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On the generalization of union of knots

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1. Introduction

In this note, we generalize the union of knots introduced by Kinoshita and Terasaka [1] and consider its relation to the problem on the primeness of knots with the unknotting number one. All of our treatments are done in the piecewise linear category. The author is grateful to all members of Topology seminars at Kobe University, Tsuda College and Hokkaido University.

1.1 First of all, we will define a set of knots that is the central concept of this note. We use the following usual notation: Let k be a knot in the 3-sphere S^3 and let C be a 3-cell in S^3 satisfying (*) and (**):

(*) k intersects with ∂C transversely in two points.

(**) For $C_0 = cl(S^3 - C)$, $(C_0, C_0 \cap k)$ is the trivial cell pair.

Then we call $(C, C \cap k)$ the cell pair associated to the knot k. Now let K_1 and K_2 be knots in S^3. For non-negative integer n, $K_1 +_n K_2$ denotes the set of knots constructed in the following manner: Let $\bar{K}$ denote the knot sum of K_1 and K_2, and S^2 be the decomposition sphere; $\bar{K} = K_1 \underset{S^2}{\#} K_2$, and let $C_i (i=1, 2)$ be the 3-cell bounded by S^2 such that the cell pair $(C_i, C_i \cap \bar{K})$ is equivalent to the one associated to K_i. Let Γ be an arc in S^3 which satisfies the conditions (1) and (2):

(1) $\Gamma \cap \bar{K} = \partial\Gamma \cap \bar{K} = \{a, b\}$ (=two points)$\subset \bar{K} - S^2$ and $a \in C_1$,

(2) Γ and S^2 intersect transversely in n points.

We put $\Gamma \cap S^2 = \{a_1, \cdots, a_n\}$, where the ordering is from a_1 to a_n counting from nearer point to a. Next we choose a regular n. b. d. of Γ, say $\bar{B}$, satisfying (3) and (4):

(3) Each component of $\bar{B} \cap S^2$ is a disc containing exactly one point of $\Gamma \cap S^2$.

Let D_0^i denote the disc in (3) containing a_i. Then $\bar{B}$ is decomposed into $(n+1)$ 3-cells by $D_0^1 \cup \cdots \cup D_0^n$. Let $B_j (0 \leq j \leq n)$ denote the j-th cell counting from the side of a. Then:

(4) $\bar{B} \cap \bar{K} = (B_0 \cap \bar{K}) \cup (B_n \cap \bar{K})$ and, $(B_0, B_0 \cap \bar{K})$ and $(B_n, B_n \cap \bar{K})$ are trivial cell pairs.

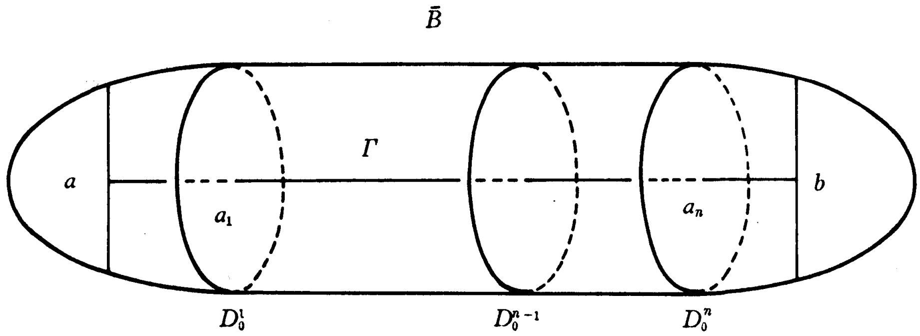


Fig. 1.

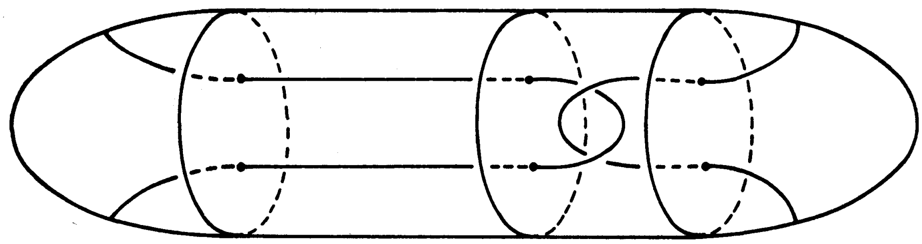


Fig. 2.

Now $(\bar{B}, \bar{B}\cap\bar{K})$ can be seen as in Fig. 1 abstractly. Taking off $\bar{B}\cap\bar{K}$ from $\bar{K}$ and adding two arcs as in Fig. 2, we obtain a knot $K=K_\Gamma$ (Note that there is an ambiguity to twist $\bar{B}$ around Γ, but we need not cause it for our problem). $K_1+_nK_2$ is the set of all knots constructed in the above way using Γ's satisfying above conditions. Specially $K_1+_1K_2$ is the union of K_1 and K_2 in the sense of Kinoshita-Terasaka.

Here we settle two facts used later (§ 5). Let $K=K_\Gamma$ be a knot contained in $K_1+_nK_2$. Then:

PROPOSITION 1.1 *Suppose there exists a disc Δ such that*:

(1) $\Delta\cap\Gamma=\widehat{a_i a_{i+1}}$ $(1\leq i\leq n-1)$.

(2) $\Delta\cap S^2$ *is an arc in* $S^2-(\Gamma-\Delta\cap\Gamma)$ *whose boundary is* $\{a_i, a_{i+1}\}$. *Then K is also contained in $K_1+_{n-2}K_2$. (See Fig. 3)*

In fact let Γ' be the arc described in Fig. 3'. Then K_Γ is equivalent to $K_{\Gamma'}$ and this shows 1.1.

PROPOSITION 1.2 *Suppose there exists a disc Δ such that*:

(1) $\Delta\cap\Gamma=\widehat{a\ a_1}$ *(or* $\widehat{a_n\ b}$*)*.

(2) $\Delta\cap S^2$ *is an arc in* $S^2-(\Gamma-\Delta\cap\Gamma)$ *whose boundary is* $\{a_1$ *(resp.* a_n*), a point, say Q, in* $\bar{K}\cap S^2\}$.

(3) $\Delta\cap\bar{K}$ *is an arc whose boundary is* $\{a$ *(resp. b)*$, Q\}$. *Then K is also contained in $K_1+_{n-1}K_2$. (See Fig. 4).*

In fact let Γ' be the arc described in Fig. 4'. Then K_Γ is equivalent to $K_{\Gamma'}$ and this shows 1.2.

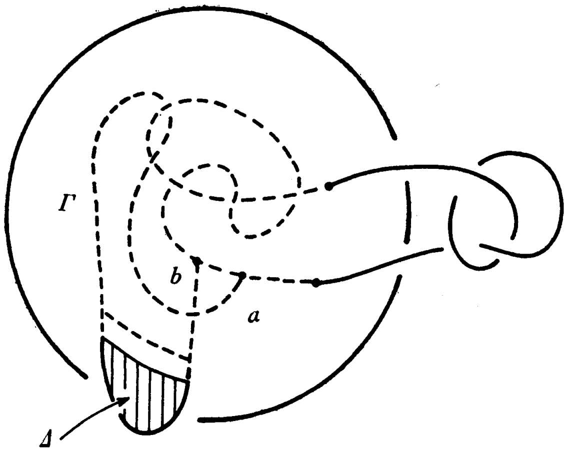


Fig. 3.

