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Root systems and orthogonal groups of root lattices

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0. Introduction.

The theory of root systems attached to finite dimensional complex semisimple Lie algebras has been developed much deeply (cf. [1], [3]). As a natural generalization of these Lie algebras and the corresponding root systems, the notion of Lie algebras defined by (generalized) Cartan matrices has recently been introduced (cf. [4], [10]), and the structure of associated root systems has been studied (cf. [5], [12], [13], [14]).

On the other hand, in [6] the root lattice, which is corresponding to a finite, Euclidean or low rank hyperbolic Cartan matrix, and its orthogonal group are discussed. For example, it has been confirmed that in the case when a Cartan matrix is $\begin{pmatrix} 2 & -3 & -1 \\ -1 & 2 & -1 \\ -1 & -3 & 2 \end{pmatrix}$ the orthogonal group of the associated root lattice is strictly greater than the subgroup generated by its Weyl group, diagram automorphism group and $-I$ (minus identity). Indeed the group index is 2 (cf. [6]).

The starting point of this paper is the following observation:

(♯) *If Δ is a root system of type C_4, and if Γ and $O(\Gamma)$ are the root lattice and its orthogonal group respectively, then the set of all elements in $O(\Gamma)$-orbit of Δ is just a root system of type F_4.*

One can easily see this by looking at the list of root systems in [1] (cf. Section 3). In this paper we shall show the following:

(♯♯) *If Δ is a root system associated with a finite, Euclidean or hyperbolic Cartan matrix, and if Γ and $O(\Gamma)$ are the root lattice and its orthogonal group respectively, then the set of all elements in $O(\Gamma)$-orbit of Δ forms again a root system (cf. Section 2, Theorem A).*

If an original Cartan Matrix is $\begin{pmatrix} 2 & -1 & 0 \\ -4 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix}$, for example, then we get $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -4 & 2 \end{pmatrix}$ as the Cartan matrix corresponding to the new root system (cf. Section 3).

1. Preliminary.

In this section, we will review the theory of Kac-Moody Lie algebras (cf. [4], [7], [10]).

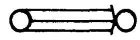
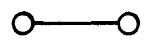
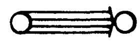
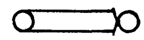
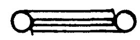
An $\ell \times \ell$ integral matrix $A=(a_{ij})$ is called a (generalized) Cartan matrix if the following three conditions hold.

(C1) $a_{ii}=2$ for all $i \in J$,

(C2) $a_{ij} \leq 0$ for distinct $i, j \in J$,

(C3) $a_{ij}=0$ implies $a_{ji}=0$ for $i, j \in J$,

where $J=\{1, 2, \cdots, \ell\}$. If $a_{ij} \cdot a_{ji} \leq 4$, we draw a diagram, called a Dynkin diagram. The Dynkin diagram of A is a diagram having ℓ vertices, the i-th joined to the j-th $(i \neq j)$ by edges or arrows according to the following rule.

a_{ij},	a_{ji}	i j	a_{ij},	a_{ji}	i j
0	0		-1	-3	
-1	-1		-1	-4	
-1	-2		-2	-2	

For any Cartan matrix A and for any field F of characteristic zero, we let by $\mathfrak{g}=\mathfrak{g}_F(A)$ denote the Lie algebra over F generated by 3ℓ generators e_i, h_i, f_i $(i \in J)$ with the defining relations $[h_i, h_j]=0$, $[e_i, f_j]=\delta_{ij}h_i$, $[h_i, e_j]=a_{ij}e_j$, $[h_i, f_j]=-a_{ij}f_j$ for all $i, j \in J$, and $(\mathrm{ad}\, e_i)^{-a_{ij}+1}e_j=0$, $(\mathrm{ad}\, f_i)^{-a_{ij}+1}f_j=0$ for distinct $i, j \in J$. We call this algebra $\mathfrak{g}$ the (standard) Kac-Moody Lie algebra over F associated with A. Let Γ be a free $\boldsymbol{Z}$-module of rank ℓ, and choose a free basis $\Pi=\{\alpha_1, \cdots, \alpha_\ell\}$ of Γ. By defining $\deg(e_i)=\alpha_i$, $\deg(h_i)=0$, $\deg(f_i)=-\alpha_i$ for all $i \in J$, we can view $\mathfrak{g}$ as a Γ-graded Lie algebra $\mathfrak{g}=\bigoplus_{\alpha \in \Gamma} \mathfrak{g}^\alpha$, where $\mathfrak{g}^\alpha$ is the subspace of $\mathfrak{g}$ corresponding to α. Put $\Delta=\{\alpha \in \Gamma \mid \mathfrak{g}^\alpha \neq 0\}$, called the root system of $\mathfrak{g}$. We may say that $\Delta=(\Delta, \Pi)$ is a root system of A. Since $\mathfrak{g}^{\alpha_i}=Fe_i$, $\mathfrak{g}^{-\alpha_i}=Ff_i$ and $\mathfrak{g}^0=\bigoplus_{i \in J} Fh_i$, we have $\{\pm\alpha_i \mid i \in J\} \cup \{0\} \subseteq \Delta$. We call $\Pi=\{\alpha_1, \cdots, \alpha_\ell\}$ a fundamental root system of Δ. Let $Z_+=Z_+(\Pi)$ be the set of nonzero elements $\sum c_i \alpha_i \in \Gamma$ satisfying c_i is nonnegative for all $i \in J$, and let $Z_-=-Z_+$ and $Z=Z(\Pi)=Z_+ \cup \{0\} \cup Z_-$. Then $\Delta \subseteq Z$, which leads to a decomposition $\Delta=\Delta_+ \cup \{0\} \cup \Delta_-$. Let w_i be a $\boldsymbol{Z}$-module automorphism of Γ defined by $w_i(\alpha_j)=\alpha_j-a_{ij}\alpha_i$, and let W be the subgroup of $GL(\Gamma)$ generated by w_i for all $i \in J$. We call W the Weyl