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On measures which are continuous by certain translation

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§ 1. Introduction.

Let G be a LCA group with the dual group $\hat{G}$. We denote by m_G the Haar measure on G. For $x\in G$, δ_x denotes the point mass at x. Let $M(G)$ and $L^1(G)$ be the measure algebra and the group algebra respectively. For a subset E of $\hat{G}$, $M_E(G)$ denotes the space of measures in $M(G)$ whose Fourier-Stieltjes transforms vanish off E. For a closed subgroup H of G, $H^{\perp}$ means the annihilator of H. Let μ be a measure in $M(G)$. Then, as well known, the fact that $\mu\in L^1(G)$ can be caracterized by

(1. 1) $$\lim_{x\to 0}\|\mu-\mu*\delta_x\|=0 .$$

Our first purpose in this note is to characterize the class of measures μ in $M(G)$ with the following property

(1. 2) $$\lim_{\substack{y\to 0\\ y\in H}}\|\mu-\mu*\delta_y\|=0 .$$

When there is a continuous homomorphism ϕ from the reals R into G, deLeeuw and Glicksberg proved in [2] that ϕ-analytic measures $\mu\in M(G)$ satisfy $\lim_{t\to 0}\|\mu-\mu*\delta_{\phi(t)}\|=0$. The second purpose in this paper is to give a theorem corresponding to theirs under our setting. As an extension of a theorem of Bochner, one of the authors proved in [8] that the product set of a Riesz set and a small p set is a small p set. As a corollary our second theorem, we shall prove that the product set of a small p set and a small q set is a small $\max(p, q)$ set. In section 2 we state our results, and we give their proofs in sections 3 and 4.

§ 2. Notations and Results.

DEFINITION 2. 1. *Let G be a LCA group and H a closed subgroup of G. A Borel set E in G is called a H-null set if $m_H(\{t\in H: t+x\in E\})=0$ for all $x\in G$.*

DEFINITION 2. 2. *For a positive integer p, a closed set E in $\hat{G}$ is called* *a small p set if the following is satisfied: For each $\mu \in M_E(G)$, $\mu^p(=\mu*\mu*\cdots*\mu$ (p times)) belongs to $L^1(G)$. In particular, a small 1 set is called a Riesz set.*

We denote by $M^+(G)$ the set of positive measures in $M(G)$. For a closed subgroup H of G, let $M_{aH}(G)$ be the smallest L-ideal in $M(G)$ containing all measures of the form $\rho*\nu$, where $\rho \in L^1(H)$ and $\nu \in M(G)$. By the definition, we note

$$(2.1) \qquad M_{aH}(G)=\left\{\mu \in M(G): \begin{array}{l} \mu \ll \rho*\nu \text{ for some } \rho \in L^1(H) \cap M^+(H) \\ \text{and } \nu \in M^+(G) \end{array}\right\}.$$

A measure $\mu \in M(G)$ is called H-absolutely continuous if $\mu(E)=0$ for any Borel set E in G that is H-null. Let $L_H(G)$ be a L-ideal in $M(G)$ consisting of all measures that are H-absolutely continuous. We shall say that $\mu \in M(G)$ translates H-continuously if $\lim_{\substack{y \to 0 \\ y \in H}} \|\mu-\mu*\delta_y\|=0$. Our results are as follows:

THEOREM I. *Let G be a LCA group and H a closed subgroup of G. Then, for $\mu \in M(G)$, the following are equivalent:*
(I) $\mu \in M_{aH}(G)$;
(II) $\mu \in L_H(G)$;
(III) *μ translates H-continuously.*

THEOREM II. *Let G and H be as in Theorem I. Let p be a positive integer and $\tilde{E}$ a small p set in $\hat{G}/H^\perp$. Put $E=\pi^{-1}(\tilde{E})$, where $\pi: \hat{G} \to \hat{G}/H^\perp$ is the natural homomorphism. Then, for each $\mu \in M_E(G)$, μ^p translates H-continuously.*

When $p=1$, Theorem II can be considered as a theorem corresponding to ([2], Theorem 3.1, p. 186) because, by the classical F. and M. Riesz theorem, the set of nonngeative real numbers is a Riesz set. Using Theorem II, we obtain the following corollary, which is a slight extension of ([8], Theorem 2, p. 277).