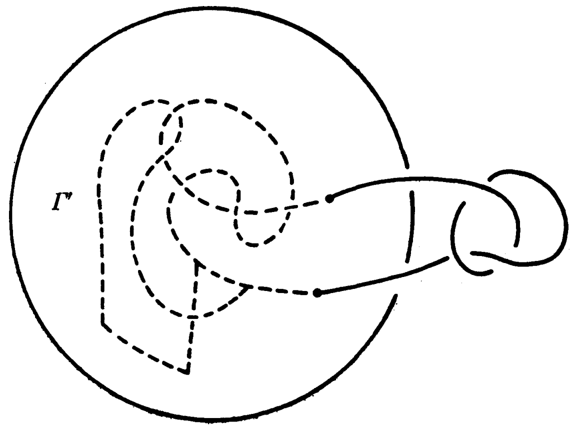


Fig. 3'.

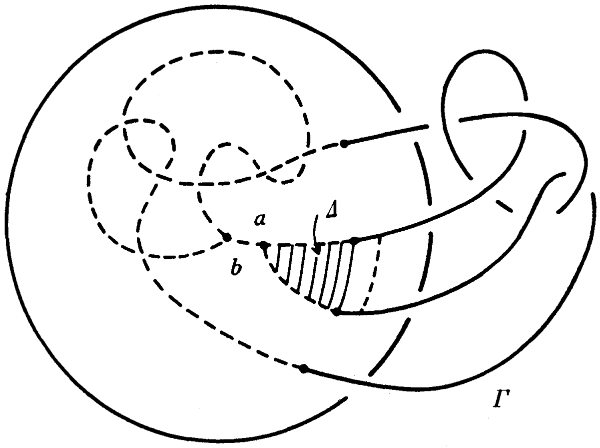


Fig. 4.

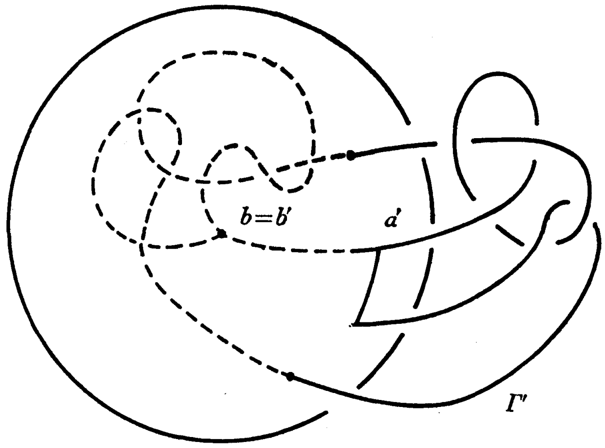


Fig. 4'.

1. 2 We will show the reason why we consider $K_1 +_n K_2$. Let $U(n)$ be the following statement:

$U(n)$: *For non-trivial knots K_1 and K_2, $K_1 +_n K_2$ does not contain the trivial knot.*

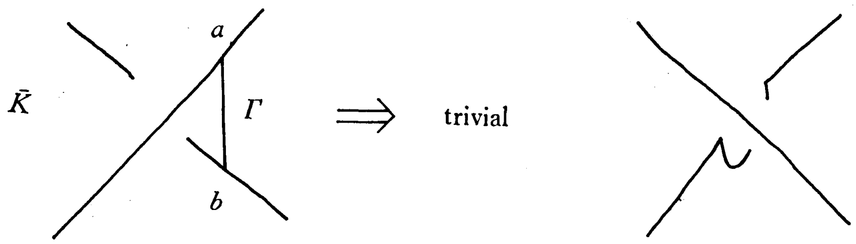
Then the following holds:

PROPOSITION. (1) *and* (2) *are equivalent.*

(1) *For all n, $U(n)$ is true.*

(2) *Knots with the unknotting number one are prime.*

PROOF: (1)$\Rightarrow$(2): Let $\bar{K}$ be a knot with the unknotting number one and suppose that there exist knots K_1 and K_2 such that $\bar{K} = K_1 \underset{S^2}{\#} K_2$, where S^2 is the decomposition sphere. Then we obtain the trivial knot by exchanging over and under crossing at some crossing point for a suitable knot projection of $\bar{K}$. Let Γ be the vertical arc which connects the over-crossing point a to the under-crossing point b:



We may assume that Γ and S^2 are in general position. Let n denote $\#(\Gamma \cap S^2)$. Then the trivial knot is contained in $K_1 +_n K_2$. So K_1 or K_2 must be trivial. The converse is similar.

1.3 Known results on $U(n)$ are as follows:

THEOREM 0. *$U(0)$ is true.*

This is a corollary to Schubert's result in [2].

THEOREM 1. ([1], Terasaka [3]). *$U(1)$ is true.*

Our result in this note is:

THEOREM 2. *$U(2)$ is true.*

Our idea of the proof is similar in spirit to that of Terasaka's in [3].

2. Outline of Proof.

First we will describe the specialized situation explicitly. Since $n=2$, $\partial\Gamma = \{a, b\} \subset C_1$. Let B denote B_1 and let B' denote C_1 and V be the solid torus $B \cup B'$. And let D_i $(i=1, 2)$ denote the disc D_0^i. See Fig. 5 and 6.

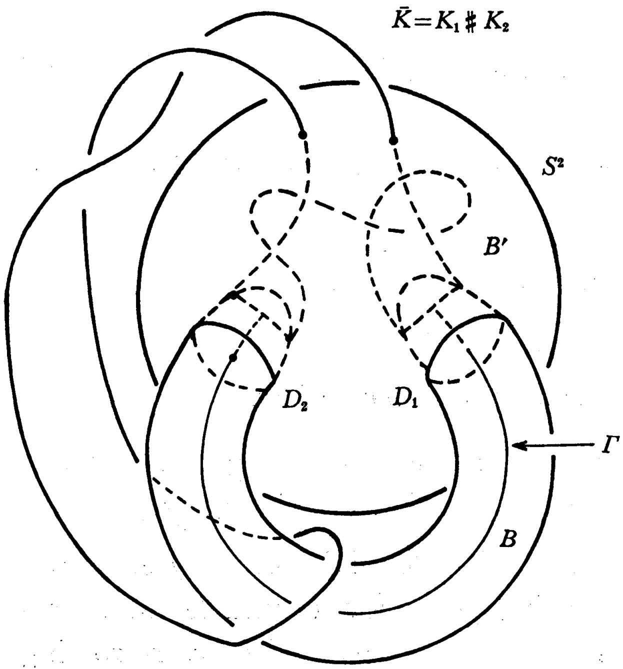


Fig. 5.

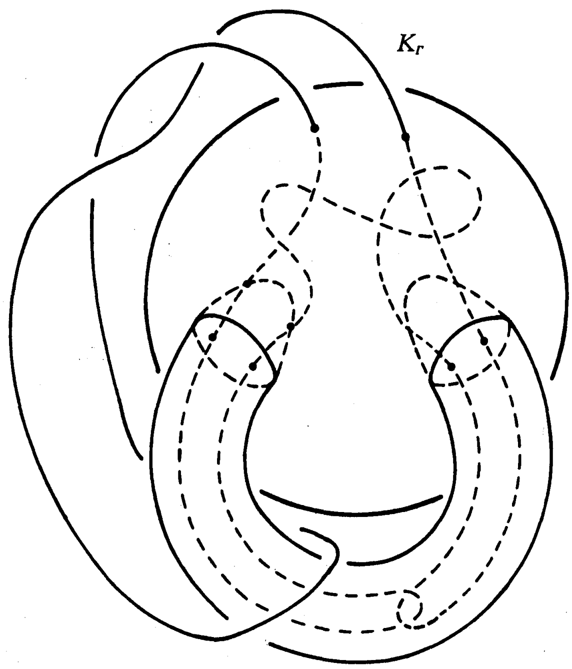


Fig. 6.

