thus we have

$$\frac{\mathscr{R}e A_p}{(p-1)!}\leq\overline{\lim_{y\to\infty}}\frac{b(y)}{y^{p-1}}\,.$$

This completes the proof.

PROPOSITION 5.3. *Let* $\sum_{k=1}^{\infty}\lambda_k^z$ *be convergent for* $\mathscr{R}e z<s_0(<0)$, *hence analytic. Assume that there exist complex numbers* $A_1, A_2, \cdots, A_p$ *such that*

$$\sum_{k=1}^{\infty}\lambda_k^z-\sum_{j=1}^{p}\frac{A_j}{(z-s_0)^j}$$

is continuous on $\mathscr{R}e z\leq s_0$. *Then we have*

$$\lim_{\lambda\to\infty}\frac{(-1)^{p-1}s_0N(\lambda)\,\lambda^{s_0}}{(\log\lambda)^{p-1}}=\frac{\mathscr{R}e A_p}{(p-1)!}\,.$$

PROOF OF PROPOSITON 5.3.

Let $s_0<0$ and

$$f(z)=\int_{1}^{\infty}x^{-\frac{z}{s_0}}d\alpha(x)$$

where $\alpha(x)$ is the number of eigenvalues such that $(\lambda_k)^{-s_0}\leq x$. Then $\alpha(x)$ is monotone increasing and $f(z)=\sum_{k=1}^{\infty}\lambda_k^z$. By the hypotheses, $f(z)$ is analytic on $\mathscr{R}e z<s_0$ and

$$f(z)-\sum_{j=1}^{p}\frac{A_j}{(z-s_0)^j}$$

is continuous on $\mathscr{R}e z\leq s_0$. If we put $\mu(x)=\alpha(e^x)$, $\dfrac{z}{s_0}=w$ and $F(w)=f(z)$, we see that

$$F(x)=\int_{0}^{\infty}e^{-wx}d\mu(x)$$

is analytic on $\mathscr{Re}w>1$ and

$$F(w)-\sum_{j=1}^{p}\frac{B_j}{(w-1)^j}\quad\left(B_j=\frac{A_j}{s_0^{\,j}}\right)$$

is continuous on $\mathscr{Re}w\geq 1$. Thus if we apply Lemma 5.2, we see

$$\lim_{x\to\infty}\frac{\alpha(x)}{x(\log x)^{p-1}}=\frac{\mathscr{Re}A_p}{(p-1)!\,s_0^{\,p}}\,.$$

Taking $x=\lambda^{-s_0}$, we have

$$\lim_{\lambda\to\infty}\frac{(-1)^{p-1}s_0N(\lambda)\,\lambda^{s_0}}{(\log\lambda)^{p-1}}=\frac{\mathscr{Re}A_p}{(p-1)!}\,.$$

This completes the proof of Proposition 5.3.

PROOF OF THEOREM 5.1.

The case (I): Since

$$\int_{S^*\Omega}p_m(\omega)^{-\frac{n}{m}}d\tilde{\omega}=\frac{m}{\Gamma\left(\frac{n}{m}\right)}\int e^{-p_m(x,\xi)}\,dx\,d\xi=n\int_{p_m(x,\xi)\leq 1}dx\,d\xi\,,$$

it is easy from Proposition 5.3.

The case (II): If we put

$$\nu(t)=\int_{\sum_{|\alpha|=M}a_\alpha(\omega)X^\alpha<t}dX_\omega\,,$$

and let $\lambda^{-1/M}t\to t$, then we have $\nu(t)=t^{d/M}\nu(1)=t^{d/M}$. On the other hand we have

$$\begin{aligned}
&\int\exp\Big(-\sum_{|\alpha|=M}a_\alpha(\omega)\,X^\alpha\Big)\,dX_\omega\\
&\quad=\int\exp\left(-|X|_\omega^M\sum_{|\alpha|=M}a_\alpha(\omega)\left(\frac{X}{|X|_\omega}\right)^\alpha\right)dX_\omega\\
&\quad=\frac{1}{M}\int_0^\infty e^{-s}s^{\frac{d}{M}-1}ds\int_{SN_\omega\Sigma}\Big(\sum_{|\alpha|=M}a_\alpha(\omega)\,Y^\alpha\Big)^{-\frac{d}{M}}dY_\omega\\
&\quad=\frac{1}{M}\Gamma\left(\frac{d}{M}\right)\int_{SN_\omega\Sigma}\Big(\sum_{|\alpha|=M}a_\alpha(\omega)\,Y^\alpha\Big)^{-\frac{d}{M}}dY_\omega\,.
\end{aligned}$$

Since

$$\int\exp\Big(-\sum_{|\alpha|=M}a_\alpha(\omega)\,X^\alpha\Big)\,dX_\omega=\int_0^\infty e^{-t}d\nu(t)=\frac{d}{M}\Gamma\left(\frac{d}{M}\right),$$

we have

$$\int_{SN_\omega \Sigma}\Big(\sum_{|\alpha|=M} a_\alpha(\omega)\, Y^\alpha\Big)^{-\frac{d}{M}} dY_\omega = d\,.$$

If we note $\dfrac{d}{M}=\dfrac{n}{m}$ and apply Proposition 5. 2, we see that (II) holds.