COROLLARY. *Let G_1 and G_2 be LCA groups. Let E_1 be a small p set in $\hat{G}_1$ and E_2 a small q set in $\hat{G}_2$. Put $r=\max(p, q)$. Then $E_1 \times E_2$ is a small r set in $\widehat{G_1 \oplus G_2}$.*

PROOF. Evidently E_1 and E_2 are small r sets. Hence, for $\mu \in M_{E_1 \times E_2}(G_1 \oplus G_2)$, it follows from Theorem II that μ^r translates G_i-continuously ($i=1, 2$). Then we have $\lim_{(x_1, x_2) \to 0} \|\mu^r-\mu^r*\delta_{(x_1, x_2)}\|=0$, which yields $\mu^r \in L^1(G_1 \oplus G_2)$. This completes the proof.

§ 3. Proof of Theorem I.

In this section, we prove Theorem I.

PROPOSITION 3. 1. *For* $\mu \in M(G)$, *the following are equivalent*:
(I) $\mu \in M_{aH}(G)$;
(II) μ *translates* H*-continuously.*

PROOF. (I)$\Rightarrow$(II): For $\mu \in M_{aH}(G)$, it follows from (2. 1) that there exist $\nu \in M^+(G)$ and $\rho \in L^1(H) \cap M^+(H)$ such that $\mu \ll \nu * \rho$. Put $\eta = \nu * \rho$. Since η translates H-continuously, we can verify that

(1) $p\eta$ translates H-continuously

for all $p \in \mathrm{Trig}(G)$. On the other hand, since $\mu \ll \eta$, there exists a sequence $\{p_n\}$ in $\mathrm{Trig}(G)$ such that $\lim_{n\to 0} ||\mu - p_n \eta|| = 0$. Hence it follows from (1) that μ translates H-continuously.

(II)$\Rightarrow$(I): For each natural number n, there exists a symmetric open neighborhood V_n of 0 in H such that $\sup_{y \in V_n} ||\mu - \mu * \delta_y|| < \frac{1}{n}$. Let ω_n be a probability measure in $L^1(H)$ with $\mathrm{supp}(\omega_n) \subset V_n$. Then we can verify that

$$||\mu - \mu * \omega_n|| \leq \sup_{y \in V_n} ||\mu - \mu * \delta_y|| < \frac{1}{n},$$

which, together with $\mu * \omega_n \in M_{aH}(G)$, yields $\mu \in M_{aH}(G)$. This completes the proof.

LEMMA 3. 2. *Let G be a LCA group and H a closed subgroup of G. Let G_0 be an open subgroup of G. Then, for $\mu \in L_H(G)$ with* $\mathrm{supp}(\mu) \subset G_0$, *we have* $\mu \in L_{H \cap G_0}(G_0)$.

PROOF. Let $K \subset G_0$ be a compact set that is $H \cap G_0$-null in G_0. We note that K is $H \cap G_0$-null (in G). For each $x \in G$, since $H \cap (K - x)$ is a is a compact set in H, there exist $y_1, \cdots, y_n \in H$ such that $H \cap (K-x) \subset \bigcup_{i=1}^{n} (H \cap G_0 + y_i)$. Put $J_i = H \cap (K-x) \cap (H \cap G_0 + y_i)$. Then $H \cap (K-x) = \bigcup_{i=1}^{n} J_i$. Since $J_i - y_i \subset (H \cap G_0) \cap (K - x - y_i)$, we have $m_{H \cap G_0}(J_i - y_i) = 0$, so that $m_H(J_i) = m_H(J_i - y_i) = 0$. Hence we have $m_H(H \cap (K-x)) = m_H\left(\bigcup_{i=1}^{n} J_i\right) = 0$. This shows that K is H-null, hence $\mu(K) = 0$. By regularity of μ, we can verify that $\mu(E) = 0$ for any Borel set $E \subset G_0$ which is $H \cap G_0$-null. This completes the proof.