We say V the solid torus associated to $K=K_{\Gamma}$. Actually we will show Theorem A and B:

THEOREM A. *Suppose K_1 and K_2 is non-trivial. Let K be a knot in $K_1+_2K_2$ and V be the solid torus associated to K. Then, if V is knotted in S^3, K is non-trivial.*

THEOREM B. *Let K_1, K_2, K and V be as in Theorem A. Then, if V is unknotted in S^3, K is non-trivial.*

Theorem A will be proved in section 3 and the remaining of this note is the proof of Theorem B. To prove Theorem B, we assume K is trivial and deduce a contradiction. To do so, first, we choose a disc bounded by K that is "nice" with respect to $\partial V\cup D_1\cup D_2$ (§ 4). Next we define a set Σ as follows: Let Σ be the set of all components of int $D-(\partial V\cup D_1\cup D_2)$ that are not open 2-cells. Now we will complete the proof in § 6 in case of $\Sigma=\phi$ and in § 7 in case of $\Sigma\neq\phi$.

3. Proof of Theorem A.

LEMMA 3.1 *Suppose the trivial knot K is contained in $K_1+_2K_2$ and let V be the solid torus associated to K. If there is a pair (W, D) satisfying (1) to (3) below, K_1 is trivial.*

(1) *D is a disc bounded by K.*

(2) *D is contained in int W.*

(3) *W is a solid torus containing V in its interior and for some meridian disc of W, say M, $M\cap V$ is a meridian disc of V.*

REMARK: In this lemma, we do not assume that V is knotted.

PROOF: Let $p: (\tilde{W}, \tilde{V})\to(W, V)$ be the universal cover of (W, V) and let W_0 be the closure of a connected component of $p^{-1}(W-V\cap M)$ and $V_0=\tilde{V}\cap W_0$. Let $\tilde{D}$ be a lift of D in $\tilde{W}$. Then $\tilde{K}=\partial\tilde{D}$ intersects with $\partial\tilde{V}$ in two points. Let t be a covering translation which generates $\mathrm{Cov}(\tilde{W}/W)$. Put $V_i=t^{-i}V_0\cup\cdots\cup V_0\cup\cdots\cup t^iV_0$. Then V_i is a 3-cell and $\partial\tilde{V}\cap\tilde{D}=\partial V_i\cap\tilde{D}$ for some i. Here the cell pair $(V_i, V_i\cap\tilde{K})$ is equivalent to the one associated to K_1. Also $(V_i, V_i\cap\tilde{K})$ is trivial since $(\tilde{W}, \tilde{K})$ is trivial. Thus K_1 is trivial.

LEMMA 3.2 *Suppose the trivial knot K is contained in $K_1+_2K_2$ and let V be the solid torus associated to K. If there is a disc D bounded by K and a meridian curve m of V such that m does not intersect with D, then K_1 is trivial.*

PROOF: In fact there exists a solid torus W such that (W, D) satisfies the conditions in 3.1 as follows: Let W be the closed complement of sufficiently small tubular $n.b.d.$ of the loop that is a slight push of m out of $V\cup D$. This is a desired one.

PROOF OF THEOREM A: Assume K is trivial. Then there exists a disc D bounded by K. Without loss of generality, we may assume D and ∂V are in general position. Since V is knotted and any cable of non-trivial knot is non-trivial, all loops in $D\cap\partial V$ are homologous to zero in V. So we can find a meridian curve m of V which does not intersect with $D\cap\partial V$. By Lemma 3.2, K_1 must be trivial and this is a contradiction.

4. Preliminary Part of Proof of Theorem B.

We will proceed to the proof of Th. B. In this section, we will choose a disc D bounded by K which is in "nice" position with respect to $\partial V\cup D_1\cup D_2$.

REMARK. In this section, we need not assume that V is unknotted.

It is trivial that there exists a disc D bounded by K which satisfies (1) and (2):

(1) D and $\partial V\cup D_1\cup D_2$ are in general position.

(2) $(\#(D\cap(D_1\cup D_2)), \sum_{k=1,2}\#(\text{components of } D\cap D_k), \#(\text{loops of } D\cap A'), \#(\text{loops of } D\cap A))$ is minimal with respect to the lexicographic order among the discs satisfying (1), where A denotes $\partial B-(\mathring{D}_1\cup\mathring{D}_2)$ and A' denotes $\partial B'-(\mathring{D}_1\cup\mathring{D}_2)$. For D above the followings hold:

(4.3) There is no arc among the components of $D\cap(D_1\cup D_2)$ whose boundary is $\partial D\cap D_k$ for some k, $k=1, 2$.

(4.4) There is no meridian curve of V among the components of $D\cap A$.

(4.5) $\#(\text{loops of } D\cap A)=0$.

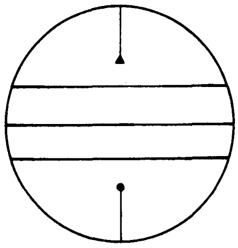
(4.6) Let x be any component of $D\cap A$. Then x is a proper arc in A and the points of ∂x are contained in the different components of $\partial D_1\cup\partial D_2$ (i.e. x runs from ∂D_1 to ∂D_2.).

(4.7) Let y be any component of $D\cap A'$ such that ∂y is contained in ∂D_i for some $i=1, 2$. Then y separates $K\cap\partial V$ on A'.

(4.8) $\#(D\cap\partial D_1)=\#(D\cap\partial D_2)$.

(4.9) Let y, y' be any components of $D\cap A'$ such that $\partial y\cup\partial y'$ is contained in ∂D_i for some $i=1, 2$. Let ω (resp. ω') denote the disc in A' bounded by y (resp. y') and a subarc of ∂D_i. Then $\omega\subset\omega'$ or $\omega'\subset\omega$ (i.e. y is parallel to y' in $A'-K\cap\partial V$).

(4.10) $(D_i, D_i\cap D)$ $(i=1, 2)$ is viewed as follows:



PROOFS: (4. 3) Suppose $D\cap D_i$ contains such a component as in (4. 3). Then we may assume that $D\cap D_i$ contains only one arc whose boundary is $\partial D\cap D_i$. So there exists a meridian of V which does not intersect with D. By Lemma 3. 2, K_1 must be trivial and this is a contradiction. (4. 4) is similar. (4. 5) Suppose there exists a loop component c in $D\cap A$. Since A is an annulus, c is parallel to a boundary component of A or homotopic to a point in A. By (4. 4), the first case cannot occur. So c must bound a disc in A and this contradicts to the minimality condition (2) by cut and paste argument.

(4. 6) By (4. 5), x must be an arc. Suppose ∂x is contained in ∂D_i. Then there exists a disc ω in A bounded by x and a subarc of D_i. So we can decrease $\#(D\cap(\partial D_1\cup\partial D_2))$ by pushing D along ω and this contradicts to (2).

(4. 7) This is proved similarly to (4. 6).

(4. 8) This follows from (4. 6).

(4. 9) It is sufficient to prove $\omega\cap\omega'\neq\phi$. Suppose $\omega\cap\omega'=\phi$ and $\partial y\cup\partial y'\subset\partial D_1$. Then, by (4. 7), $\omega\cup\omega'\supset K\cap\partial V$. This means that there is no component y'' of $D\cap A'$ such that $\partial y''\subset\partial D_2$. This contradicts to (4. 8).