The case (III): In this case we have

$$\int_{\mu(\rho)\geq 1} d\rho$$
$$=\int_{r(\rho)^{(md-Mn)/M}\mu(\omega)\geq 1} r(\rho)^{-(md-Mn)/M-1}dr(\rho)\, d\omega$$
$$=\frac{M}{Mn-md}\int_{S^*\Sigma}[r^{-(md-Mn)/M}]_0^{\mu(\omega)^{M/(Mn-md)}}d\omega$$
$$=\frac{M}{Mn-md}\int_{S^*\Sigma}\mu(\omega)\, d\omega\,.$$

Thus applying Proposition 5. 3 we see that (III) holds. This completes the proof of Theorem 5. 1.

If we take $\lambda=\lambda_k$ in Theorem 5. 1, we can also give the asymptotic formula which is an extension of [15] to the hypoelliptic case.

COROLLARY 5. 4. (I) *If $md>Mn$, then we have*

$$\lim_{k\to\infty} k\lambda_k^{-\frac{n}{m}}=(2\pi)^{-n}\int_{p_m(x,\xi)\leq 1} dx\, d\xi\,.$$

(II) *If $md=Mn$, then we have*

$$\lim_{k\to\infty}\frac{k\lambda_k^{-\frac{n}{m}}}{\log\lambda_k}=\frac{n}{m(n-d/2)}(2\pi)^{-n}\int_{S^*\Sigma} d\omega\,.$$

(III) *If $md<Mn$, then we have*

$$\lim_{k\to\infty} k\lambda_k^{-\frac{2n-d}{2m-M}}=\frac{Mn-md}{M(n-d/2)}(2\pi)^{-n}\int_{\mu(\rho)\geq 1} d\rho\,.$$

EXAMPLE. Let Ω be a compact C^∞ Riemannian manifold of dimension $n>1$ with the metric $\sum_{j,k=1}^{n} g_{jk}(x)\,dx^j dx^k$ and its volume element $d\Omega=g^{1/2}dx$ ($g=\det(g_{jk})$). Let $\phi_i\in C^\infty(\Omega)$ $i=1, 2, \cdots, d$ ($d<n$) such that ϕ_i are real valued and $d\phi_1, d\phi_2, \cdots, d\phi_d$ are linearly independent at $\Omega_1=\{x\in\Omega\,;\ \phi_i(x)=0,\ i=1, 2, \cdots, d\}$. Define

$$\Delta_\phi=-\sum_{j,k=1}^{n} g^{-1/2}\frac{\partial}{\partial x_j}(\phi g^{1/2}g^{jk})\frac{\partial}{\partial x_k}$$

where $\phi=\sum_{i=1}^{d}\phi_i^2$ and $(g^{jk})=(g_{jk})^{-1}$. We consider the operator

$$P=\Delta_{\phi}+\sqrt{-\Delta}$$

where Δ is the Laplace-Beltrami operator on Ω (c. f. Nordin [14]). Then for $\rho\in\Sigma=\{(x,\xi)\in T^*\Omega\backslash 0\,;\ x\in\Omega_1\}=\pi^{-1}(\Omega_1)$, we have

$$\sigma_{\rho}(P)(y,D_y)=\Big(\sum_{j,k=1}^{n}\frac{\partial^2\phi}{\partial x_j\partial x_k}\big(\pi(\rho)\big)y_jy_k\Big)\Big(\sum_{j,k=1}^{n}g^{jk}\big(\pi(\rho)\big)\xi_j\xi_k\Big)+\sigma_1(\sqrt{-\Delta})(\rho).$$

where π is the natural projection $T^*\Omega\backslash 0\rightarrow\Omega$. Thus $\sigma_{\rho}(P)(y,D_y)$ is an isomorphism from $\mathscr{S}$ onto $\mathscr{S}$ and satisfies (H. 1)$\sim$(H. 5). Therefore we have

$$\lim_{\lambda\leftarrow\infty}N(\lambda)\,\lambda^{-(n-d/2)}=\frac{1}{n-d/2}(2\pi)^{-n}\int_{S^*\Sigma}\int_{N_{\omega}\Sigma}\Big(Hess\,\phi\big(\pi(\omega)\big)(X)+1\Big)^{-(n-d/2)}dX_{\omega}d\omega$$