PROPOSITION 3. 3. *Let G be a LCA group and H a closed subgroup of G. Then* $L_H(G) = M_{aH}(G)$.

PROOF. We first prove $M_{aH}(G)\subset L_H(G)$. Let $\mu\in M_{aH}(G)$. Then by (2. 1) there exist $\nu\in M^+(G)$ and $\rho\in L^1(H)\cap M^+(H)$ such that $\mu\ll\rho*\nu$. Let E be a Borel set in G that is H-null. Then

$$\begin{aligned}\rho*\nu(E) &= \int_G \rho\big((E-x)\cap H\big)\, d\nu(x)\\ &=0\,,\end{aligned}$$

which yields $\mu(E)=0$. Thus we have $m_{aH}(G)\subset L_H(G)$.

Next we prove $L_H(G)\subset M_{aH}(G)$. Let $\mu\in L_H(G)$. We may assume that μ is a positive measure because $L_H(G)$ and $M_{aH}(G)$ are L-spaces. Since μ is regular, there exists a σ-compact open subgroup G_0 of G with supp$(\mu)\subset G_0$. Put $H_0=G_0\cap H$. Then there exists a positive measure $\omega\in L^1(H)$ with supp$(\omega)\subset H_0$ such that $\omega\approx m_{H_0}$.

Clam A. $\quad \mu\ll\omega*\mu\,.$

In fact, let E be a Borel set in G with $\omega*\mu(E)=0$. We have to show $\mu(E)=0$. Since μ and $\omega*\mu$ are concentrated on G_0, we may assume that $E\subset G_0$. Then, since $\int_{G_0}\omega(E-x)\,d\mu(x)=0$, there exists a σ-compact set K in G_0 with the following properties:

(1) $\quad \mu(K^c)=0\,;$

(2) $\quad \omega(E-y)=0$ for all $y\in K$.

Put $V=H_0+K$ and $E_0=V\cap E$. Then

(3) $\quad E_0$ is H_0-null in G_0.

In fact, if $x\in G_0\backslash V$, then $(E_0-x)\cap H_0=\emptyset$. If $x=h+y\in V(h\in H_0,\ y\in K)$, then $\{t\in H_0: t+x\in E_0\}\subset\{t\in H_0: t+y\in E\}-h$. Hence (2) yields $m_{H_0}((E_0-x)\cap H_0)=0$. Thus (3) follows. On the other hand, it follows from Lemma 3. 2 that $\mu\in L_{H_0}(G_0)$. Hence by (1) and (3) we have

$$\mu(E)=\mu(V^c\cap E)+\mu(V\cap E)=0\,,$$

which shows that the claim is satisfied. By Claim A we get $\mu\in M_{ah}(G)$ and the proof is complete.

Theorem I is obtained from Propositions 3. 1 and 3. 3.

§ 4. Proof of Theorem II.

In this section, we prove Theorem II.

LEMMA 4. 1. *Let G be a LCA group. Let μ and ν be measures in $M(G)$ with $\mu\perp\nu$. Then there exists a σ-compact open subgroup Γ_0 of $\hat{G}$ such that*

(1) $\pi_{\Gamma^\perp}(\mu) \perp \pi_{\Gamma^\perp}(\nu)$

for all open subgroups Γ of $\hat{G}$ with $\Gamma \supset \Gamma_0$, where $\pi_{\Gamma^\perp}: G \mapsto G/\Gamma^\perp$ is the natural homomorphism.