(4. 10) By (4. 3), $(D_i, D_i\cap D)$ contains four sorts of curves in general:

(a) various loops in int D_i,

(b) two arcs which connect a point in $D_i\cap\partial D$ to a point in ∂D_i,

(c) various arcs connecting two points in ∂D_i and separating $D_i\cap\partial D$ on D_i,

(d) various arcs connecting two points in ∂D_i and not separating $D_i\cap\partial D$ on D_i.

But the curves in (a) and (d) cannot be contained by the minimality condition (2). In fact the curves in (a) can be removed by usual cut and paste arguments and the ones in (d) by "push out along the outermost disc" arguments. This shows (4. 10).

5. Some Lemmas.

LEMMA 5. 1 *Let $K=K_r$ be the trivial knot in $K_1+_2K_2$ and let V be the solid torus associated to K. Suppose there is a disc Δ properly embedded*

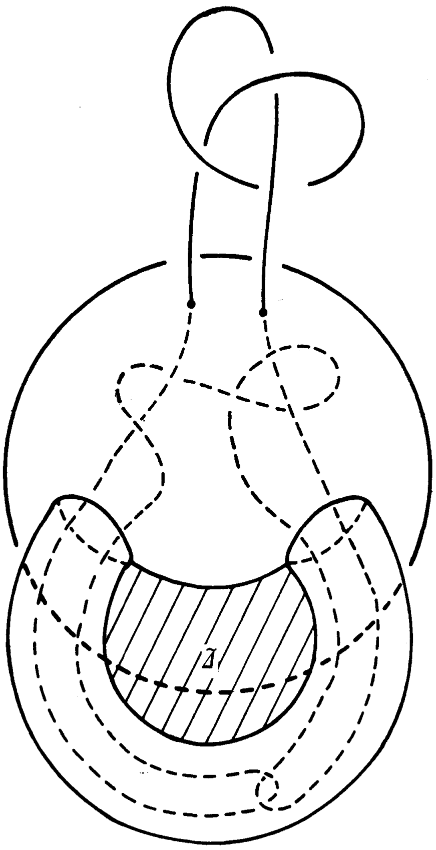


Fig. 7.

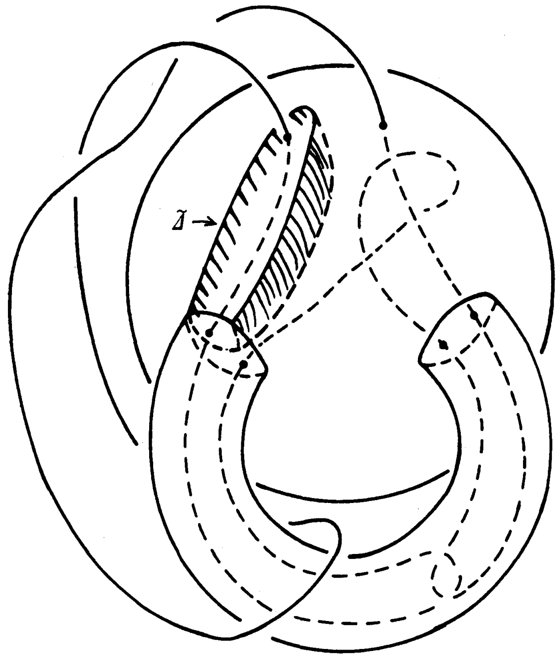


Fig. 8.

in S^3-int V such that $\partial\tilde{\Delta}$ intersects with $\partial D_i\,(i=1, 2)$ in one point and $\tilde{\Delta}$ does not intersect with K. Then K_1 or K_2 is trivial.

PROOF: Trivially there exists a disc Δ which satisfies the conditions in Proposition 1. 1. Thus K is also contained in $K_1+_0K_2$. So, by Theorem 0, 5. 1 has been proved.

LEMMA 5. 2 *Let $K=K_r$ be the trivial knot in $K_1+_2K_2$ and let V be the solid torus associated to K. Suppose there is a disc $\tilde{\Delta}$ such that:*

(1) *$\tilde{\Delta}$ is properly embedded in $B'-K$.*

(2) *$\tilde{\Delta}\cap(D_1\cup D_2)$ is an arc.*

(3) *$\partial\tilde{\Delta}$ separates $K\cap\partial V$ on $\partial B'$.*

Then K_1 or K_2 is trivial.

PROOF: Suppose $\tilde{\Delta}\cap D_2=\phi$. Then B' is decomposed into two 3-cells by $\tilde{\Delta}$. Let C denote the one that does not contain D_2. Since K is trivial, $(C, C\cap K)$ is the trivial cell pair. So we can find a disc Δ which satisfies the conditions in Proposition 1. 2. Thus K is also contained in $K_1+_1K_2$. So, by Theorem 1, 5. 2 has been proved.

6. Completion of Proof in case of $\Sigma=\phi$.

As in § 2, let Σ denote the set of all components of int $D-(\partial V\cup D_1\cup D_2)$ that are not open 2-cells. The key point is that inequalities (6. 1) to (6. 5) hold except in cases that we can deduce contradictions comparatively easily.

6. 1 Let D_0 denote $D\cap V$ and consider $D_0^*=\text{int}\, D_0-(D_0\cap(D_1\cup D_2))$. By the assumption $\Sigma=\phi$, we know the followings easily:

(6. 1. 1) D_0 is connected.

(6. 1. 2) Each component of D_0^* is an open 2-cell.

Now we set:

$$D\cap\partial V=l\cup c_1\cup\cdots\cup c_\mu$$

where l is an arc whose boundary is $K\cap\partial V$ and c_i's are loops. Note that, by (6. 1. 1):

(6. 1. 3) Each c_i is innermost in D.

So:

(6. 1. 4) $\partial D_0=(V\cap K)\cup(D\cap\partial V)$

See Fig. 9 and Fig. 10 for (6. 1. 1) to (6. 1. 4).

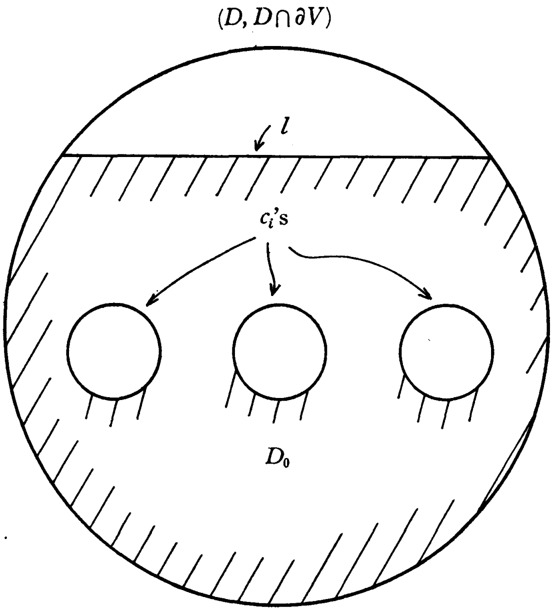


Fig. 9.