where $S^*\Sigma=\Big\{\rho=(x,\xi)\in\Sigma\,;\ r(x,\xi)=\sqrt{\sum_{j,k=1}^{n}g^{jk}(x)\xi_j\xi_k}=1\Big\}$. Since $|X|_{\omega}=\{Hess\,\phi(\pi(\omega))(X)+1\}^{1/2}$, the right hand side is equal to

$$\frac{1}{n-d/2}(2\pi)^{-n}\int_{S^*\Sigma}\int_{|X|_{\omega}=1}\int_0^{\infty}(s^2+1)^{-(n-d/2)}s^{d-1}ds\,dX_{\omega}d\omega$$
$$=\frac{1}{n-d/2}(2\pi)^{-n}\frac{\Gamma(d/2)\,\Gamma(n-d)}{2\Gamma(n-d/2)}\int_{S^*\Sigma}\int_{|X|_{\omega}=1}dX_{\omega}d\omega\,.$$

By the definitions of dX_{ω} and $d\omega$, we see

$$\int_{|X|_{\omega}=1}dX_{\omega}=d$$

and

$$\int_{S^*\Sigma}d\omega=(\text{the volume of the unit sphere in } \boldsymbol{R}^d)\times$$
$$(\text{the surface area of the unit sphere in } \boldsymbol{R}^n)\times\int_{\Omega_1}d\Omega|_{\Omega_1}\,.$$

Thus we have

$$\lim_{\lambda\rightarrow\infty}N(\lambda)\,\lambda^{-(n-d/2)}=\frac{2^{-(n-1)}\pi^{-(n-d)/2}\Gamma(n-d)}{\Gamma(n/2)\,\Gamma(n+1-d/2)}\int_{\Omega_1}d\Omega|_{\Omega_1}\,.$$

References

[1] BOUTET DE MONVEL, L.: Hypoelliptic operators with double characteristics and related pseudodifferential operators, Comm. Pure and Appl. Math., 27 (1974), 585-639.

[2] BOUTET DE MONVEL, L., GRIGIS, A. and HELFFER, B.: Paramétrices d'opérateurs pseudo-différentiels a caractéristiques multiples, Astérisque, 34–35 (1976), 93–121.

[3] DONOGHUE, W.: Distributions and Fourier transforms, Acad. Press, New York (1969).

[4] DUISTERMAAT, J. J. and HÖRMANDER, L.: Fourier integral operators II, Acta Math. 128 (1972), 183–269.

[5] HELFFER, B.: Invariant associés à une classe d'operateurs pseudo-différentiels et applications à L'hypoellipticite, Ann. Inst. Fourier, Grenoble, 26 (1976), 55–70.

[6] HÖRMANDER, L.: Pseudodifferential operators and hypoelliptic equations, Amer. Math. Soc. Symp. Pure Math. 10 (1967), 138–183.

[7] HÖRMANDER, L.: Fourier integral operators I, Acta Math., 127 (1971), 79–183.

[8] IWASAKI, C. and IWASAKI, N.: Parametrix for a degenerate parabolic equation and its application to the asymptotic behavior of spectral functions for stationary problems, Publ. of RIMS, Kyoto Univ. 17 (1981), 577–655.

[9] KUMANO-GO, H. and TSUTSUMI, C.: Complex powers of hypoelliptic pseudo-differential operators with applications, Osaka J. Math. 10 (1973), 147–174.

[10] MENIKOFF, A. and SJÖSTRAND, J.: On the eigenvalues of a class of hypoelliptic operators, Math. Ann. 235 (1978), 55–85.

[11] MENIKOFF, A. and SJÖSTRAND, J.: On the eigenvalues of a class of hypoelliptic operators II, Springer Lecture note 755 (1978), 201–247.

[12] MENIKOFF, A. and SJÖSTRAND, J.: The eigenvalues of hypoelliptic operators III. The non semibounded case, J. Analyse Math. 35 (1979), 123–150.

[13] MOHAMED, A.: Etude spectrale d'opérateurs hypoelliptiques a caractéristiques multiple, Jour. "Equations aux derivees partielles" Saint Jean de Monts, juin 1981.

[14] NORDIN, C.: The asymptotic distribution of the eigenvalues of a degenerate elliptic operator, Ark. Mat. 10 (1972), 9–21.

[15] SEELEY, R. T.: Complex powers of a elliptic operators, Singular iutegrals, Proc. Symposia Pure Math. 10 (1967), 288–307.

[16] SJÖSTRAND, J.: Parametrices for pseudodifferential operators with multiple characteristics, Arkiv för Mat. 12 (1974), 85–130.

[17] SJÖSTRAND, J.: On the eigenvalues of a class of hypoelliptic operators IV, Ann. Inst. Fourier, Grenoble, 30–2 (1980), 109–169.

[18] SMAGIN, S. A.: Fractional powers of hypoelliptic operators in R^n, Soviet Math. Dokl. 14 (1973), No 2, 585–588.

[19] WIENER, N.: Tauberian theorems, Ann. of Math., 33 (1932), 1–100.

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