PROOF. We may assume that μ and ν are positive measures without loss of generality. Since $\mu \perp \nu$, there exist compact sets $E_n \uparrow$ and $F_n \uparrow$ in G with $E_n \cap F_n = \emptyset$ such that

(2) $$\mu\left(G \setminus \bigcup_{n=1}^{\infty} E_n\right) = 0 \quad \text{and} \quad \nu\left(G \setminus \bigcup_{n=1}^{\infty} F_n\right) = 0 .$$

Then there exists a symmetric open neighborhood V_n of 0 in G such that

(3) $$(V_n + E_n) \cap (V_n + F_n) = \emptyset \qquad (n = 1, 2, 3, \cdots) .$$

Then, by the definition of compact-open topology, there exists a σ-compact open subgroup Γ_0 of $\hat{G}$ such that

(4) $$\Gamma_0^\perp \subset \bigcap_{n=1}^{\infty} V_n .$$

Put $E = \bigcup_{n=1}^{\infty} E_n$ and $F = \bigcup_{n=1}^{\infty} F_n$. Then, for each open subgroup Γ of $\hat{G}$ with $\Gamma \supset \Gamma_0$, (3) and (4) yield $(\Gamma^\perp + E) \cap (\Gamma^\perp + F) = \emptyset$. Hence $\pi_{\Gamma^\perp}(\mu)$ and $\pi_{\Gamma^\perp}(\nu)$ are mutually singular because $\pi_{\Gamma^\perp}(\mu)$ and $\pi_{\Gamma^\perp}(\nu)$ are concentrated on $\pi_{\Gamma^\perp}(E)$ and $\pi_{\Gamma^\perp}(F)$ respectively. This completes the proof.

LEMMA 4.2. *Let G be a metrizable LCA group and H a closed subgroup of G. Let p be a positive integer and $\tilde{E}$ a small p set in $\hat{G}/H^\perp$. Put $E = \pi^{-1}(\tilde{E})$, where $\pi: \hat{G} \mapsto \hat{G}/H^\perp$ is the natural homomorphism. Then for $\mu \in M_E(G)$, μ^p translates H-continuously.*

PROOF. Since μ is regular, there exists a σ-compact open subgroup G' of G with $\operatorname{supp}(\mu) \subset G'$. We define a map $\tau: (G'+H)/H \mapsto G'/G' \cap H$ by $\tau(x+H) = X + G' \cap H$ $(x \in G')$. Then τ is a topological isomorphism (cf. [5], (5.33) Theorem, p. 44). Let $\beta: G' \mapsto G'/G' \cap H$ be the natural homomorphism, and put $\eta' = \beta(|\mu|)$. Then, by the theory of disintegration (cf. [1], Théorème 1, p. 58), there exists a family $\{\lambda_{\dot{y}}\}_{\dot{y} \in G'/G' \cap H}$ of measures in $M(G')$ with the following properties:

(1) $\dot{y} \mapsto \lambda_{\dot{y}}(f)$ is a Borel measurable function for each bounded Borel function f on G';

(2) $\|\lambda_{\dot{y}}\| \leq 1$;

(3) $\operatorname{supp}(\lambda_{\dot{y}}) \subset \beta^{-1}(\dot{y})$;

(4) $$\mu(f)=\int_{G'/G'\cap H}\lambda_{\dot y}(f)\,d\eta'(\dot y) \quad \text{for each bounded Borel } f.$$

We define measures $\mu_{\dot x}\in M(G)$ $(\dot y\in G/H)$ as follows:

(5) $$\mu_{\dot x}=\begin{cases}\lambda_{\tau(\dot x)} & \text{for } \dot x\in(G'+H)/H\\ 0 & \text{for } \dot x\notin(G'+H)/H.\end{cases}$$

We define a measure $\eta\in M^+((G'+H)/H)$ by $\tau(\eta)=\eta'$, and we regard η as a measure in $M^+(G/H)$. Let α: $G\mapsto G/H$ be the natural homomorphism. Then by (1)-(4) the following are satisfied:

(6) $\dot x\mapsto\mu_{\dot x}(f)$ is a Borel measurable function for each bounded Borel function f on G;

(7) $\|\mu_{\dot x}\|\leq 1$;

(8) $\mathrm{supp}\,(\mu_{\dot x})\subset\alpha^{-1}(\dot x)\cap G'$;

(9) $$\mu(f)=\int_{G/H}\mu_{\dot x}(f)\,d\eta(\dot x) \quad \text{for each bounded Borel function } f \text{ on } G.$$

Since G' is σ-compact metrizable, there exists a countable dense set $\mathscr{A}=\{f_m\}$ in $C_0(G')$. Then by (6) and Lusin's theorem, for each natural number n, there exists a compact set $K_n\subset\mathrm{supp}\,(\eta)$ with the following properties:

(10) $\eta(K_n^c)<\frac{1}{n}$;

(11) $\dot x\mapsto\mu_{\dot x}(f_m)$ is continuous on K_n for all $f_m\in\mathscr{A}$;

(12) $\eta(V\cap K_n)>0$ for each $\dot x\in K_n$ and each neighborhood V of $\dot x$.

Since $C_0(G)|_{G'}=C_0(G')$ and $\mathscr{A}$ is dense in $C_0(G')$, it follows from (8) and (11) that

(13) $\dot x\mapsto\mu_{\dot x}(f)$ is continuous on K_n for all $f\in C_0(G)$.

Claim B. $\mu_{\dot x}\in M_E(G)$ for all $\dot x\in K_n$ $(n=1, 2, 3, \cdots)$.

In fact, let f be a function in $L^1(\hat G)$ with $\mathrm{supp}\,(f)\subset E^c$. Then

(14) $$\begin{aligned}0&=\int_{\hat G}\hat\mu(\gamma)f(\gamma)\,d\gamma\\ &=\int_G\hat f(x)\,d\mu(x)\\ &=\int_{G/H}\mu_{\dot x}(\hat f)\,d\eta(\dot x). \qquad \text{(by (9))}\end{aligned}$$

For $\gamma_*\in H^\perp$, we define $f_{\gamma_*}\in L^1(\hat G)$ by $f_{\gamma_*}(\gamma)=f(\gamma-\gamma_*)$. Then since $H^\perp+$

$E^c \subset E^c$, we have $\operatorname{supp}(f_{\gamma_*}) \subset E^c$. Hence (14) yields

$$0 = \int_{G/H} \mu_{\dot{x}}(\hat{f}_{\gamma_*})\, d\eta(\dot{x})$$
$$= \int_{G/H}\int_G (-x, \gamma_*) \hat{f}(x)\, d\mu_{\dot{x}}(x)\, d\eta(\dot{x})$$
$$= \int_{G/H} (-\dot{x}, \gamma_*) \int_G \hat{f}(x)\, d\mu_{\dot{x}}(x)\, d\eta(\dot{x}) \qquad \text{(by (8))}$$
$$= \int_{G/H} (-\dot{x}, \gamma_*)\, \mu_{\dot{x}}(\hat{f})\, d\eta(\dot{x})\,.$$

Since γ_* is an arbitrary element in $H^\perp$, we have

(15) $$0 = \int_{G/H} p(\dot{x})\, \mu_{\dot{x}}(\hat{f})\, d\eta(\dot{x}) \qquad \text{for all} \quad p \in \operatorname{trig}(G/H)\,.$$

Since $\operatorname{Trig}(G/H)$ is dense in $L^1(\eta)$ and $\dot{x} \mapsto \mu_{\dot{x}}(\hat{f})$ is a bounded Bore function, we have

$$\mu_{\dot{x}}(\hat{f}) = 0 \qquad \eta-\text{a. a. } \dot{x} \in G/H\,,$$

which, together with (13), yields

$$\int_{\hat{G}} \hat{\mu}_{\dot{x}}(\gamma) f(\gamma)\, d\gamma = \mu_{\dot{x}}(\hat{f}) = 0 \qquad \text{for all} \quad \dot{x} \in K_n$$

Since f is any function in $L^1(\hat{G})$ with $\operatorname{supp}(f) \subset E^c$, we have $\hat{\mu}_{\dot{x}} = 0$ on E^c for all $\dot{x} \in K_n$ and the claim follows.