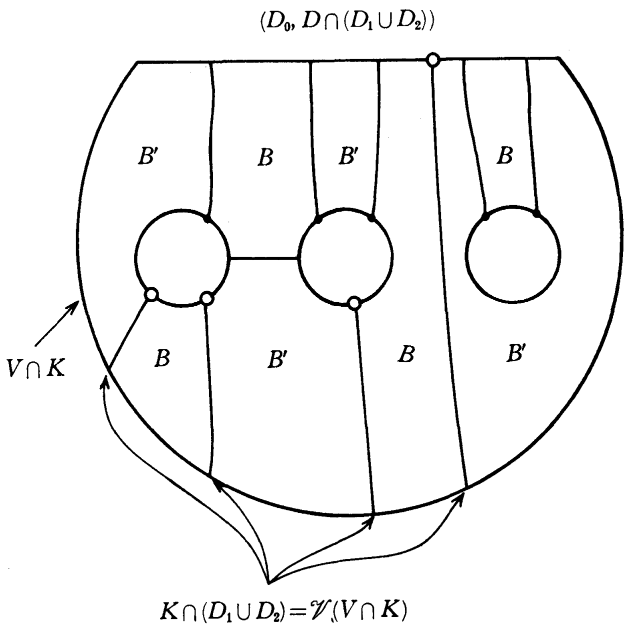


Fig. 10.

Let $m_i\,(i=1, 2)$ denote $\#(l\cap\partial D_i)$. By (4. 6), $m_1=m_2$, so we put $m\equiv m_1$. By similar consideration, we put $p_j=\#(c_j\cap\partial D_1)=\#(c_j\cap\partial D_2)$ and $p\equiv p_1+\cdots+p_\mu$. Then:

(6. 1) $m\geqq 2$ and $p_j\geqq 3$ for all j.

PROOF: Suppose $m\leqq 1$ or $p_j\leqq 2$ for some j. Then there exists a disc

$\tilde{\Delta}$ satisfying all conditions in Lemma 5.1, so K_2 must be trivial. In fact, suppose $p_j=2$ for example. Let d_j denote the disc on D bounded by c_j (see (6.1.3)). Note c_j is homologous zero on ∂V since p_j is even and $d_j \subset S^3$-int V. Let d'_j denote the disc bounded by c_j on ∂V and let C denote the 3-cell in S^3-int V bounded by the 2-sphere $d_j \cup d'_j$. Let W denote the union of V and C. Then K is contained in W and W is a solid torus which is unknotted in S^3 (Recall the assumption V is unknotted in S^3). Since $p_j=2$, $d'_j \cap \partial D_i$=an subarc of ∂D_i for $i=1, 2$. So there exists a longitude c of V that is contained in $\partial V - d_j = \partial W - d'_j$ and intersects with ∂D_i in one point for $i=1, 2$. Since W is unknotted, there is a disc $\tilde{\Delta}$ in S^3-int W such that $\partial\tilde{\Delta}=c$. Then $\tilde{\Delta}$ is a desired one. Other cases are similar.

6.2 We consider $(D_0, D\cap(D_1\cup D_2))$. Let Ω_0 denote the set of points in $\partial(D\cap(D_1\cup D_2))$. We call a element of Ω_0 a vertex (of D_0). The points of Ω_0 devide $\partial D_0 \cup (D\cap(D_1\cup D_2))$ into N_1 subarcs $\Omega_1=\{\lambda_1, \cdots, \lambda_{N_1}\}$. Let $\Omega_2=\{O_1, \cdots, O_{N_2}\}$ denote the set of closures of components of D_0^*. Then:

(6.2.1) For each vertex v of D_0, there are exactly three elements of Ω_1 each of which contains v as one of its end points.

(6.2.2) Let ε be any component of $D\cap(D_1\cup D_2)$. Then there are exactly two elements of Ω_2, say O_i and O_j, such that $O_i\cap O_j \supset \varepsilon$ and if $O_i \subset B$, then $O_j \subset B'$.

By (6.1.2) and (6.2.2), O_i's are closed 2-cells. Therefore we obtain the cell decomposition of D_0 by considering Ω_0, Ω_1 and Ω_2 as 0, 1 and 2-cells respectively.

The following property is used later.

(6.2.3) For any 1-cell one of whose end points is contained in $V\cap K$, another end point is contained in $D\cap\partial V$.

In fact, if not so, it contradicts to (4.3).

REMARK: For (6.2.1) to (6.2.3), we do not need the assumption $\Sigma=\phi$. We will use these facts in §7 again.

In the following, we consider to estimate the number of 2-cells. First of all: $N_0=2m+2p+4$. By (6.2.1), $N_1=3N_0/2$. By calculating Euler characteristic number of D_0:

$$N_0-N_1+N_2=1-\mu$$

Thus we obtain:

(6.2) $N_2=m+p+3-\mu$

6.3 We use the following notation. For any subset X of D_0, let $\mathscr{V}(X)$ denote the set of all vertices contained in X and for any subset Y of Ω_2: $\mathscr{V}(Y)=\bigcup_{O\in Y}$ {vertices contained in ∂O}. A 2-cell O is called k-gonal if its boundary is the union of k 1-cells (or equivalently $\#(\mathscr{V}(O))=k$). Note that k is always even.

Now we devide Ω_2 into the disjoint union of subsets. First:

$$\Omega_2=\Omega_2(B)\cup\Omega_2(B')$$

where $\Omega_2(B)=\{O\in\Omega_2|O\subset B\}$ and $\Omega_2(B')=\{O\in\Omega_2|O\subset B'\}$. Moreover:

$$\Omega_2(B)=\Omega_2^0(B)\cup\Omega_2^1(B)$$

$$\Omega_2(B')=\Omega_2^0(B')\cup\Omega_2^1(B')$$

where $\Omega_2^0(B)=\{O\in\Omega_2(B)|\mathscr{V}(O)\cap\mathscr{V}(V\cap K)\neq\phi\}$ and $\Omega_2^1(B)=\Omega_2(B)-\Omega_2^0(B)$. $\Omega_2^0(B')$ and $\Omega_2^1(B')$ are similar. Then:

(i) $\Omega_2=\Omega_2^0(B)\cup\Omega_2^0(B')\cup\Omega_2^1(B)\cup\Omega_2^1(B')$ (disjoint)

(ii) $\Omega_2^0(B)\cup\Omega_2^0(B')=\{O\in\Omega_2|\mathscr{V}(O)\cap\mathscr{V}(V\cap K)\neq\phi\}$

Note that $\mathscr{V}(V\cap K)=K\cap(D_1\cup D_2)$ (See Fig. 10). So $\#(\mathscr{V}(V\cap K))=4$. Thus, by (ii):

(iii) $\#(\Omega_2^0(B)\cup\Omega_2^0(B'))\leqq 5$

We put:

$$\alpha=\#\left(\Omega_2^1(B)\right)$$

$$\beta=\#\left(\Omega_2^1(B')\right)$$

Then, by (i) and (iii) (recall $N_2=\#(\Omega_2)$):

(6.3) $\alpha+\beta+5\geqq N_2$

6.4 Now consider $\Omega_2(B)$

(iv) Let O_i and O_j be different elements of $\Omega_2(B)$. Then $O_i\cap O_j$ is empty.

PROOF: This follows easily from (6.2.1) and (6.2.2).

(v) $\mathscr{V}(\Omega_2^1(B))=\cup\{\mathscr{V}(O)|O\in\Omega_2^1(B)\}$ (disjoint)

PROOF: This is trivial by (iv).

(vi) For $O\in\Omega_2^1(B)$, suppose O is $2m$-gonal. Then $m\geqq 4$.

PROOF: By (4.6), m cannot be odd. Suppose $m=2$. Then, by (4.6) and (4.10), O must have the form as indicated in Fig. 11. and so, $O\cap(K\cap B)\neq\phi$. This is impossible since $O\subset\operatorname{int} D$.