By (10) and Claim B we have

(16) $$\mu_{\dot{x}} \in M_E(G) \qquad \eta\text{-a. a. } \dot{x} \in G/H\,.$$

On the other hand, by (8), there exist $\xi_{\dot{x}} \in M(H)$ and $x \in G$ with $\alpha(x) = \dot{x}$ such that $\mu_{\dot{x}} = \xi_{\dot{x}} * \delta_x$. Then (16) yields

$$\xi_{\dot{x}} \in M_{\tilde{E}}(H) \qquad \eta\text{-a. a. } \dot{x} \in G/H\,,$$

hence it follows from ([8], Lemma 1) that

(17) $$\xi_{\dot{x}_1} * \cdots * \xi_{\dot{x}_p} \in L^1(H) \qquad (\eta \times \cdots \times \eta)\text{-a. a. } (x_1, \cdots, x_p) \in (G/H)^p\,.$$

We note the following (cf. [8], Claims 2 and 3 in Theorem 1):

(18) $$\mu^p(f) = \int_{G/H} \cdots \int_{G/H} \mu_{\dot{x}_1} * \cdots * \mu_{\dot{x}_p}(f)\, d\eta(\dot{x}_1) \cdots d\eta(\dot{x}_p)$$

for $f \in C_0(G)$. Let $\{t_n\}$ be a sequence in H which converges to 0. Then (17) yields $\lim_{n\to\infty} \|\mu_{\dot{x}_1} * \cdots * \mu_{\dot{x}_p} - \mu_{\dot{x}_1} * \cdots * \mu_{\dot{x}_p} * \delta_{t_n}\| = 0$ $(\eta \times \cdots \times \eta)$-a. a. $(\dot{x}_1, \cdots, \dot{x}_p) \in (G/H)^p$. Hence, by (18) and Lebesgue's convergence theorem, we have

$$\begin{aligned}&\lim_{n\to\infty}\|\mu^p-\mu^p*\delta_{t_n}\|\\&\leq\lim_{n\to\infty}\int_{G/H}\cdots\int_{G/H}\|\mu_{\dot{x}_1}*\cdots*\mu_{\dot{x}_p}-\mu_{\dot{x}_1}*\cdots*\mu_{\dot{x}_p}*\delta_{t_n}\|\,d\eta(\dot{x}_1)\cdots d\eta(\dot{x}_p)\\&=0\,.\end{aligned}$$

Since H is metrizable, this shows that μ^p translates H-continuously and the proof is complete.

Now we prove Theorem II. Suppose μ^p does not translate H-continuously. Then by Theorem I we have $\mu^p\notin M_{aH}(G)$. Hence there exist $\xi_1\in M_{aH}(G)$ and a nonzero measure $\xi_2\in M_{aH}(G)^{\perp}$ such that $\mu^p=\xi_1+\xi_2$. Let ω be a positive measure in $L^1(H)$ such that $\omega(V)>0$ for all neighborhoods V of 0 in H. Then $|\xi_2|\perp\omega*|\xi_2|$. Hence by Lemma 4.1 there exists a σ-compact open subgroup Γ of $\hat{G}$ such that

$$(1)\qquad \pi_{\Gamma^{\perp}}(\xi_2)\neq 0\quad\text{and}\quad \pi_{\Gamma^{\perp}}(|\xi_2|)\perp\pi_{\Gamma^{\perp}}(\omega)*\pi_{\Gamma^{\perp}}(|\xi_2|)\,,$$

where $\pi_{\Gamma^{\perp}}: G\mapsto G/\Gamma^{\perp}$ is the natural homomorphism. Put $\tilde{H}=\pi_{\Gamma^{\perp}}(H)$ and $\tilde{G}=\pi_{\Gamma^{\perp}}(G)$. Then, since $\Gamma^{\perp}$ is compact, $\tilde{H}$ is a closed subgroup of $\tilde{G}$.

Claim 1. $\qquad \pi_{\Gamma^{\perp}}\big(M_{aH}(G)\big)\subset M_{a\tilde{H}}(\tilde{G})\,.$

In fact, the claim follows from the fact that $\pi_{\Gamma^{\perp}}(L^1(H))\subset L^1(\tilde{H})$.

Claim 2. $\qquad \pi_{\Gamma^{\perp}}(\mu^p)\notin M_{a\tilde{H}}(\tilde{G})\,.$

By Claim 1, it is sufficient to prove that $\pi_{\Gamma^{\perp}}(\xi_2)\notin M_{a\tilde{H}}(\tilde{G})$. Suppose $\pi_{\Gamma^{\perp}}(\xi_2)\in M_{a\tilde{H}}(\tilde{G})$. Then, by Theorem I, $|\pi_{\Gamma^{\perp}}(\xi_2)|$ translates $\tilde{H}$-continuously. On the other hand, by the choice of ω, we note that $\pi_{\Gamma^{\perp}}(\omega)(\tilde{V})>0$ for any neighborhood $\tilde{V}$ of 0 in $\tilde{H}$. Hence we have

$$\left|\pi_{\Gamma^{\perp}}(\xi_2)\right|\ll\pi_{\Gamma^{\perp}}(\omega)*\left|\pi_{\Gamma^{\perp}}(\xi_2)\right|,$$

which contradicts (1). Thus the claim follows.

Put $E_0=E\cap\Gamma$. Let $\alpha: \Gamma+H^{\perp}\to(\Gamma+H^{\perp})/H^{\perp}$ and $\beta: \Gamma\to\Gamma/\Gamma\cap H^{\perp}$ be the natural homomorphisms, and let $\tau: (\Gamma+H^{\perp})/H^{\perp}\to\Gamma/\Gamma\cap H^{\perp}$ be a topological isomorphism given by $\tau(\gamma+H^{\perp})=\gamma+\Gamma\cap H^{\perp}$ for $\gamma\in\Gamma$ (cf. [5], (5.33) Theorem, p. 44). We note $\tau\circ\alpha|_{\Gamma}=\beta$.

Claim 3. $\qquad \beta(E_0)$ is a small p set in $\Gamma/\Gamma\cap H^{\perp}$.

In fact, since $\beta^{-1}(\beta(E_0))=E_0+\Gamma\cap H^{\perp}=E_0$, $\beta(E_0)$ is a closed set, hence $\alpha(E_0)=\tau^{-1}(\beta(E_0))$ is also closed. Thus, since $\alpha(E_0)\subset\pi(E)=\tilde{E}$, $\alpha(E_0)$ is a small p set in $(\Gamma+H^{\perp})/H^{\perp}$. Hence $\beta(E_0)=\tau(\alpha(E_0))$ is a small p set in $\Gamma/\Gamma\cap H^{\perp}$, and the claim follows.

We note that $\tilde{G}$ is metrizable and the annihilator of $\Gamma \cap H^{\perp}$ in $\tilde{G}$ coincides with $\tilde{H}$. Thus, since $\pi_{\Gamma^{\perp}}(\mu) \in M_{E_0}(\tilde{G})$ and $E_0 = \beta^{-1}(\beta(E_0))$, it follows from Claim 3 and Lemma 4. 2 that $\pi_{\Gamma^{\perp}}(\mu^p) = \pi_{\Gamma^{\perp}}(\mu)^p$ translates $\tilde{H}$-continuously. This contradicts Claim 2 and Theorem I, and the proof is complete.

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