By (v) and (vi), we obtain:

(a) $$\#\left(\mathscr{V}\left(\Omega_2^1(B)\right)\right)=\sum_{O\in\Omega_2^1(B)}\#\left(\mathscr{V}(O)\right)\geqq 8\alpha$$

The following is trivial by the definition of $\Omega_2^1(B)$.

(vii) For $O\in{}_2^1(B)$, the vertices of ∂O is contained in $D\cap\partial V$.

By (iv) and (vii):

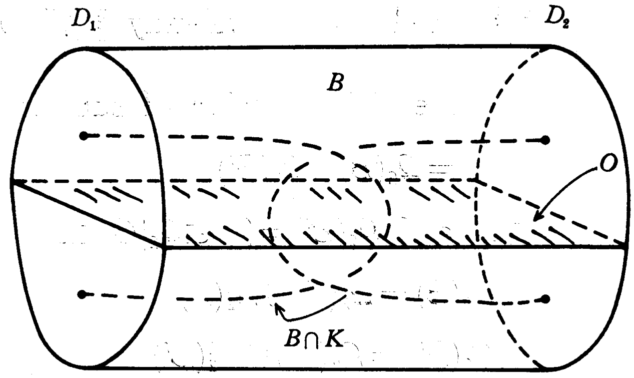


Fig. 11.

(viii) For $O \in \Omega_2^1(B)$;

$$\mathscr{V}(O) \subset (D \cap \partial V) - \mathscr{V}\left(\Omega_2^0(B)\right)$$
$$= \mathscr{V}(D \cap \partial V) - \left(\mathscr{V}(D \cap \partial V) \cap \mathscr{V}\left(\Omega_2^0(B)\right)\right)$$

Here, by (6.2.3) and the fact $\#(V \cap K) = 4$, we have:

(ix) $\mathscr{V}(D \cap \partial V) \cap \mathscr{V}(\Omega_2^0(B))$ contains at least four points.

Therefore, by (v), (viii) and (ix):

$$\text{(b)} \qquad \#\left(\mathscr{V}\left(\Omega_2^1(B)\right)\right) = \#\left(\cup\left\{\mathscr{V}(O) \middle| O \in \Omega_2^1(B)\right\}\right)$$
$$= \#\left(\mathscr{V}(D \cap \partial V)\right) - \#\left(\mathscr{V}(D \cap \partial V) \cap \mathscr{V}\left(\Omega_2^0(B)\right)\right)$$
$$\leqq 2m + 2p - 4$$

Combining (a) and (b), we obtain:

$$8\alpha \leqq 2m + 2p - 4$$

Thus we have the estimate for α:

(6.4) $4\alpha \leqq m + p - 2$

6.5 We consider $\Omega_2(B')$. As in 6.4, we know:

(iv′) Let O_i and O_j be different elements of $\Omega_2(B')$. Then $O_i \cap O_j$ is empty.

(v′) $\mathscr{V}(\Omega_2^1(B')) = \cup\{\mathscr{V}(O) | O \in \Omega_2^1(B')\}$ (disjoint)

(vii′) For $O \in \Omega_2^1$(B′), the vertices of ∂O is contained in $D \cap \partial V$.

(viii′) For $O \in \Omega_2^1(B')$;

$$\mathscr{V}(O) \subset \mathscr{V}(D \cap \partial V) - \left(\mathscr{V}(D \cap \partial V) \cap \mathscr{V}\left(\Omega_2^0(B')\right)\right)$$

(ix′) $\mathscr{V}(D \cap \partial V) \cap \mathscr{V}(\Omega_2^0(B')$ contains at least four points.

(b′) $\#(\mathscr{V}(\Omega_2^1(B') \leqq 2m + 2p - 4$

6.6 Next we consider $\Omega_2^1(B')$. First we may assume that no 2-cell in

$\Omega_2^1(B')$ is 2-gonal (in fact: Suppose there exists a 2-gonal 2-cell $\tilde{\Delta}$ in $\Omega_2^1(B')$. Then $\partial\tilde{\Delta}$ is the union of two 1-cells y and z, where y is contained in A' and z is contained in D_i, $i=1$ or 2 and $\partial y \subset \partial D_i$. By (4.7) and (4.10), $\tilde{\Delta}$ satisfies the conditions in Lemma 5.2. Thus K_2 must be trivial, so a contradiction.)

(6.6.1)′ There are 4-gonal 2-cells in $\Omega_2^1(B')$.

PROOF: Suppose the contrary, i.e. $\#(\mathscr{V}(O)) \geqq 6$ for every 2-cell O in $\Omega_2^1(B')$. Then by (v′) and (b′) in 6.5, we have:

$$(*) \qquad 6\beta \leqq 2m+2p-4$$

Then we can deduce a contradiction from (6.1) to (6.4) and (*). In fact: By (6.3), $12\alpha+12\beta \geqq 12(N_2-5)$. Using (6.2), (6.4) and (*); $12\mu+10 \geqq 5m+5p$ By (6.1); $12\mu+10 \geqq 10+15\mu$. This is a contradiction.

Next we show that 4-gonal 2-cells and 6-gonal 2-cells in $\Omega_2^1(B')$ must have restricted forms ((6.6.2) and (6.6.3) below). In general let O be an $2n$-gonal 2-cell in $\Omega_2^1(B')$. Then $\partial O(\subset \partial B')$ is built up from n 1-cells $\{\lambda_1', \cdots, \lambda_n'\}$ which are contained in A' and n 1-cells $\{\lambda_1'', \cdots, \lambda_n''\}$ which are contained in $D_1 \cup D_2$. Now let O be a 4-gonal 2-cell and $\partial O = \lambda_1' \cup \lambda_2' \cup \lambda_1'' \cup \lambda_2''$. Then:

(6.6.2) $\lambda_1'' \cup \lambda_2''$ cannot be contained in the same component of $D_1 \cup D_2$. That is, O must have the form as indicated in Fig. 12.

PROOF: Suppose $\lambda_1'' \cup \lambda_2'' \subset D_1$. Then $\partial\lambda_1' \cup \partial\lambda_2' \subset \partial D_1$. So, by (4.9), λ_1' and λ_2' are parallel in $A' - (K \cap \partial V)$. Let δ be a component of $B' \cap K$ such

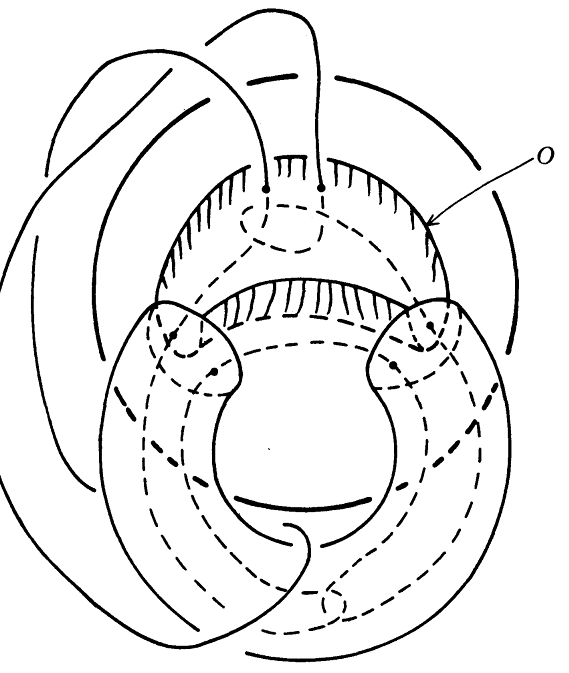


Fig. 12.

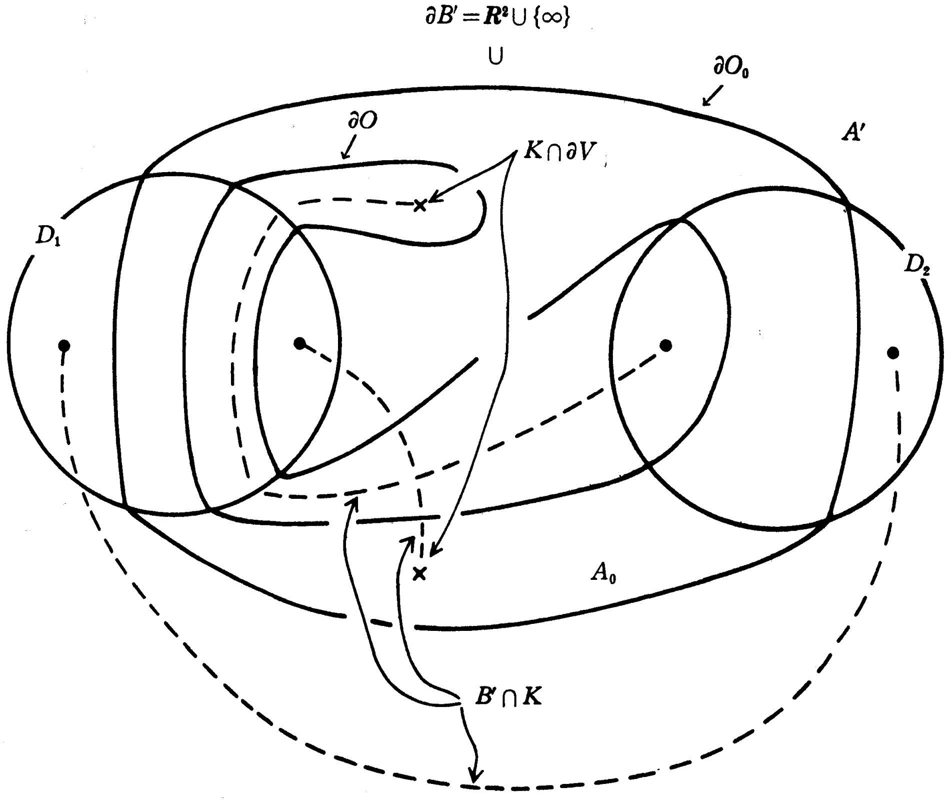


Fig. 13.

that $\delta\cap D_1=\phi$. Then ∂O separates $\partial\delta$ on $\partial B'$ and this is impossible since $O\cap\delta=\phi$.

Let O_0 be a fixed 4-gonal 2-cell and A_0 denote the disc bounded by ∂O_0 on $\partial B'$ that contains $K\cap\partial V$ in its interior. Let O be any 6-gonal 2-cell and $\partial O=\lambda'_1\cup\lambda'_2\cup\lambda'_3\cup\lambda''_1\cup\lambda''_2\cup\lambda''_3$. Then:

(6. 6. 3) (i) ∂O is contained in A_0.

(ii) Two of λ'_j's run from ∂D_1 to ∂D_2.

That is any 6-gonal 2-cell must have the form as indicated in Fig. 13.

Proof: (i) Since O is 6-gonal, for some $j=1, 2, 3$, λ'_j is a component of $D\cap A'$ such that $\partial\lambda'_j\subset\partial D_i$, $i=1$ or 2. Thus, by (4. 7), $\lambda'_j\subset A_0$ and so $\partial O\subset A_0$.

(ii) Suppose $\lambda''_1\cup\lambda''_2\cup\lambda''_3\subset D_1$. Then, by (i) and (4. 9), λ'_j's are parallel in $A_0\cap(A'-(K\cap\partial V))$. Also, by (4. 10), λ''_j's are parallel in $D_1-(K\cap D_1)$. These facts contradict to each other (See Fig. 14).

6. 7 Now let β_1 denote the number of 4-gonal 2-cells in $\Omega^1_2(B')$ and β_2 denote the number of 6-gonal 2-cells in $\Omega^1_2(B')$ and $\beta_3=\beta-\beta_1-\beta_2$. And let $\tilde{\Omega}$ denote the set $\{\varepsilon\in\Omega_1|\varepsilon\subset\partial O,\ O\in\Omega^1_2(B')\}$ and let Ω denote $\{\varepsilon\in\tilde{\Omega}|\varepsilon\not\subset\partial O$ for any 4-gonal 2-cell O in $\Omega^1_2(B')\}$. Recall $\#(\tilde{\Omega})=\#(\mathscr{V}(\Omega^1_2(B')))$. By (b′) in 6. 5:

$$\#(\tilde{\Omega})\leqq 2m+2p-4$$

Also:
$$\#(\tilde{\Omega})=4\beta_1+\#(\Omega)$$

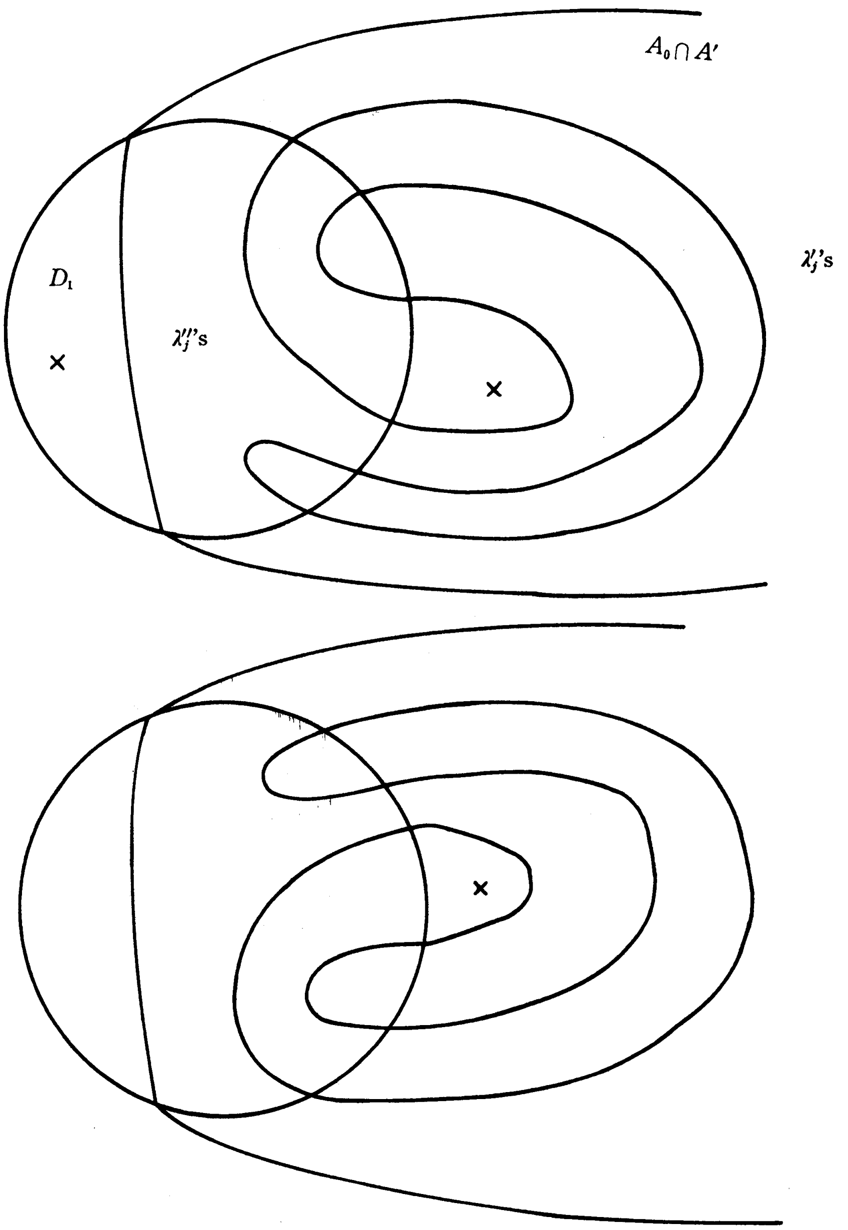


Fig. 14.

We will count $\#(\Omega)$ to estimate β. Now Ω contains the following three sorts of 1-cells:

(a) 1-cells contained in $D_1 \cup D_2$ that are contained in the boundaries of $2n$-gonal 2-cells, where $n \geqq 3$. $\#$(this sort of 1-cells) is at least $3\beta_2 + 4\beta_3$.

(b) 1-cells contained in A' $(=\partial B' - (\mathring{D}_1 \cup \mathring{D}_2))$ each of which connects two points belonging to the same component of $\partial D_1 \cup \partial D_2$. (Recall (6. 6. 2)). $\#$(this sort of 1-cells) is at least 2μ (Proof: Recall that each c_j, a loop component of $D \subset \partial V$, is a longitude of V or homologous zero on ∂V. Thus, by the fact $p_j \geqq 3$ (by (6. 1)) there are at least two such 1-cells as above that are contained in c_j.).

(c) 1-cells contained in A' that run from ∂D_1 to ∂D_2 and are contained in the boundaries of $2n$-gonal 2-cells, where $n \geqq 3$. $\#$(this sort of 1-cells) is at least $2\beta_2$ by (ii) of (6. 6. 3). Thus $\#(\Omega) \geqq 2\mu + 5\beta_2 + 4\beta_3$. Therefore:

$$4\beta_1 + (2\mu + 5\beta_2 + 4\beta_3) \leqq 4\beta_1 + \#(\Omega) = \#(\tilde{\Omega}) \leqq 2m + 2p - 4$$

I. e: $$4(\beta_1 + \beta_2 + \beta_3) \leqq 4\beta_1 + 5\beta_2 + 4\beta_3 \leqq 2m + 2p - 4 - 2\mu$$

Thus we obtain the following estimate for β:

(6. 5) $2\beta \leqq m + p - 2 - \mu$

6. 8 Now we can deduce a contradiction from (6. 1) to (6. 5) as follows: By (6. 3):

$$4\alpha + 4\beta \geqq 4(N_2 - 5)$$

Using (6. 2), (6. 4) and (6. 5):

$$m + p - 2 + 2(m + p - 2 - \mu) \geqq 4(m + p - 2 - \mu)$$

I. e. $2 + 2\mu \geqq m + p$. By (6. 1):

$$2 + 2\mu \geqq m + p \geqq 2 + 3\mu$$

This is a contradiction.

7. Completion of Proof in case of $\Sigma \neq \phi$.

In this section, we will find a suitable subset of D (X_0 below) and do similar arguments to the previous section. First we consider $(D, D \cap \partial V)$. Let Φ denote the set of all loops of $D \cap \partial V$ that are innermost in D and let θ denote the set of discs in D bounded by the loops in Φ. Note that Φ is non-empty (In fact, if not so, applying Lemma 3. 2 again, K_1 must be trivial). Now:

(*) *Every disc in θ is contained in S^3-int V.*

PROOF: Let c be a loop in Φ and d be the disc bounded by c. Assume d is contained in V. If $d \cap (D_1 \cup D_2) = \phi$, by (4. 5), c must be contained in A'. Since c does not separate D_1 and D_2 on $\partial B'$, c must bound a disc d' in A'. Since $d' \supset K \cap \partial V$ means K_2 is trivial, $d' \not\supset K \cap \partial V$. So we can remove c. This contradicts to (2) in § 4. Thus we may assume $d \cap (D_1 \cup D_2) \neq \phi$. Then by usual outermost disc argument on d, we have a disc $\tilde{\Delta}$ satisfying all conditions in Lemma 5. 2, so a contradiction.

Now we define a order $>$ on Σ as follows: Let σ and σ' be elements of Σ. Then $cl(\sigma)$ is a submanifold of D homeomorphic to the disc with holes. Let $X_1, \cdots, X_n$ denote the discs in D bounded by the innerboundaries of σ. We define $\sigma > \sigma'$ iff σ' is contained in X_j for some j. Since $\Sigma \neq \phi$, there is a minimal element σ_0 in Σ with respect to this order. Let X_0 be a disc bounded by some inner boundary of σ_0. Then we obtain a cell decomposition for X_0 as follows: Let Ω_0 be the set of points in $X_0 \cap (\partial D_1 \cup \partial D_2)$ and Ω_1 be the set of subarcs of $X_0 \cap (\partial V \cup D_1 \cup D_2)$ devided by the points in Ω_0. Let $\tilde{\Omega}_2 = \{O_1, \cdots, O_{N_2}\}$ denote the set of closures of all components of $D_0^* (= \mathrm{int}\, D_0 - (D_0 \cap (D_1 \cup D_2)))$ that are contained in X_0. Then, by the choice of σ_0, O_i's are closed 2-cells (Recall § 6. 2). Let θ_0 denote the set of all discs belonging to θ and contained in X_0. Then, by (*) above, X_0 is the union of all O_i's in $\tilde{\Omega}_2$ and discs in θ_0. Now setting $\Omega_2 = \theta_0 \cup \tilde{\Omega}_2$ and considering Ω_0, Ω_1 and Ω_2 as 0, 1 and 2-cells, we obtain a cell decomposition for X_0.

Let N_0 denote $\#(\Omega_0)$ and N_1 denote $\#(\Omega_1)$ and let $\Phi_0 = \{c_1, \cdots, c_\mu\}$ denote the set of all loops in Φ that are contained in X_0. Of course $\#(\Phi_0) = \#(\theta_0)$. Then as in § 6. 2:

$$N_0 - N_1 + (N_2 + \mu) = 1 \quad \text{and} \quad N_1 = 3N_0/2$$

Thus we have:

($) $\quad N_2 = N_0/2 + 1 - \mu$

Let p_j denote $\#(c_j \cap \partial D_1) = \#(c_j \cap \partial D_2)$ and $p = p_1 + \cdots + p_\mu$. As in § 6. 1:

(7. 1) $p_j \geqq 3$ for all j.

Let α (resp. β) denote the number of 2-cells of $\tilde{\Omega}_2$ that are contained in B (resp. B'). We must consider two cases:

(I) $\partial X_0 = \tilde{c}$ is a single loop of $D \cap \partial V$.

(II) Otherwise.

(I) In this case $N_0 = 2(p + \tilde{p})$, where $\tilde{p} = \#(\tilde{c} \cap \partial D_1)$. So, by ($) above, we have:

(7. 2) $N_2 = p + \tilde{p} + 1 - \mu$

Trivially:

(7. 3) $N_2 = \alpha + \beta$

As in § 6. 4:

(7. 3) $4\alpha \leqq p + \tilde{p}$

As in § 6. 5 to § 6. 7:

(7. 5) $2\beta \leqq p + \tilde{p} - \mu$

In case (II), the correspondings are the ones that are obtained from (7. 1) to (7. 5) by setting $\tilde{p}$ to be zero.

Now the computations similar to those in § 6. 8 complete the whole proof of our theorem.

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